Some new results on the right units of special inverse monoids

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Remark

$M = \mathrm{Inv}\langle A \mid w_i = 1 \ (i \in I)\rangle$ is E-unitary
$\Longrightarrow \mathrm{RU}(M) \cong$ the prefix monoid of $\mathrm{Gp}\langle A \mid w_i = 1 \ (i \in I)\rangle$.

What did we know thus far? (1)

Theorem (IgD, RDG, 2023):
For every group-embeddable recursively presented monoid M there is a natural number μ_M such that

$$M * \Sigma_k^*$$

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Let $\mathcal{RU}$ be the class of all RU-monoids (of f.p. SIMs).

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Theorem (IgD, RDG, 2023):
Every finitely RC-presented monoid is an RU-monoid.

Hence, $\mathcal{RU} \not\subseteq \mathcal{P}$.

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Analogous results hold for monoids and inverse monoids. However, at present there is no HET for RC-presentations, and so we don't know if the containment $\mathcal{RC}_1 \subseteq \mathcal{RC}_2$ is proper or an equality.

A result on free products as RU-monoids

Theorem (IgD, RDG, 2025):
M – a f.p. SIM, U – the group of units of M. If

$$\mathrm{RU}(M) \cong U * T$$

for a f.g. monoid T with a trivial group of units
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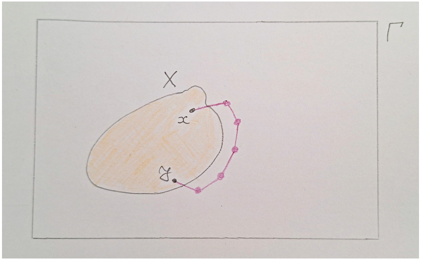
- $\mathcal{RC}_1 \not\subseteq \mathcal{RU}$.
- $\mathcal{RU}$ is a proper subclass of $\mathcal{RC}_2$.

Boundary width and ball covers in graphs

Boundary pair:



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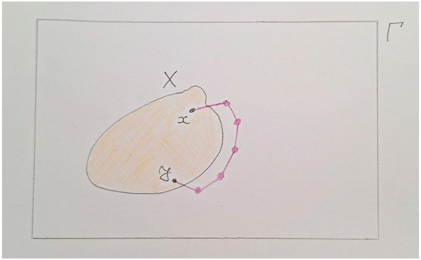

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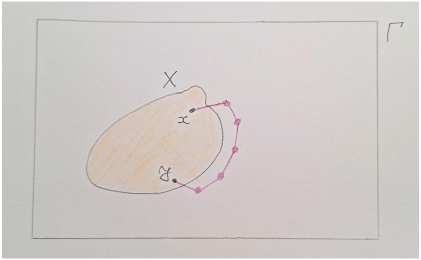

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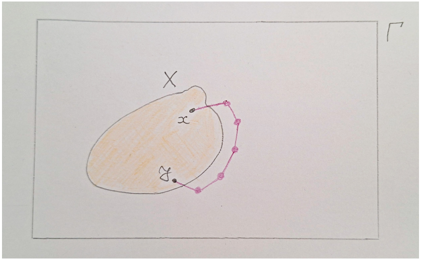

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Finite ball cover of finite boundary width: $\exists r \geq 0$ such that Δ_r has finite boundary width.

Graphs as metric spaces: a bit of geometry

(λ, ϵ, μ)-quasi-isometry $f : (X, d) \to (X', d')$ ($\lambda \geq 1$, $\epsilon, \mu \geq 0$):

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- *Δ has a connected finite ball cover in $\Gamma \implies f(\Delta) \subseteq \Gamma'$ has a connected finite ball cover in Γ'.*

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H is f.p. $\iff$ H is f.g. and $\mathrm{Rips}_r(\Delta)$ is simply connected for large enough r.

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This then applies to our free product result, as U has finite boundary width in $U * T$. 😊

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Method: The "generalised Gray-Kambites" construction! ❤️

Adapting the GK for right cancellative monoids (1)

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$M_{S,T}$ – the f.p. SIM presented by $\Sigma = A \cup \{p_0, p_1, \ldots, p_k, z, d\}$ &

$$p_i a p_i^{-1} p_i a^{-1} p_i^{-1} = 1 \qquad (a \in A,\ i = 0, 1, \ldots, k)$$

$$p_i u_i d^{-1} v_i^{-1} p_i^{-1} = 1 \qquad (i = 1, \ldots, k)$$

$$p_0 d p_0^{-1} = 1$$

$$zbz^{-1}zb^{-1}z^{-1} = 1 \qquad (b \in B)$$

$$z \left(\prod_{i=0}^{k} p_i^{-1} p_i \right) z^{-1} = 1.$$

Adapting the GK for right cancellative monoids (1)

Let $S = \mathrm{MonRC}\langle A \mid u_i = v_i \ (1 \leq i \leq k)\rangle$ and
$T = \langle B \rangle$, $B \subseteq A$ – a f.g. submonoid ($T \in \mathcal{RC}_1$).

$M_{S,T}$ – the f.p. SIM presented by $\Sigma = A \cup \{p_0, p_1, \dots, p_k, z, d\}$ &

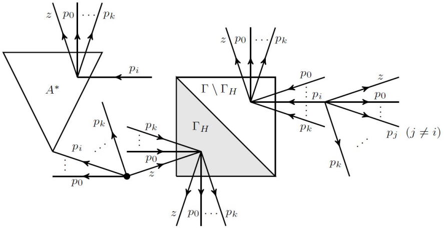
$$p_i a p_i^{-1} p_i a^{-1} p_i^{-1} = 1 \qquad (a \in A,\ i = 0, 1, \dots, k)$$

$$p_i u_i d^{-1} v_i^{-1} p_i^{-1} = 1 \qquad (i = 1, \dots, k)$$

$$p_0 d p_0^{-1} = 1$$

$$z b z^{-1} z b^{-1} z^{-1} = 1 \qquad (b \in B)$$

$$z \left(\prod_{i=0}^{k} p_i^{-1} p_i \right) z^{-1} = 1.$$



Adapting the GK for right cancellative monoids (2)

Theorem (IgD, RDG, 2025)

$\mathrm{RU}(M_{S,T})$ *is presented by generators:* p_i, q_i *(*$0 \leq i \leq k$*),* $a^{(i)}$ *(*$a \in A$*,* $0 \leq i \leq k$*),* $b^{(z)}$ *(*$b \in B$*), and relations:*

$$q_i w^{(i)} p_i = q_0 w^{(0)} p_0 \qquad (w \in A^*,\ 1 \leq i \leq k)$$

$$q_i u^{(i)} = q_i v^{(i)} \qquad (u, v \in A^* \text{ s.t. } u = v \text{ holds in } S, 0 \leq i \leq k)$$

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Corollary

The class $\mathcal{RU}$ *includes r.c. monoids that are not finitely RC-presented (and even have a trivial group of units).*

The Gray-Ruškuc construction (1)

$Q = \{r_i : \ i \in I\}, W = \{w_j : \ 1 \leq\leq k\} \subseteq (A \cup A^{-1})^*$,
$K_Q = \mathrm{Mon}\langle A \cup A^{-1} \mid r_i = 1, \ (i \in I)\rangle$ – a group,
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$M_{Q,W}$ – the (E-unitary) SIM presented by $A \cup \{t\}$ and

$$\begin{aligned} r_i &= 1 & (i \in I), \\ a_p a_p^{-1} = a_p^{-1} a_p &= 1 & (a_p \in A), \\ t w_j t^{-1} t w_j^{-1} t^{-1} &= 1 & (1 \leq j \leq k). \end{aligned}$$

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Theorem (IgD, RDG, 2025)

Let B be disjoint from $A \cup A^{-1}$, $|B| = |W|$. Then

$$\mathrm{RU}(M_{Q,W}) = \mathrm{MonRC}\langle A \cup A^{-1}, B, t \mid r_i = 1 \ (i \in I), t w_j = b_j t \ (1 \leq j \leq k)\rangle$$

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(Otto-Pride extensions...)

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- If K_Q is finitely presented $\Longrightarrow$ $\mathrm{RU}(M_{Q,W})$ is finitely RC-presented,
- So, there exists an E-unitary f.p. SIM such that its monoid of right units is finitely RC-presented but not finitely presented as a monoid (because either K_Q or T_W not f.p. $\Rightarrow$ $\mathrm{RU}(M_{Q,W})$ not f.p. as a monoid).

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- If K_Q is finitely presented $\Longrightarrow$ RU($M_{Q,W}$) is finitely RC-presented,
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- There is a finitely RC-presented monoid whose group of units is not f.p.

The Gray-Ruškuc construction (2)

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- If K_Q is finitely presented $\Longrightarrow$ $\mathrm{RU}(M_{Q,W})$ is finitely RC-presented,
- So, there exists an E-unitary f.p. SIM such that its monoid of right units is finitely RC-presented but not finitely presented as a monoid (because either K_Q or T_W not f.p. $\Rightarrow$ $\mathrm{RU}(M_{Q,W})$ not f.p. as a monoid).
- There is a finitely RC-presented monoid whose group of units is not f.p. (even though the complement of the group of units is an ideal).

Thank you! 😊 ❤️