Some new results on the right units of special inverse monoids

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Why right unit monoids?

M an (inverse) monoid: $a \in M$ is right invertible (or a right unit) if $ax = 1$ for some $x \in M$. In inverse monoids: $aa^{-1} = 1$.

Right units form a right cancellative submonoid $\mathrm{RU}(M)$ of M: $ac = bc \;\Rightarrow\; a = acc^{-1} = bcc^{-1} = b$.

Membership problem of $\mathrm{RU}(M)$ in M undecidable $\Longrightarrow$ the word problem of M undecidable. This is exactly how Gray (2020) constructed a 1-relator special inverse monoid $\mathrm{Inv}\langle A \,|\, w = 1\rangle$ with undecidable WP. (A stark contrast to groups (Magnus, 1932) and ordinary special monoids (Adyan, 1966).)

Remark

$M = \mathrm{Inv}\langle A \,|\, w_i = 1\ (i \in I)\rangle$ is E-unitary
$\Longrightarrow \mathrm{RU}(M) \cong$ the prefix monoid of $\mathrm{Gp}\langle A \,|\, w_i = 1\ (i \in I)\rangle$.

What did we know thus far? (1)

Theorem (IgD, RDG, 2023):
For every group-embeddable recursively presented monoid M there is a natural number μ_M such that

$$M * \Sigma_k^*$$

arises as a prefix monoid (with $|\Sigma_k| = k$) if and only if $k \geq \mu_M$.

- If M is group-embeddable and finitely presented $\Longrightarrow \mu_M = 0$.
- If M is a group and $\mu_M = 0 \Longrightarrow M$ is finitely presented.

Let $\mathcal{P}$ be the class of all prefix monoids (of f.p. groups).

Let $\mathcal{RU}$ be the class of all RU-monoids (of f.p. SIMs).

What did we know thus far? (2)

Fact

- *Every RU-monoid is recursively presented (as a monoid).*
- *If a group arises as an RU-monoid $\Longrightarrow$ it is finitely presented.*

RC-presentations:

$$M = \mathrm{MonRC}\langle A \,|\, \mathfrak{R} \rangle$$

$\Leftrightarrow M \cong A^*/\mathfrak{R}^{\mathrm{RC}}$, where $\mathfrak{R}^{\mathrm{RC}}$ is the intersection of all congruences ρ of A^* such that

- $\mathfrak{R} \subseteq \rho$,
- A^*/ρ is right cancellative.

Theorem (IgD, RDG, 2023):
Every finitely RC-presented monoid is an RU-monoid.

Hence, $\mathcal{RU} \not\subseteq \mathcal{P}$.

Classes of right cancellative monoids

$\mathcal{RC}_1$ = the class of finitely generated submonoids of finitely RC-presented (r.c.) monoids

$\mathcal{RC}_2$ = the class of recursively RC-presented monoids
= the class of recursively presented monoids that happen to be right cancellative

Remark
$\mathcal{RU} \subseteq \mathcal{RC}_2$.

Remark
By the Higman Embedding Theorem,

f.g. subgroups of f.p. groups = recursively presented groups.

Analogous results hold for monoids and inverse monoids. However, at present there is no HET for RC-presentations, and so we don't know if the containment $\mathcal{RC}_1 \subseteq \mathcal{RC}_2$ is proper or an equality.

A result on free products as RU-monoids

Theorem (IgD, RDG, 2025):
M – a f.p. SIM, U – the group of units of M. If

$$\mathrm{RU}(M) \cong U * T$$

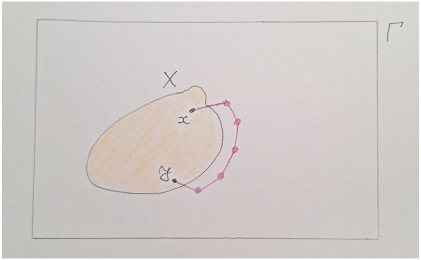
for a f.g. monoid T with a trivial group of units
$\Longrightarrow$ U is finitely presented.

Consequences:

- $\mathrm{RU}(M) \cong U_M * X^*$ for a finite $X \Longrightarrow U_M$ (and $\mathrm{RU}(M)$) is f.p.
- G a f.g. group that is not f.p. $\Longrightarrow G * X^* \notin \mathcal{RU}$ ($\forall$ finite X).
- $\mathcal{P} \nsubseteq \mathcal{RU}$.
- $\mathcal{RC}_1 \nsubseteq \mathcal{RU}$.
- $\mathcal{RU}$ is a proper subclass of $\mathcal{RC}_2$.

Boundary width and ball covers in graphs

Boundary pair:



Boundary width:

$$\beta(X) = \sup\{d(x,y) : (x,y) \text{ is a boundary pair in } X\}$$

Ball cover of $\Delta \subseteq V(\Gamma)$: $\Delta_r = \bigcup_{v\in\Delta} \mathcal{B}_r(v)$

Finite ball cover of finite boundary width: $\exists r \geq 0$ such that Δ_r has finite boundary width.

Graphs as metric spaces: a bit of geometry

(λ, ϵ, μ)-quasi-isometry $f : (X, d) \to (X', d')$ ($\lambda \geq 1$, $\epsilon, \mu \geq 0$):

$$\frac{1}{\lambda} d(x, y) - \epsilon \leq d'(f(x), f(y)) \leq \lambda d(x, y) + \epsilon$$

and $f(X) \subseteq X'$ is μ-quasi-dense.

Lemma (IgD, RDG, 2025)

$f : \Gamma \to \Gamma'$ – a quasi-isometry between graphs, $\Delta \subseteq V(\Gamma)$.

- *Δ has a finite ball cover with finite boundary width in $\Gamma \implies f(\Delta)$ has a finite ball cover with finite boundary width in Γ'.*
- *Δ has a connected finite ball cover in $\Gamma \implies f(\Delta) \subseteq \Gamma'$ has a connected finite ball cover in Γ'.*

How does this all apply in inverse monoids? (1)

$S = \langle A \rangle$ – a f.g. inverse monoid, $H \leq S$ – a subgroup

A finite cover of H: a union $\Delta = \bigcup_{i \in F} H_i \supseteq H$ of finitely many right cosets of H

Δ has finite boundary width = Δ has FBW in $S\Gamma_A(R)$, where R the $\mathscr{R}$-class containing H (connected...)

Theorem (IgD, RDG, 2025)

M – a f.g. inverse monoid, $H \subseteq M$ – a subgroup of M,
Γ – the Schützenberger graph containing H.
Then H is f.g. $\iff$ H admits a finite connected cover.
In this case the graph induced by Δ is quasi-isometric to the Cayley graph of the group H.
H is f.p. $\iff$ H is f.g. and $\mathrm{Rips}_r(\Delta)$ is simply connected for large enough r.

How does this all apply in inverse monoids? (2)

Theorem (IgD, RDG, 2025)

S – a f.g. inverse monoid, H – a subgroup of S which has a finite cover with finite boundary width. Then H is finitely generated. Moreover, if S is f.p. (as an inv. monoid) $\Longrightarrow$ the group H is f.p.

This then applies to our free product result, as U has finite boundary width in $U * T$. 😊

$\mathcal{RC}_1$ and $\mathcal{RU}$ – revisited

$\mathcal{RC}_1 \not\subseteq \mathcal{RU}$ but...

Theorem (IgD, RDG, 2025)

For any $T \in \mathcal{RC}_1$ there exists a f.p. SIM M such that $\mathrm{RU}(M)$ has a submonoid containing the group of units of M (which is also the group of units of $\mathrm{RU}(M)$) that is isomorphic to T.

...$\mathcal{RC}_1$ is "dense" in $\mathcal{RU}$

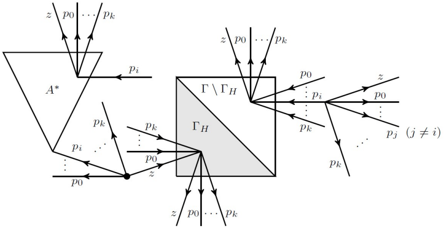
Method: The "generalised Gray-Kambites" construction! ❤️

Adapting the GK for right cancellative monoids (1)

Let $S = \mathrm{MonRC}\langle A \mid u_i = v_i \ (1 \leq i \leq k)\rangle$ and
$T = \langle B\rangle$, $B \subseteq A$ – a f.g. submonoid ($T \in \mathcal{RC}_1$).

$M_{S,T}$ – the f.p. SIM presented by $\Sigma = A \cup \{p_0, p_1, \dots, p_k, z, d\}$ &

$$
\begin{aligned}
&p_i a p_i^{-1} p_i a^{-1} p_i^{-1} = 1 && (a \in A,\ i = 0, 1, \dots, k)\\
&p_i u_i d^{-1} v_i^{-1} p_i^{-1} = 1 && (i = 1, \dots, k)\\
&p_0 d p_0^{-1} = 1 &&\\
&z b z^{-1} z b^{-1} z^{-1} = 1 && (b \in B)\\
&z \left(\prod_{i=0}^{k} p_i^{-1} p_i \right) z^{-1} = 1. &&
\end{aligned}
$$



Adapting the GK for right cancellative monoids (2)

Theorem (IgD, RDG, 2025)

$\mathrm{RU}(M_{S,T})$ *is presented by generators:* p_i, q_i *(*$0 \leq i \leq k$*),*
$a^{(i)}$ *(*$a \in A$*,* $0 \leq i \leq k$*),* $b^{(z)}$ *(*$b \in B$*), and relations:*

$$q_i w^{(i)} p_i = q_0 w^{(0)} p_0 \qquad (w \in A^*,\ 1 \leq i \leq k)$$

$$q_i u^{(i)} = q_i v^{(i)} \qquad (u, v \in A^* \text{ s.t. } u = v \text{ holds in } S, 0 \leq i \leq k)$$

$$q_i b^{(i)} = b^{(z)} q_i \qquad (b \in B,\ 0 \leq i \leq k)$$

and $T \hookrightarrow \mathrm{RU}(M_{S,T})$ *such that (the image of)* T *contains the group of units of the latter.*

Corollary

The class $\mathcal{RU}$ *includes r.c. monoids that are not finitely RC-presented (and even have a trivial group of units).*

The Gray-Ruškuc construction (1)

$Q = \{r_i : \ i \in I\}, W = \{w_j : \ 1 \leq\leq k\} \subseteq (A \cup A^{-1})^*$,
$K_Q = \mathrm{Mon}\langle A \cup A^{-1} \mid r_i = 1, \ (i \in I)\rangle$ – a group,
$T_W = \langle W \rangle \leq K_Q$ – a submonoid

$M_{Q,W}$ – the (E-unitary) SIM presented by $A \cup \{t\}$ and

$$
\begin{aligned}
r_i &= 1 && (i \in I),\\
a_p a_p^{-1} &= a_p^{-1} a_p = 1 && (a_p \in A),\\
t w_j t^{-1} t w_j^{-1} t^{-1} &= 1 && (1 \leq j \leq k).
\end{aligned}
$$

Theorem (IgD, RDG, 2025)

Let B be disjoint from $A \cup A^{-1}$, $|B| = |W|$. Then

$$\mathrm{RU}(M_{Q,W}) = \mathrm{MonRC}\langle A \cup A^{-1}, B, t \mid r_i = 1 \ (i \in I), tw_j = b_j t \ (1 \leq j \leq k)\rangle$$

(Otto-Pride extensions...)

The Gray-Ruškuc construction (2)

Some consequences:

- If K_Q is finitely presented $\Longrightarrow$ $\mathrm{RU}(M_{Q,W})$ is finitely RC-presented,
- So, there exists an E-unitary f.p. SIM such that its monoid of right units is finitely RC-presented but not finitely presented as a monoid (because either K_Q or T_W not f.p. $\Rightarrow$ $\mathrm{RU}(M_{Q,W})$ not f.p. as a monoid).
- There is a finitely RC-presented monoid whose group of units is not f.p. (even though the complement of the group of units is an ideal).

Thank you! 😊 ❤️