A small retrospective of my collaboration with Misha Volkov

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Algebra and Its Role in Computer Science

A tribute to Mikhail V. Volkov on his 70th birthday

Lisbon, Portugal, 26 June 2025

Kovačević winery, Irig, Serbia, August 2009

Temerin, Serbia, August 2009

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$\mathcal{K}$ – a class of similar algebraic structures, Σ – a set of identities $\longrightarrow$ $\mathcal{K}$ is an equational class.

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If $\mathcal{V}(\mathbf{A}) = \mathrm{Mod}(\Sigma)$ for a set of identities Σ then Σ is the equational basis of $\mathbf{A}$. The FBP asks for an algebra $\mathbf{A}$ (usually but not necessarily finite) if it has a finite (equational) basis.

Some classical positive results

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- algebras generating congruence $\wedge$-semidistributive varieties with a finite residual bound (Willard, 2000)
- ...

Some negative results

Examples of finite NFB algebras:

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- a certain 7-element semiring of binary relations (IgD, 2007)

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The Tarski-Sapir problem: Is there an algorithm to decide whether a finite semigroup is FB? This problem is still open.

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- *matrix semigroups* $\mathcal{M}_n(\mathbb{F})$ *for any* $n \geq 2$ *and any finite field* $\mathbb{F}$

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Examples

- groups
- inverse semigroups
- regular *-semigroups ($xx^*x = x$)
- matrix semigroups with transposition $\mathcal{M}_n(\mathbb{F}) = (\mathrm{M}_n(\mathbb{F}), \cdot, ^{\mathrm{T}})$

"Unary version" of Volkov's Theorem (1)

For a unary semigroup S, let $H(S)$ denote the Hermitian subsemigroup of S, generated by aa^* for all $a \in S$.

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Furthermore, let K_3 be the 10-element unary Rees matrix semigroup over a trivial group $E = \{1\}$ with the sandwich matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

while $(i, 1, j)^* = (j, 1, i)$ and $0^* = 0$.

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Fact

K_3 generates the variety of all strict combinatorial regular *-semigroups (studied by K.Auinger in 1992).

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Theorem (K.Auinger, M.V.Volkov – Oberwolfach, 1991)

Let S be a unary semigroup such that $\mathcal{V}(S)$ contains K_3.
If there exist a group G which belongs to $\mathcal{V}$ but not to $H(\mathcal{V})$
$\Longrightarrow$ *S is NFB.*

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- *matrix semigroups $(\mathrm{M}_2(\mathbb{F}), \cdot, ^{\dagger})$, where $\mathbb{F}$ is either a finite field such that $|\mathbb{F}| \equiv 3 \pmod 4$, or a subfield of $\mathbb{C}$ closed under complex conjugation, and † is the unary operation of taking the Moore-Penrose inverse.*

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Also, the following open problem was both intriguing and inviting.

Problem

Do finite INFB involution semigroups exist at all?

What the... INFB?

An algebra A is inherently nonfinitely based (INFB) if:

- $\mathcal{V}(A)$ is locally finite, and
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Therefore, problems concerning INFB algebras are in fact Burnside-type problems.

INFB algebras are a powerful tool for proving the NFB property; namely, the INFB property is "contagious":

> if $\mathcal{V}(A)$ is locally finite and contains an INFB algebra B, then A is (I)NFB.

Finite INFB semigroups: a success story

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Theorem (Sapir, 1987)

Let S be a finite semigroup. Then

$$S \text{ is INFB} \iff S \text{ does not satisfy } Z_n = W$$

for all $n \geq 1$ and all words $W \neq Z_n$.

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Sapir also found an effective structural description of finite INFB semigroups, thus proving

Theorem (Sapir, 1987)

It is decidable whether a finite semigroup is INFB or not.

Examples of finite INFB semigroups

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The same argument applies to $\mathcal{T}_n$ ($n \geq 3$), $\mathcal{R}_n$ ($n \geq 2$), $\mathcal{PT}_n$ ($n \geq 2$),...

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So, once again:

Problem

Do finite INFB involution semigroups exist at all?

An INFB criterion for involution semigroups

Yes!

An INFB criterion for involution semigroups

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Theorem (IgD, cca. Spring 2008)

Let S be an involution semigroup such that $\mathcal{V}(S)$ is locally finite. If S fails to satisfy any nontrivial identity of the form

$$Z_n = W,$$

where W is an involutorial word (a word over the "doubled" alphabet $X \cup X^*$), then S is INFB.

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Great Igor, but... how about a (finite) example?

“C’mon baby, let’s do the twist...!”

Rescue: Luckily, B_2^1 admits one more involution aside from the inverse one: define the nilpotents a, b (and, of course, $0, 1$) to be fixed by *, which results in $(ab)^* = ba$ and $(ba)^* = ab$.

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Remark

Analogously, one can also define TA_2^1, the “involutorial version” of A_2^1, which is also INFB.

One thing led to another...

K.Auinger, IgD, M.V.Volkov, Matrix identities involving multiplication and transposition, *Journal of the European Mathematical Society* **14** (2012), 937–969.

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- $\mathcal{M}_n(\mathbb{F})$ for all $n \geq 3$ and all finite fields $\mathbb{F}$.

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So, what about $\mathcal{M}_2(\mathbb{F})$ if $|\mathbb{F}| \equiv 3 \pmod{4}$?
(We already know it is NFB.)

Non-INFB results (IgD, 2010)

Theorem

Let S be a finite involution semigroup satisfying a nontrivial identity of the form $Z_n = W$ such that $B_2^1 \notin \mathcal{V}(S)$.
Then S is not INFB.

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Corollary

No finite regular *-semigroup is INFB.
(Namely, $x = x(x^*x)$ holds.)

Power involution semigroups

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For any finite group G, the involution semigroup of subsets $\mathcal{P}_G^* = (\mathcal{P}(G), \cdot, ^*)$ is not INFB.

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The ordinary power semigroup $\mathcal{P}_G = (\mathcal{P}(G), \cdot)$ is INFB if and only if G is not Dedekind.

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Theorem (IgD, Gusev, Volkov, 2024)

If S is an inverse semigroup that is either not a semilattice of groups or all subgroups are solvable and at least one is not Dedekind $\Longrightarrow$ $\mathcal{P}_S^*$ is NFB.

The (I)NFB problem for matrix involution semigroups

Two facts:

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- the involution semigroup of 2×2 matrices over a finite field $\mathbb{F}$ with transposition admits a Moore-Penrose inverse if and only if $|\mathbb{F}| \equiv 3 \pmod 4$.

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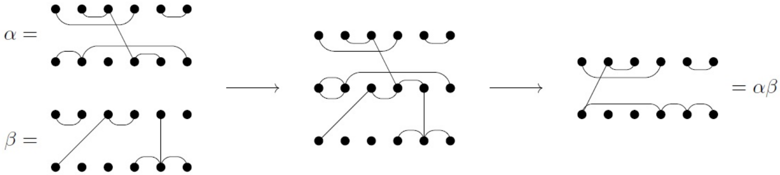
Theorem (Auinger, IgD, Volkov, 2012)

Let $n \geq 2$ and $\mathbb{F}$ be a finite field. Then

(1) $\mathcal{M}_n(\mathbb{F})$ *is not finitely based;*

(2) $\mathcal{M}_n(\mathbb{F})$ *is INFB if and only if either $n \geq 3$, or $n = 2$ and $|\mathbb{F}| \not\equiv 3 \pmod 4$.*

Applying our results to diagram monoids



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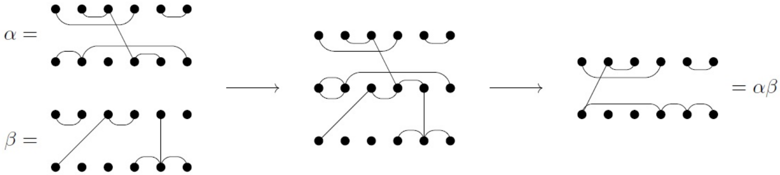

The following regular *-semigroups are NFB:

- the partition monoids $\mathcal{P}_n$ for $n \geq 2$;
- the Brauer monoids $\mathcal{B}_n$ for $n \geq 4$;
- the partial Brauer monoids $\mathcal{PB}_n$ for $n \geq 3$;
- the annular monoids $\mathcal{A}_n$ for $n \geq 4$, n even or a prime power;
- the partial annular monoids $\mathcal{PA}_n$ for $n \in \{2^k + 2, p^k, p^k + 1\}$, p prime, $k \geq 1$.

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TFAE for a regular finite semigroup with involution S*:*

1. S *is INFB,*
2. $(S, \cdot)$ *is INFB and* $TSL_3 \in \mathcal{V}(S)$*,*
3. S *fails to satisfy a nontrivial identity of the form* $Z_n = W$*.*

I hope Andy is down there.

I hope I can make it across the border.

I hope to see my friend and shake his hand.

I hope the Pacific is as blue as it has been in my dreams.

I hope.

Stephen King: Rita Hayworth and the Shawshank Redemption

Thank you! 😊 ❤️