A small retrospective of my collaboration with Misha Volkov

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Algebra and Its Role in Computer Science

A tribute to Mikhail V. Volkov on his 70th birthday

Lisbon, Portugal, 26 June 2025

Kovačević winery, Irig, Serbia, August 2009

Temerin, Serbia, August 2009

The finite basis problem (1)

A (very) large proportion of Misha's work in algebra in devoted to the Finite Basis Problem (FBP), particularly for finite semigroups and similar structures.

Let us briefly explain the concept.

Let $\mathcal{K}$ be a class of first-order structures of a given similarity type. $\mathcal{K}$ is axiomatisable if there is a set of fomulæ Σ such that $\mathcal{K} = \mathrm{Mod}(\Sigma)$.

$\mathcal{K}$ – a class of similar algebraic structures, Σ – a set of identities $\longrightarrow$ $\mathcal{K}$ is an equational class.

The finite basis problem (2)

A *variety* is a class of algebras closed for taking (a) homomorphic images, (b) subalgebras, and (c) direct products.

Theorem (Birkhoff)

A class of algebraic structures $\mathcal{V}$ is a variety if and only if it is an equational class.

$\mathcal{V}(\mathbf{A})$ – the smallest variety containing the algebra $\mathbf{A}$

If $\mathcal{V}(\mathbf{A}) = \mathrm{Mod}(\Sigma)$ for a set of identities Σ then Σ is the equational basis of $\mathbf{A}$. The FBP asks for an algebra $\mathbf{A}$ (usually but not necessarily finite) if it has a finite (equational) basis.

Some classical positive results

Each of the following algebras is FB:

- finite groups (Oates & Powell, 1964)
- commutative semigroups (Perkins, 1968)
- finite lattices and lattice-based algebras (McKenzie, 1970)
- finite (associative) rings (L'vov; Kruse, 1973)
- algebras generating congruence distributive varieties with a finite residual bound (Baker, 1977)
- algebras generating congruence modular varieties with a finite residual bound (McKenzie, 1987)
- algebras generating congruence $\wedge$-semidistributive varieties with a finite residual bound (Willard, 2000)
- ...

Some negative results

Examples of finite NFB algebras:

	0	1	2
0	0	0	0
1	0	0	1
2	0	2	2

  (Murskiĭ, 1965)
- the 6-element Brandt inverse monoid
  $$B_2^1 = \langle a, b :\ a^2 = b^2 = 0,\ aba = a,\ bab = b \rangle \cup \{1\}.$$
  (Perkins, 1968)
- a certain finite *pointed* group (Bryant, 1982)
- the full transformation semigroup $\mathcal{T}_n$ for $n \geq 3$ and the full semigroup of binary relations $\mathcal{R}_n$ for $n \geq 2$
- a certain 7-element semiring of binary relations (IgD, 2007)

Algorithmic decidability?

Tarski's Finite Basis Problem: Is there any algorithmic way to distinguish between finite FB and NFB algebras?

No!

Theorem (McKenzie, 1996)

There is no algorithm to decide whether a finite algebra is FB.

This is exactly why it is so interesting to study the (N)FB property, especially for finite algebras.

The Tarski-Sapir problem: Is there an algorithm to decide whether a finite semigroup is FB? This problem is still open.

So, what can we do?

Theorem (M.V.Volkov, 1989)

Let S be a semigroup and T a subsemigroup of S. Assume that there exist a positive integer d and a group G satisfying $x^d = e$ such that

- $a^d \in T$ *for all* $a \in S$, *and*
- $G \in \mathcal{V}(S)$, *but* $G \notin \mathcal{V}(T)$.

$A_2 = \langle a, b : \ a^2 = a = aba, \ b^2 = 0, \ bab = b \rangle \in \mathcal{V}(S) \Rightarrow$ *S is NFB.*

Corollary

The following semigroups are NFB:

- *the full transformation semigroup* $\mathcal{T}_n$ $(n \geq 3)$
- *the full semigroup of binary relations* $\mathcal{B}_n$ $(n \geq 2)$
- *the semigroup of partial transformations* $\mathcal{PT}_n$ $(n \geq 2)$
- *matrix semigroups* $\mathcal{M}_n(\mathbb{F})$ *for any* $n \geq 2$ *and any finite field* $\mathbb{F}$

Unary semigroups

Unary semigroup: a structure $(S,\cdot,^*)$ such that $(S,\cdot)$ is a semigroup and * is a unary operation on S

Involution semigroup: a unary semigroup satisfying $(xy)^* = y^*x^*$ and $(x^*)^* = x$ ("socks and shoes")

Examples

- groups
- inverse semigroups
- regular *-semigroups ($xx^*x = x$)
- matrix semigroups with transposition $\mathcal{M}_n(\mathbb{F}) = (\mathrm{M}_n(\mathbb{F}),\cdot,^{\mathrm{T}})$

“Unary version” of Volkov’s Theorem (1)

For a unary semigroup S, let $H(S)$ denote the Hermitian subsemigroup of S, generated by aa^* for all $a \in S$.

For a variety $\mathcal{V}$ of unary semigroups, let $H(\mathcal{V})$ be the subvariety of $\mathcal{V}$ generated by all $H(S)$, $S \in \mathcal{V}$.

Furthermore, let K_3 be the 10-element unary Rees matrix semigroup over a trivial group $E = \{1\}$ with the sandwich matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

while $(i, 1, j)^* = (j, 1, i)$ and $0^* = 0$.

Fact

K_3 generates the variety of all strict combinatorial regular *-semigroups (studied by K.Auinger in 1992).

“Unary version” of Volkov’s Theorem (2)

Theorem (K.Auinger, M.V.Volkov – Oberwolfach, 1991)

Let S be a unary semigroup such that $\mathcal{V}(S)$ contains K_3.
If there exist a group G which belongs to $\mathcal{V}$ but not to $H(\mathcal{V})$
$\Longrightarrow$ *S is NFB.*

Corollary

The following unary semigroups are NFB:

- *the full involution semigroup of binary relations $\mathcal{R}_n^{\vee}$ ($n \geq 2$), endowed with relational converse*
- *matrix semigroups with transposition $\mathcal{M}_n(\mathbb{F})$, where $\mathbb{F}$ is a finite field, $|\mathbb{F}| \geq 3$*
- *matrix semigroups $(\mathrm{M}_2(\mathbb{F}), \cdot, ^\dagger)$, where $\mathbb{F}$ is either a finite field such that $|\mathbb{F}| \equiv 3 \pmod 4$, or a subfield of $\mathbb{C}$ closed under complex conjugation, and † is the unary operation of taking the Moore-Penrose inverse.*

However...

The Auinger-Volkov manuscript remained unpublished, because, as Karl and Misha said at the time, it bothered them that the following question remained unsolved:

Problem

Exactly which of the involution semigroups $\mathcal{M}_n(\mathbb{F})$ are NFB, $n \geq 2$, $\mathbb{F}$ is a finite field? (i.e. what about the case $|\mathbb{F}| = 2$?)

Also, the following open problem was both intriguing and inviting.

Problem

Do finite INFB involution semigroups exist at all?

What the... INFB?

An algebra A is inherently nonfinitely based (INFB) if:

- $\mathcal{V}(A)$ is locally finite, and
- $\mathcal{V}$ is locally finite & $A \in \mathcal{V} \Longrightarrow \mathcal{V}$ is NFB.

$\Longleftrightarrow$ for any finite set of identities Σ satisfied by A, the variety defined by Σ is not locally finite.

Therefore, problems concerning INFB algebras are in fact Burnside-type problems.

INFB algebras are a powerful tool for proving the NFB property; namely, the INFB property is "contagious":

> if $\mathcal{V}(A)$ is locally finite and contains an INFB algebra B, then A is (I)NFB.

Finite INFB semigroups: a success story

M.V.Sapir, 1987: a full description of (finite) INFB semigroups.

Zimin words: $Z_1 = x_1$ and $Z_{n+1} = Z_n x_{n+1} Z_n$ for $n \geq 1$.

Theorem (Sapir, 1987)

Let S be a finite semigroup. Then

$$S \text{ is INFB} \iff S \text{ does not satisfy } Z_n = W$$

for all $n \geq 1$ and all words $W \neq Z_n$.

Sapir also found an effective structural description of finite INFB semigroups, thus proving

Theorem (Sapir, 1987)

It is decidable whether a finite semigroup is INFB or not.

Examples of finite INFB semigroups

Proposition

B_2^1 fails to satisfy a nontrivial identity of the form $Z_n = W$. Hence, it is INFB.

Corollary

For any $n \geq 2$ and any (semi)ring R, the matrix semigroup $\mathcal{M}_n(R)$ is (I)NFB.

Since $B_2^1 \in \mathcal{V}(A_2^1)$, where A_2 is the 5-element semigroup from Volkov's theorem, we have that A_2^1 is (I)NFB as well.

The same argument applies to $\mathcal{T}_n$ ($n \geq 3$), $\mathcal{R}_n$ ($n \geq 2$), $\mathcal{PT}_n$ ($n \geq 2$),...

What a difference an involution makes? Well...

How on Earth can be the case of unary semigroups different?

For example, an involution * can be defined on B_2^1 by $a^* = b$, $b^* = a$, the remaining 4 elements (which are idempotents: $0, 1, ab, ba$) being fixed. This turns B_2^1 into an inverse semigroup.

Surprise...!!!

Theorem (Sapir, 1993)

B_2^1 is not INFB as an inverse semigroup.
In fact, there is no finite INFB inverse semigroup at all! 🤯

Still, the inverse semigroup B_2^1 is NFB (Kleiman, 1979).

So, once again:

Problem

Do finite INFB involution semigroups exist at all?

An INFB criterion for involution semigroups

Yes!

Theorem (IgD, cca. Spring 2008)

Let S be an involution semigroup such that $\mathcal{V}(S)$ is locally finite. If S fails to satisfy any nontrivial identity of the form

$$Z_n = W,$$

where W is an involutorial word (a word over the “doubled” alphabet $X \cup X^*$), then S is INFB.

Great Igor, but... how about a (finite) example?

“C’mon baby, let’s do the twist...!”

Rescue: Luckily, B_2^1 admits one more involution aside from the inverse one: define the nilpotents a, b (and, of course, $0, 1$) to be fixed by *, which results in $(ab)^* = ba$ and $(ba)^* = ab$.

In this way we obtain the twisted Brandt monoid TB_2^1.

Proposition

TB_2^1 fails to satisfy a nontrivial identity of the form $Z_n = W$. Hence, it is INFB.

Similarly to B_2^1, this little guy is quite powerful.

Remark

Analogously, one can also define TA_2^1, the “involutorial version” of A_2^1, which is also INFB.

One thing led to another...

K.Auinger, IgD, M.V.Volkov, Matrix identities involving multiplication and transposition, *Journal of the European Mathematical Society* **14** (2012), 937–969.

K.Auinger, IgD, M.V.Volkov, Equational theories of semigroups with involution, *Journal of Algebra* **369** (2012), 203–225.

IgD, On identities of finite involution semigroups, *Semigroup Forum* **80** (2010), 105–120.

K.Auinger, IgD, T.V.Pervukhina, M.V.Volkov, Unary enhancements of inherently non-finitely based semigroups, *Semigroup Forum* **89** (2014), 41–51.

IgD, S.V.Gusev, M.V.Volkov, Semiring and involution identities of powers of inverse semigroups, *Communications in Algebra* **52** (2024), 1922–1929.

Examples of finite INFB involution semigroups

- $\mathcal{R}_n^\vee$, the involution semigroup of binary relations on an n-element set, $n \geq 2$,
  - ☞ TB_2^1 embeds into $\mathcal{R}_2^\vee$
- $\mathcal{M}_2(\mathbb{F})$ when $|\mathbb{F}| \not\equiv 3 \pmod 4$,
  - ☞ TB_2^1 embeds into $\mathcal{M}_2(\mathbb{F}) \iff x^2 + 1$ has a root in $\mathbb{F}$
- $\mathcal{M}_n(\mathbb{F})$ for all $n \geq 3$ and all finite fields $\mathbb{F}$.
  - ☞ TB_2^1 embeds into $\mathcal{M}_n(\mathbb{F})$ as a consequence of the Chevalley-Warning theorem from algebraic number theory (!).

So, what about $\mathcal{M}_2(\mathbb{F})$ if $|\mathbb{F}| \equiv 3 \pmod 4$?
(We already know it is NFB.)

Non-INFB results (IgD, 2010)

Theorem

Let S be a finite involution semigroup satisfying a nontrivial identity of the form $Z_n = W$ such that $B_2^1 \notin \mathcal{V}(S)$. Then S is not INFB.

Theorem

Let S be a finite semigroup satisfying an identity of the form $Z_n = Z_n W$. Then S is not INFB.

☞ Margolis & Sapir (1995) + certain semigroup quasiidentities can be "encoded" into unary semigroup identities.

Corollary

No finite regular *-semigroup is INFB.
(Namely, $x = x(x^* x)$ holds.)

Power involution semigroups

Corollary

For any finite group G, the involution semigroup of subsets $\mathcal{P}_G^* = (\mathcal{P}(G), \cdot, {}^*)$ is not INFB.
(Namely, $\mathcal{P}_G^*$ satisfies $Z_n = Z_n x_1^* x_1$ for $n = |G| + 2$.)

Remark

The ordinary power semigroup $\mathcal{P}_G = (\mathcal{P}(G), \cdot)$ is INFB if and only if G is not Dedekind.

Theorem (Gusev, Volkov, 2023)

G solvable and not Dedekind $\Longrightarrow$ $\mathcal{P}_G^*$ is NFB.

Theorem (IgD, Gusev, Volkov, 2024)

If S is an inverse semigroup that is either not a semilattice of groups or all subgroups are solvable and at least one is not Dedekind $\Longrightarrow$ $\mathcal{P}_S^*$ is NFB.

The (I)NFB problem for matrix involution semigroups

Two facts:

- (Crvenković, 1982) if a finite involution semigroup S admits a Moore-Penrose inverse †, then the inverse is term-definable in S; (So, S satisfies $x = x \cdot w(x, x^*) \cdot x$ for some $w \Longrightarrow$ it is not INFB.)
- the involution semigroup of 2×2 matrices over a finite field $\mathbb{F}$ with transposition admits a Moore-Penrose inverse if and only if $|\mathbb{F}| \equiv 3 \pmod 4$.

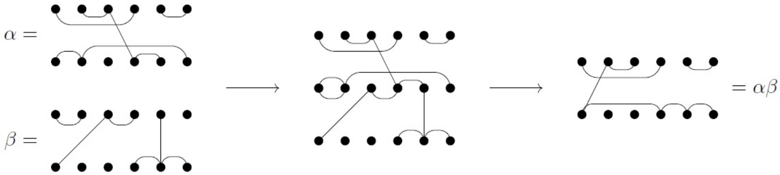
Theorem (Auinger, IgD, Volkov, 2012)

Let $n \geq 2$ and $\mathbb{F}$ be a finite field. Then

(1) *$\mathcal{M}_n(\mathbb{F})$ is not finitely based;*

(2) *$\mathcal{M}_n(\mathbb{F})$ is INFB if and only if either $n \geq 3$, or $n = 2$ and $|\mathbb{F}| \not\equiv 3 \pmod 4$.*

Applying our results to diagram monoids



The following regular *-semigroups are NFB:

- the partition monoids $\mathcal{P}_n$ for $n \geq 2$;
- the Brauer monoids $\mathcal{B}_n$ for $n \geq 4$;
- the partial Brauer monoids $\mathcal{PB}_n$ for $n \geq 3$;
- the annular monoids $\mathcal{A}_n$ for $n \geq 4$, n even or a prime power;
- the partial annular monoids $\mathcal{PA}_n$ for $n \in \{2^k + 2, p^k, p^k + 1\}$, p prime, $k \geq 1$.

INFB finite regular semigroups with involution

Let $TSL_3 = \langle a \mid aa^* = a^*a = 0 \rangle$.

Theorem (Auinger, IgD, Pervukhina, Volkov, 2014)

If $(S, \cdot)$ is a finite INFB semigroup and $TSL_3 \in \mathcal{V}(S)$
$\Longrightarrow$ *$(S, \cdot, ^*)$ is INFB.*

Theorem (Auinger, IgD, Pervukhina, Volkov, 2014)

TFAE for a regular finite semigroup with involution S:

1. *S is INFB,*
2. *$(S, \cdot)$ is INFB and $TSL_3 \in \mathcal{V}(S)$,*
3. *S fails to satisfy a nontrivial identity of the form $Z_n = W$.*

I hope Andy is down there.

I hope I can make it across the border.

I hope to see my friend and shake his hand.

I hope the Pacific is as blue as it has been in my dreams.

I hope.

Stephen King: Rita Hayworth and the Shawshank Redemption

Thank you! 😊 ❤️