Recent developments in combinatorial inverse semigroup theory

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Most of the original results presented here...

...are obtained in collaboration with
Robert D. Gray (University of East Anglia, Norwich, UK)

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- (orientable) surface groups
  $\mathrm{Gp}\langle a_1, \dots, a_g, b_1, \dots, b_g \mid [a_1, b_1] \dots [a_g, b_g] = 1 \rangle$

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"Da sind Sie also blind gegangen!"

Max Dehn (Magnus' PhD advisor)

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NB. These presentations define right cancellative monoids.

Inverse semigroups / monoids [???]

Structures $(S, {}^{-1})$ where S is a semigroup / monoid, and the unary operation satisfies the laws:

$$(x^{-1})^{-1} = x, \qquad (xy)^{-1} = y^{-1}x^{-1},$$
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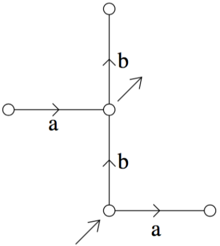
Free inverse monoid $FIM(X)$: Munn, Scheiblich (1973/4)



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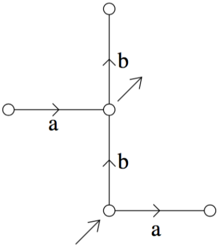

Elements of $FIM(X)$ are represented as Munn trees = birooted finite subtrees of the Cayley graph of $FG(X)$. The Munn tree on the left illustrates the equality

$$aa^{-1}bb^{-1}ba^{-1}abb^{-1} = bbb^{-1}a^{-1}ab^{-1}aa^{-1}b.$$

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Hence, the WP for one-relator monoids reduces to the WP for one-relator (special) inverse monoids.

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- Then, $A(P_4)$ embeds into a one-relator group $G = \mathrm{Gp}\langle a, b \mid \ldots \rangle$;
- Finally, a one-relator SIM $M = \mathrm{Inv}\langle a, b, t \mid \ldots \rangle$ is constructed so that $u \in \{a, b, a^{-1}, b^{-1}\}^*$ represents an element of the "critical" undecidable f.g. submonoid of G

$$\Longleftrightarrow$$

tut^{-1} is right invertible in M (i.e. $tut^{-1}tu^{-1}t^{-1} = 1$).

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Further evidence that PMP is very relevant

Guba (1997):
For any monadic $M = \mathrm{Mon}\langle a, b \mid aUb = a\rangle$ constructs $G_M = \mathrm{Gp}\langle x, y, A \mid xWyx^{-1} = 1\rangle$ (for some $W \in (A \cup \{x, y\})^*$ related, but not trivially, to U) such that the WP for M reduces to PMP for G_M.

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However, there are groups $G = \mathrm{Gp}\langle A \mid w = 1\rangle$ with:

- w reduced and undecidable PMP for G (IgD, RDG, 2021);
- $w = uv^{-1}$ reduced ($u, v \in A^+$) and undecidable PMP for G (Foniqi, RDG, Nyberg-Brodda, to appear);

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- w reduced and undecidable PMP for G (IgD, RDG, 2021);
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Problem: What about the case when w is cyclically reduced?

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Two questions

All results presented thus far very much justify the study of prefix monoids in f.p. groups and (because of Gray's counterexample) of right unit monoids (RU-monoids) in f.p. SIMs in their own right.

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Recursive stuff

A group G is recursively presented if

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where A is finite and $\{w_i : \ i \in I\}$ is a r.e. language over $A \cup A^{-1}$.

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- ☞ A finitely generated monoid embeds into a f.p. group if and only if it is group-embeddable and recursively presented.
- ☞ Every prefix monoid (of a f.p. group) is f.g. $\Longrightarrow$ it is recursively presented.

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Theorem (IgD, RDG, 2023):
For every group-embeddable recursively presented monoid M there is a natural number μ_M such that

$$M * \Sigma_k^*$$

is a prefix monoid (with $|\Sigma_k| = k$) if and only if $k \geq \mu_M$.

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So, there is evidence that the (open) problem of characterising RU-monoids might be actually quite difficult.

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In a way, this is a generalisation of the Ivanov-Margolis-Meakin result.

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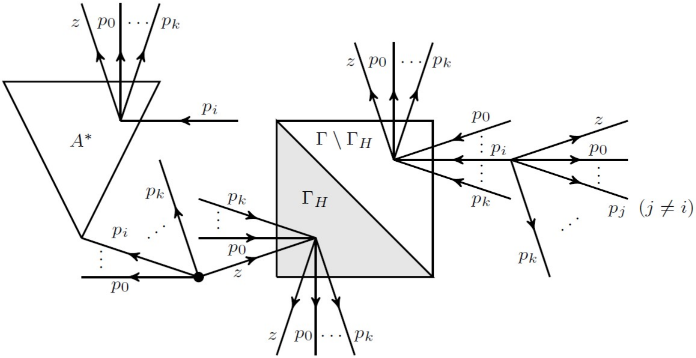
Their construction takes a f.g. subgroup H of a f.p. group G and produces a f.p. SIM $M_{G,H}$ such that $U(M_{G,H}) \cong H$.

The Gray-Kambites construction

RDG, Kambites (2023/24, JEMS, to appear): The groups of units of f.p. SIMs are precisely the recursively presented groups.

Their construction takes a f.g. subgroup H of a f.p. group G and produces a f.p. SIM $M_{G,H}$ such that $U(M_{G,H}) \cong H$.



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IgD, RDG (2024): A generalisation to the situation when we take a f.g. submonoid S of a finitely RC-presented (right cancellative) monoid T.

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Conclusion 2: Right cancellative monoids and RC-presentations are strange animals!

Thank you! 😊 ❤️