Recent developments in combinatorial inverse semigroup theory

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Most of the original results presented here...

...are obtained in collaboration with
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The word problem (in groups, monoids,...)

Assume we have given a (finitely generated) group $G = \langle A \rangle$ (e.g. by a presentation, $G = \mathrm{Gp}\langle A \,|\, \mathfrak{R} \rangle$, etc.).
So, elements of G are represented by words over $\overline{A} = A \cup A^{-1}$.

The word problem for G is the following decision (algorithmic) problem:
INPUT: A word $w \in \overline{A}^*$.
QUESTION: Does w represent the identity element 1 in G?

Similarly, one can ask about the word problem for semigroups / monoids / inverse monoids / ..., with the difference being that the input requires two words u, v (over A^* or $\overline{A}^*$, respectively), and then we want to decide if $u = v$ holds in the corresponding semigroup / monoid.

One-relator groups

Some easy word problems:

- $\mathrm{Mon}\langle a, b \mid ab = ba \rangle = \mathbb{N} \times \mathbb{N}$
- $\mathrm{Gp}\langle a, b \mid a^{-1}b^{-1}ab = 1 \rangle = \mathrm{Gp}\langle a, b \mid ab = ba \rangle = \mathbb{Z} \times \mathbb{Z}$

H. H. Wilhelm Magnus (1932):

The word problem for every one-relator group $\mathrm{Gp}\langle A \mid r = 1 \rangle$ *is decidable.*

Further examples:

- Baumslag-Solitar groups
  $B(m, n) = \mathrm{Gp}\langle a, b \mid b^{-1}a^m ba^{-n} = 1 \rangle$
- (orientable) surface groups
  $\mathrm{Gp}\langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid [a_1, b_1] \ldots [a_g, b_g] = 1 \rangle$

Magnus' result: The strategy

- Uses a result from Magnus' PhD thesis (1930), the famous Freiheitssatz, to identify certain free subgroups in a one-relator group $G = \mathrm{Gp}\langle A \mid r = 1\rangle$;
- This gives rise to a (very "controlled") embedding of G into an HNN-extension of its subgroup $L = \mathrm{Gp}\langle A' \mid r' = 1\rangle$ w.r.t. a pair of free subgroups of L, where $|r'| < |r|$;
- Such an embedding suffices to reduce the WP for G to that of L;
- Eventually, we end up with a free group of finite rank, where we trivially solve the WP.

"Da sind Sie also blind gegangen!"

Max Dehn (Magnus' PhD advisor)

The one-relator monoid problem

Open problem (as of 14 November 2024):

Does every one-relator monoid
Mon$\langle A \mid u = v \rangle$ *have a decidable WP?*

S.I.Adian (1966) – The word problem for Mon$\langle A \mid u = v \rangle$ is decidable for:

- *special monoids – the def. relation is of the form* $u = 1$,
- *the case when both* u, v *are non-empty, and have different initial letters and different terminal letters.*

Adian & Oganessian (1987) – The general problem reduces to two particular cases:

- Mon$\langle a, b \mid aUb = aVa \rangle$,
- Mon$\langle a, b \mid aUb = a \rangle$ (the “monadic” case).

NB. These presentations define right cancellative monoids.

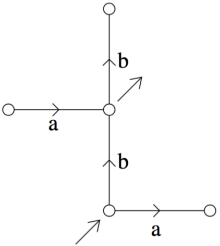
Inverse semigroups / monoids [???]

Structures $(S, {}^{-1})$ where S is a semigroup / monoid, and the unary operation satisfies the laws:

$$(x^{-1})^{-1} = x, \qquad (xy)^{-1} = y^{-1}x^{-1},$$
$$xx^{-1}x = x, \qquad xx^{-1}yy^{-1} = yy^{-1}xx^{-1}.$$

Just as groups capture the concept of a symmetry of mathematical objects, so do inverse semigroups for their partial symmetries.

Free inverse monoid $FIM(X)$: Munn, Scheiblich (1973/4)



Elements of $FIM(X)$ are represented as Munn trees = birooted finite subtrees of the Cayley graph of $FG(X)$. The Munn tree on the left illustrates the equality

$$aa^{-1}bb^{-1}ba^{-1}abb^{-1} = bbb^{-1}a^{-1}ab^{-1}aa^{-1}b.$$

Enter: Combinatorial inverse semigroup theory

The existence of $FIM(A)$ caters for inverse semigroup/monoid presentations $\mathrm{Inv}\langle A \mid \mathfrak{R} \rangle$. When all defining relators are of the form $w = 1$, we have special inverse monoids.

Ivanov, Margolis & Meakin (JPAA, 2001):
The (right cancellative) monoid $\mathrm{Mon}\langle A \mid aUb = aVc \rangle$ $(b \neq c)$ embeds (as the monoid of right units) into

$$\mathrm{Inv}\langle A \mid aUbc^{-1}V^{-1}a^{-1} = 1 \rangle.$$

Similarly, $\mathrm{Mon}\langle A \mid aUb = a \rangle$ embeds into $\mathrm{Inv}\langle A \mid aUba^{-1} = 1 \rangle$.

Hence, the WP for one-relator monoids reduces to the WP for one-relator (special) inverse monoids.

Surprise, surprise...!

	$\mathrm{Gp}\langle X \mid w = 1 \rangle$	$\mathrm{Mon}\langle X \mid w = 1 \rangle$	$\mathrm{Inv}\langle X \mid w = 1 \rangle$
decidable WP	✓ (Magnus, 1932)	✓ (Adyan, 1966)	? ✗ (Gray, 2019)

RDG (Inventiones Math, 2020):

There exists a one-relator special inverse monoid with an undecidable WP. [!!!]

This still does not invalidate the IMM-approach, as the counterexample is of a different from (e.g. the relator word is not reduced). But it does show that there are great difficulties ahead.

Gray's Anatomy :-)

- At the heart of the proof is Lohrey-Steinberg's result (2008) that the right-angled Artin group $A(P_4)$ has a fixed finitely generated submonoid with undecidable membership problem;
- Then, $A(P_4)$ embeds into a one-relator group $G = \mathrm{Gp}\langle a, b \mid \ldots \rangle$;
- Finally, a one-relator SIM $M = \mathrm{Inv}\langle a, b, t \mid \ldots \rangle$ is constructed so that $u \in \{a, b, a^{-1}, b^{-1}\}^*$ represents an element of the "critical" undecidable f.g. submonoid of G

$$\Longleftrightarrow$$

tut^{-1} is right invertible in M (i.e. $tut^{-1}tu^{-1}t^{-1} = 1$).

The importance of being E-unitary

It is a foundational result of inverse semigroup theory that every inverse semigroup S has a maximum group image G. Let $\phi : S \to G$ be the corresponding natural homomorphism. Clearly, for any idempotent $e \in S$ we must have $\phi(e) = 1$.

However, if the converse holds: $\phi(s) = 1 \implies s^2 = s$, then S is said to be E-unitary.

For example, $M = \mathrm{Inv}\langle A \mid w = 1 \rangle$ is E-unitary if:

- $w = 1$ holds in any group (i.e. w is a Dyck word),
- w is cyclically reduced (IMM, 2001).

IMM (2001): If $M = \mathrm{Inv}\langle A \mid w = 1 \rangle$ is E-unitary then the WP for M reduces to the prefix membership problem (PMP) for its greatest group image $G = \mathrm{Gp}\langle A \mid w = 1 \rangle$ = the membership problem for the submonoid of G generated by all prefixes of w.

Further evidence that PMP is very relevant

Guba (1997):
For any monadic $M = \mathrm{Mon}\langle a, b \mid aUb = a \rangle$ constructs $G_M = \mathrm{Gp}\langle x, y, A \mid xWyx^{-1} = 1 \rangle$ (for some $W \in (A \cup \{x, y\})^*$ related, but not trivially, to U) such that the WP for M reduces to PMP for G_M.

However, there are groups $G = \mathrm{Gp}\langle A \mid w = 1 \rangle$ with:

- w reduced and undecidable PMP for G (IgD, RDG, 2021);
- $w = uv^{-1}$ reduced ($u, v \in A^+$) and undecidable PMP for G (Foniqi, RDG, Nyberg-Brodda, to appear);
- $w \in A^+$ and undecidable submonoid membership problem for G (again, F+G+NB).

Problem: What about the case when w is cyclically reduced?

Some one-relator groups with decidable PMP

IgD, RDG (TransAMS, 2021): Theorems providing sufficient conditions for decidability of certain f.g. submonoids of (1) amalgamated free products and (2) HNN-extensions of groups.

Applications:

- Assume a conservative factorisation $w \equiv w_1 \cdots w_k$;
- Unique marker letters: pieces axb, ayb,

$$\mathrm{Gp}\langle a, b, x, y \mid (axb)(ayb)(ayb)(axb)(ayb)(axb) = 1 \rangle;$$

- Sometimes, the application is not immediate, e.g. in the O'Hare example:

$$\mathrm{Gp}\langle a, b, c, d \mid (abcd)(acd)(ad)(abbcd)(acd) = 1 \rangle;$$

  but the same group (and resulting with the same prefix monoid!) is defined by

$$\mathrm{Gp}\langle a, b, c, d \mid (aba^{-1})(aca^{-1})(ad)(aca^{-1})(ad)(ad) \\ (aba^{-1})(aba^{-1})(aca^{-1})(ad)(aca^{-1})(ad) = 1 \rangle$$

Some one-relator groups with decidable PMP

- Disjoint alphabets:

  $\mathrm{Gp}\langle a, b, c, d \mid (abab)(cdcd)(abab)(cdcd)(cdcd)(abab) = 1\rangle$;

- Exponent sum zero: $G = \mathrm{Gp}\langle A, t \mid w = 1\rangle$, where the sum of exponents of t in w is 0. Then (by Moldavanskiĭ, 1967) G is an HNN extension of a group $G_0 = \mathrm{Gp}\langle A' \mid w' = 1\rangle$ where $|w'| < |w|$. If G_0 is free and w is prefix t-positive $\Rightarrow$ G has decidable PMP;

- Cyclically pinched groups: $\mathrm{Gp}\langle A, B \mid uv^{-1} = 1\rangle$ ($u \in \overline{A}^*$, $v \in \overline{B}^*$)
  - Orientable surface groups (known):

    $\mathrm{Gp}\langle a_1, \dots, a_n, b_1, \dots, b_n \mid [a_1, b_1] \dots [a_n, b_n] = 1\rangle$;
  - Non-orientable surface groups (new):

    $\mathrm{Gp}\langle a_1, \dots, a_n \mid a_1^2 \dots a_n^2 = 1\rangle$;

- Conjugacy pinched groups: $\mathrm{Gp}\langle X, t \mid t^{-1}utv^{-1} = 1\rangle$ ($u, v \in \overline{X}^*$ non-empty and reduced) – include the Baumslag-Solitar groups;

- Some Adian-type groups: $\mathrm{Gp}\langle X \mid uv^{-1} = 1\rangle$, $u, v \in X^*$ are positive words such that the first letters of u, v are different and also the last letters of u, v are different.

Two questions

All results presented thus far very much justify the study of prefix monoids in f.p. groups and (because of Gray's counterexample) of right unit monoids (RU-monoids) in f.p. SIMs in their own right.

(1) What can the prefix monoids of f.p. groups be?

(2) What can the RU-monoids of f.p. SIMs be?

Recursive stuff

A group G is recursively presented if

$$G = \mathrm{Gp}\langle A \mid w_i = 1 \ (i \in I)\rangle$$

where A is finite and $\{w_i : \ i \in I\}$ is a r.e. language over $A \cup A^{-1}$.

Similarly, a monoid is recursively presented if

$$M = \mathrm{Mon}\langle A \mid u_i = v_i \ (i \in I)\rangle$$

where A is finite and $\{(u_i, v_i) : \ i \in I\}$ is a r.e. subset of $A^* \times A^*$.

The Higman Embedding Theorem: A finitely generated group embeds into a f.p. group if and only if it is recursively presented.

☞ A finitely generated monoid embeds into a f.p. group if and only if it is group-embeddable and recursively presented.

☞ Every prefix monoid (of a f.p. group) is f.g.
$\Longrightarrow$ it is recursively presented.

The characterisation of prefix monoids (of f.p. groups)

Two (easy) facts:

- Every group-embeddable f.p. monoid arises as a prefix monoid.
- If a group arises as a prefix monoid then it is f.p. So, not all group-embeddable recursively presented monoids are prefix monoids.

Theorem (IgD, RDG, 2023):
For every group-embeddable recursively presented monoid M there is a natural number μ_M such that

$$M * \Sigma_k^*$$

is a prefix monoid (with $|\Sigma_k| = k$) if and only if $k \geq \mu_M$.

RU-monoids

M – inverse monoid, $r \in M$ is a right unit (or right invertible) if

$$rr^{-1} = 1.$$

Right units form a (plain) submonoid of M that is always right cancellative. Any right cancellative monoid isomorphic to the monoid of right units of a f.p. SIM is called an RU-monoid.

- RU-monoids are recursively presented (as monoids);
- if a group G arises as an RU-monoid $\Rightarrow$ G is finitely presented;
- quite recently it seems we (IgD, RGD, Sept 2024) have shown that if $G * \Sigma^*$ is an RU-monoid $\Rightarrow$ G is finitely presented.

So, there is evidence that the (open) problem of characterising RU-monoids might be actually quite difficult.

RC-presentations

$$M = \mathsf{MonRC}\langle A \,|\, \mathfrak{R} \rangle$$

$\Leftrightarrow M \cong A^*/\mathfrak{R}^{\mathrm{RC}}$, where $\mathfrak{R}^{\mathrm{RC}}$ is the intersection of all congruences σ of A^* such that

- $\mathfrak{R} \subseteq \sigma$,
- A^*/σ is right cancellative.

Theorem (IgD, RDG, 2023):
Every finitely RC-presented monoid is an RU-monoid.

In a way, this is a generalisation of the Ivanov-Margolis-Meakin result.

The Gray-Ruškuc construction (2016, published in 2024)

Ingredients: A group G and a f.g. submonoid $T \leq G$.
Constructs: An E-unitary SIM $M_{G,T}$ (which is f.p. if G is such).

Effects:

- a one-relator SIM whose group of units is not one-relator;
- a one-relator SIM whose group of units is f.p. but whose RU-monoid is not f.p.;
- a f.p. SIM whose group of units is not f.p.

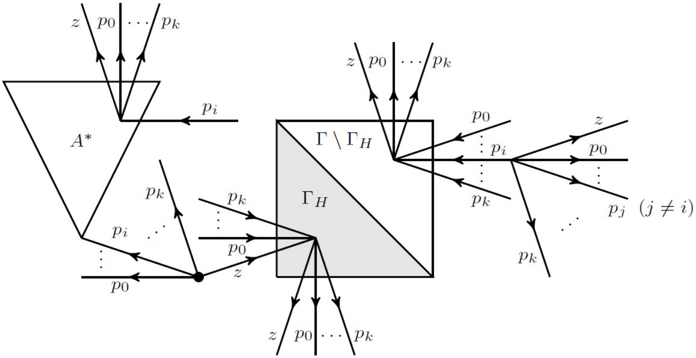
IgD, RDG (2024): The RU-monoid of $M_{G,T}$ is always finitely RC-presented(!) (even though it might be not f.p. as a monoid, and the group of units might be not f.p.)

☞ If U is the group of units of a monoid M and $M \setminus U$ is an ideal (which is always the case when M is right cancellative) M f.p. as a monoid $\Rightarrow$ U f.p. as a group

The Gray-Kambites construction

RDG, Kambites (2023/24, JEMS, to appear): The groups of units of f.p. SIMs are precisely the recursively presented groups.

Their construction takes a f.g. subgroup H of a f.p. group G and produces a f.p. SIM $M_{G,H}$ such that $U(M_{G,H}) \cong H$.



The Gray-Kambites construction

IgD, RDG (2024): A generalisation to the situation when we take a f.g. submonoid S of a finitely RC-presented (right cancellative) monoid T.

We have determined an RC-presentation for the right units of $M_{T,S}$.

- This monoid is practically never finitely RC-presented;
- The group of units might or might not be f.p., it might even be trivial.

Conclusion 1: There are non-finitely RC-presented RU-monoids out there!

Conclusion 2: Right cancellative monoids and RC-presentations are strange animals!

Thank you! 😊 ❤️