Prefix monoids of groups and right units of special inverse monoids

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- NB. There is an older approach (Magnus' original) using amalgamated free products.

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NB. These presentations define right cancellative monoids.

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IgD & RDG (TrAMS, 2021): A kaleidoscope of sufficient conditions (via amalgamated products and HNN extensions) ensuring decidability for the PMP

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- If M happens to be E-unitary, the restriction of this map to R is a monoid isomorphism.
- Consequently, the RU-monoid of any E-unitary special inverse monoid (SIM) is group-embeddable.

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The (right cancellative) monoid $\mathrm{Mon}\langle A \mid aUb = aVc \rangle$ ($b \neq c$) embeds (as the monoid of right units) into

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- $\mathrm{Inv}\langle A \mid w = 1 \rangle$ is E-unitary;
- the WP for $\mathrm{Inv}\langle A \mid w = 1 \rangle$ reduces to the PMP for $\mathrm{Gp}\langle A \mid w = 1 \rangle$.

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- finally, a one-relator SIM $M = \mathrm{Inv}\langle a, b, t \mid \dots \rangle$ is constructed so that $u \in \{a, b, a^{-1}, b^{-1}\}^*$ represents an element of the "critical" undecidable f.g. submonoid of $G \iff tut^{-1}$ is a right unit in M.

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Still, this does not invalidate the IMM aprroach.

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Guba (1997):
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- $w \in A^+$ and undecidable submonoid membership problem for G (again, FGNB).

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- other maximal subgroups of M can be wildly different from U.

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(4) What are the possible Schützenberger groups of these monoids?

Recursive stuff

A group G is recursively presented if

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- ☞ A finitely generated monoid embeds into a f.p. group if and only if it is group-embeddable and recursively presented.
- ☞ Every prefix monoid (of a f.p. group) is f.g. $\Longrightarrow$ it is recursively presented.

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For every group-embeddable recursively presented monoid M there is a natural number μ_M such that

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Also:
The class of groups of units of prefix monoids is precisely the recursively presented groups.

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- For every r.e. subgroup H of a f.p. group G there is a f.p. overgroup $G_1 \geq G$ and and $t \in G_1$ such that $G \cap t^{-1}Gt = H$.

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Theorem 2 (RDG, Kambites):
A group arises as a maximal subgroup (i.e. as a group $\mathscr{H}$-class) of a f.p. SIM $\Longleftrightarrow$ it arises as a r.e. subgroup of a f.p. group.

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In a way, this is a generalisation of the Ivanov-Margolis-Meakin result.

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Open Problem: Characterise the class of all RU-monoids.

In the remainder of the talk, I'll present two interesting phenomena in this vein discovered by IgD+RDG during this Spring's online sessions.

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Conclusion: RC-presentations are strange animals!

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$M_{T,S}$ – a f.p. SIM gen. by A and $p_0, p_1, \dots, p_k, z, d$ subject to

$$p_i a p_i^{-1} p_i a^{-1} p_i^{-1} = 1 \qquad (a \in A, \ i = 0, 1, \dots, k)$$
$$p_i u_i d^{-1} v_i^{-1} p_i^{-1} = 1 \qquad (i = 1, \dots, k)$$
$$p_0 d p_0^{-1} = 1$$
$$z b z^{-1} z b^{-1} z^{-1} = 1 \qquad (b \in B)$$
$$z \left(\prod_{i=0}^{k} p_i^{-1} p_i \right) z^{-1} = 1.$$

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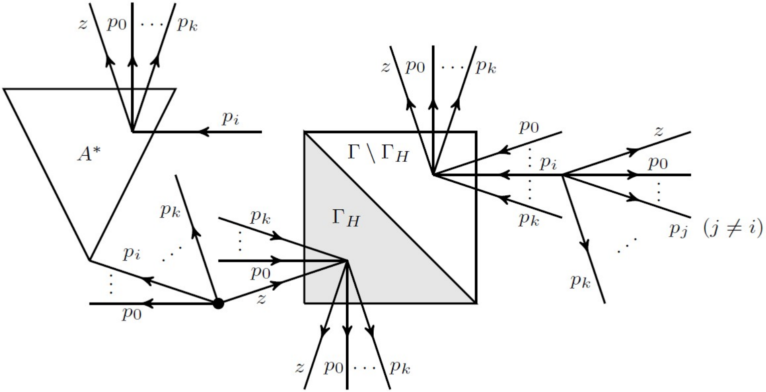

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NB. For all $u, v \in B^*$ s.t. $u = v$ holds in S, $u^{(z)} = v^{(z)}$ can be RC-derived. In fact, $\langle b^{(z)} :\ b \in B\rangle \cong S$.

The Gray-Kambites construction (4)

For example, when we take $T = \{a\}^*$ and $S = \langle \varnothing \rangle = \{1\}$ (and a silly presentation for T, say $a = a$, to have $k = 1$) we get the RU-monoid

$$\mathrm{MonRC}\langle a_1, a_1, p_0, p_1, q_0, q_1 \mid q_0 a_0^n p_0 = q_1 a_1^n p_1 \ (n \geq 0)\rangle.$$

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Conclusion: There are non-finitely RC-presented RU-monoids out there!

Thank you! 😊 ❤️

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