The word problem for free idempotent generated semigroups: An Italian symphony in IG($\mathcal{E}$) major

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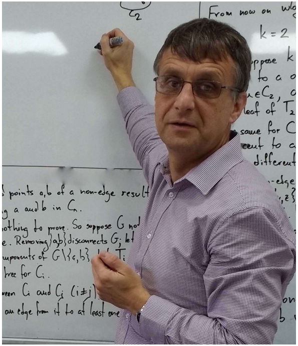
Joint work with...

Victoria Gould (York)

Dandan Yang (Xi'an)

...but also with motifs from previous collaborations with...

Robert D. Gray
(UEA Norwich)



Nik Ruškuc (St Andrews)

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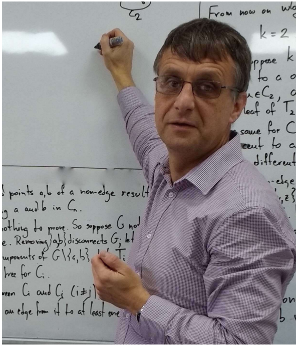
Happy Birthday, my friend!



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Introduzione

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Hence:

What can we say about the structure of the free-est idempotent-generated (IG) semigroup with a fixed structure/configuration of idempotents ???

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Biordered set of a semigroup S = the partial algebra

$$\mathcal{E}_S = (E(S), *)$$

obtained by retaining the products of basic pairs (in S).

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A big chunk of the axioms are expressed in terms of the quasi-orders

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(hence the name, "bi-ordered set"). From these, we can read off many relevant semigroup-theoretical relationships:

$$\leq = \leq^{(l)} \cap \leq^{(r)}, \quad \mathscr{L} = \leq^{(l)} \cap (\leq^{(l)})^{-1}, \quad \mathscr{R} = \leq^{(r)} \cap (\leq^{(r)})^{-1},$$

$$\mathscr{D} = \mathscr{L} \vee \mathscr{R}.$$

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Objects: Pairs (S, ϕ) where S is a semigroup and $\phi : \mathcal{E} \to \mathcal{E}_S$ is an isomorphism of biordered sets;

Morphisms: $\theta : (S, \phi) \to (T, \psi)$ – semigroup homomorphisms $\theta : S \to T$ such that $\phi\theta = \psi$.

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A more accessible definition:

$$\mathrm{IG}(\mathcal{E}) = \langle\, \overline{E} :\ \overline{e}\,\overline{f} = \overline{e * f} \text{ whenever } \{e, f\} \text{ is a basic pair} \,\rangle.$$

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So, understanding $\mathsf{IG}(\mathcal{E})$ is essential in understanding the structure of arbitrary IG semigroups.

I. Allegro vivace

A joyous quest for maximal subgroups

The (breakdown of the) freeness conjecture

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- Finally, Gray and Ruškuc (2012) proved that every group arises as a maximal subgroup of some free idempotent generated semigroup (!!!). If the group in question is finitely generated, the biordered set may be assumed to arise from a finite semigroup.

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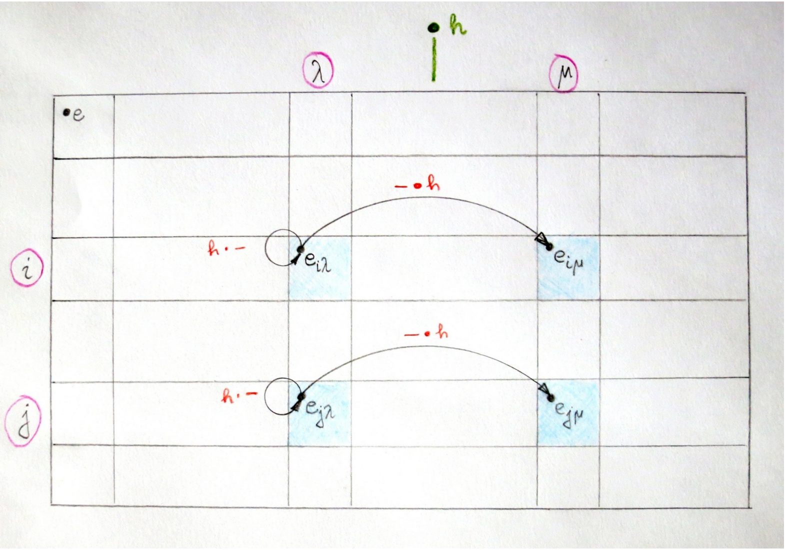
  - Some generators are equal ($f_{i\lambda} = f_{i\mu}$);
  - $f_{i\lambda}^{-1} f_{i\mu} = f_{j\lambda}^{-1} f_{j\mu}$ whenever $(i, j; \lambda, \mu)$ is a singular square.

Singular squares



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Alternatively, if the undelying biordered set $\mathcal{E}$ comes from an idempotent generated regular semigroup, Brittenham, Margolis & Meakin (2009) showed that the maximal subgroups of $\mathsf{IG}(\mathcal{E})$ are precisely the fundamental groups of connected components ($=$ $\mathscr{D}$-classes) of the Graham-Houghton complex of $\mathcal{E}$:

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This provides an alternative presentation for these groups; a clever choice of a spanning tree may speed up computations.

Refinements of the Gray-Ruškuc universality result

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- Gould & Yang, 2014: G arises as a maximal subgroup of $\mathrm{IG}(\mathcal{E}_S)$, where S is the endomorphism monoid of a free G-act.

Computing some natural examples

Goal

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- Partial transformation monoids: IgD, 2013 (symmetric groups again);
- Full matrix monoid over a skew field: IgD & Gray, 2014 (general linear groups, if rank $< n/3$, otherwise...);

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- Endomorphism monoid of a free G-act: IgD, Gould & Yang, 2015 (wreath products of G by symmetric groups).

II. Andante con moto

A taste of the word problem: the good and the ‘bad’

Recognising regular elements

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There exists an algorithm which, given $w \in E^+$, decides whether $\overline{w}$ is a regular element of $\mathrm{IG}(\mathcal{E})$, and if so, returns $f, g \in E$ such that $\overline{f} \,\mathscr{R}\, \overline{w} \,\mathscr{L}\, \overline{g}$.

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Namely, $\overline{w}$ is regular if and only if there is a factorisation

$$w = uev$$

such that $\overline{ue} \mathscr{L} \overline{e} \mathscr{R} \overline{ev}$.

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such that $\overline{ue} \mathscr{L} \overline{e} \mathscr{R} \overline{ev}$. In such a case, $\overline{e} \mathscr{D} \overline{w}$, and e is called the seed of w. (The decidability of this condition ultimately harks back to the Howie-Lallement Lemma.)

The word problem for regular elements of $\mathsf{IG}(\mathcal{E})$

Theorem (DGR, 2017)

(i) *There exists an algorithm which, given a finite biorder $\mathcal{E}$, computes the presentations of all maximal subgroups of* $\mathsf{IG}(\mathcal{E})$.

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(ii) *If all these presentations have soluble word problems, then there exists an algorithm which, for $u, v \in E^+$ representing regular elements of $\mathsf{IG}(\mathcal{E})$, decides whether $\overline{u} = \overline{v}$.*

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Method II (IgD, Gould, Yang, 2019):
Rees matrix 'coordinatisation' (via an effective version of an old result of FitzGerald) – wait for Mov.3

However... the 'bad' news

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(ii) *The solubility of the word problem of $\mathrm{IG}(\mathcal{B}_{G,H})$ implies the decidability of the membership problem of H in G.*

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(i) *Every maximal subgroup of $\mathsf{IG}(\mathcal{B}_{G,H})$ is either trivial or isomorphic to G;*

(ii) *The solubility of the word problem of $\mathsf{IG}(\mathcal{B}_{G,H})$ implies the decidability of the membership problem of H in G.*

Therefore, there exists a finite band B such that $\mathsf{IG}(\mathcal{E}_B)$ has undecidable word problem even though the word problems of all of its maximal subgroups are decidable.

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Therefore, there exists a finite band B such that $\mathsf{IG}(\mathcal{E}_B)$ has undecidable word problem even though the word problems of all of its maximal subgroups are decidable. (Because $G = F_2 \times F_2$ and the Mihailova construction.)

However... the 'bad' news (synopsis)

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- It allows for encoding the membership problem of H in G into equalities of products of certain pairs of regular elements $a(g), b(g)$, $g \in G$. In fact, we get

$$a(1)b(1) = a(g^{-1})b(g)$$

if and only if $g \in H$.

III. Con moto moderato

Working the way: factorisations, fingerprints, coordinates

(Minimal) r-factorisations

r-factorisation = a factorisation $w = p_1 \dots p_m$ such that all of $\overline{p_1}, \dots, \overline{p_m}$ are regular elements of $\mathsf{IG}(\mathcal{E})$

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We are, however, interested in the minimal r-factorisations = coarsest factorisations into regular-element-inducing factors.

As it turns out, all minimal factorisations of a word are pretty 'similar' w.r.t. $\mathrm{IG}(\mathcal{E})$.

$\approx$ and $\sim$

For two sequences of words over E^+ we define

$$(p_1, \ldots, p_m) \approx (q_1, \ldots, q_s)$$

if and only if $m = s$ and one of the three following conditions hold:

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(ii) $\overline{p_i} = \overline{q_i e}$ and $\overline{q_{i+1}} = \overline{e p_{i+1}}$ for some $1 \leq i < m$ and $e \in E$, and $p_j = q_j$ for all $j \notin \{i, i+1\}$;

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$\sim$ is the transitive closure of $\approx$.

The $\mathscr{D}$-fingerprint

Theorem

$u, v \in E^+$ such that $\overline{u} = \overline{v}$. Also, let $u = p_1 \dots p_m$ and $v = q_1 \dots q_s$ be minimal r-factorisations. Then $m = s$ and $\overline{p_i} \; \mathscr{D} \; \overline{q_i}$ $(1 \leq i \leq m)$.

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So, given $w \in E^+$, the sequence of $\mathscr{D}$-classes

$$(D_{\overline{p_1}}, \dots, D_{\overline{p_m}})$$

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is an invariant of w (where $w = p_1 \dots p_m$ is a minimal r-factorisation). This is the $\mathscr{D}$-fingerprint of w. Two words must share the same $\mathscr{D}$-fingerprint to stand any chance to represent the same element of $\mathrm{IG}(\mathcal{E})$.

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(2) *There exists an integer $m \geq 1$ such that all minimal r-factorisations of u and v, respectively, have precisely m factors, and whenever $u = p_1 \dots p_m$ and $v = q_1 \dots q_m$ are such factorisations we have*

$$(p_1, \dots, p_m) \sim (q_1, \dots, q_m).$$

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Can this representation be performed effectively? Yes.
What about $\sim$? Yup, that too.

The partial maps σ_e and τ_e

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Let $(i, g, \lambda) \in D$ *and* $e \in E$ *such that* $D \leq D_{\overline{e}}$.

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Define $\sigma_e : i \mapsto i'$ if $\overline{e}(i, g, \lambda) = (i', h, \lambda)$ for some $g, h \in G$, $\lambda \in \Lambda$.

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Analogously, let $\tau_e : \lambda \mapsto \lambda'$ if $(i, g, \lambda)\overline{e} = (i, h, \lambda')$ for some $i \in I$, $g, h \in G$.

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It follows already from the results of [DGR17] that all of these partial maps are effectively computable from $\mathcal{E}$.

The ‘effective’ FitzGerald

Lemma (Des FitzGerald, 1972)

Let S be an idempotent generated semigroup and $a \in S$ a regular element. Then $a = e_1 \dots e_n$ for some idempotents $e_1, \dots, e_n \in D_a$.

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$$w' = e_{i_1\mu_1} \dots e_{i_k\mu_k} e_{i\lambda} e_{j_1\lambda_1} \dots e_{j_l\lambda_l},$$

so that $\overline{w} = \overline{w'}$;

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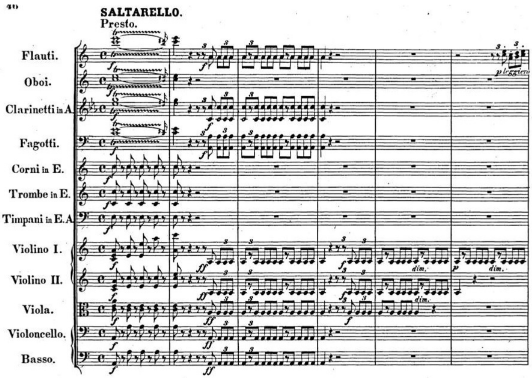
so that $\overline{w} = \overline{w'}$; hence,

$$\overline{w} = \left(i_1, f_{i_1\mu_1} f_{i_2\mu_1}^{-1} \dots f_{i_k\mu_k} f_{i\mu_k}^{-1} f_{i\lambda} f_{j_1\lambda}^{-1} f_{j_1\lambda_1} \dots f_{j_l\lambda_{l-1}}^{-1} f_{j_l\lambda_l}, \lambda_l\right).$$

IV. Saltarello: Presto

WP for IG is a CSP in FGG



Idempotent actions: the full story

If $\overline{e}(i, g, \lambda) \mathscr{D} (i, g, \lambda)$ (i.e. if $\sigma_e i$ is defined) then

$$\overline{e}(i, g, \lambda) = (\sigma_e i, f_{\sigma_e i, \lambda_0} f_{i, \lambda_0}^{-1} g, \lambda),$$

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Similarly, if $\lambda\tau_e$ is defined then

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Thus we finally get to fiddle with automata (yay!!!) with group-labelled transitions.

Contact automata

We want to capture the following transformation:

$$(\ldots, g, \lambda)(i, h, \ldots)$$

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Let D_1, D_2 be two regular $\mathscr{D}$-classes of $\mathsf{IG}(\mathcal{E})$, coordinatised by $I_1 \times G \times \Lambda_1$ and $I_2 \times H \times \Lambda_2$, respectively. We define the contact automaton $\mathcal{A}(D_1, D_2)$, a two-way NFA with states $\Lambda_1 \times I_2$ and alphabet E, where the transitions are defined and labelled by elements of $G \times H^\partial$ as follows:

Contact automata

We want to capture the following transformation:

$$(\dots, g', \mu)(j, h', \dots)$$

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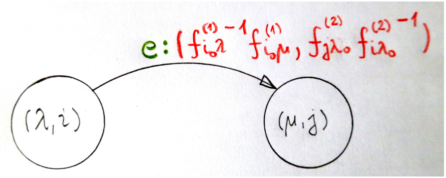

if $(\lambda = \mu\tau_e^{(1)}$ and $\sigma_e^{(2)} i = j)$ or $(\lambda\tau_e^{(1)} = \mu$ and $i = \sigma_e^{(2)} j)$.

The CSP

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INPUT: $a_k, b_k \in G_k$ $(1 \leq k \leq m)$.
OUTPUT: Decide if there exist $x_t \in G_t$, $2 \leq t \leq m-1$, such that

$$\begin{aligned} (a_1^{-1} b_1, x_2) &\in \rho_1, \\ (a_r^{-1} x_r^{-1} b_r, x_{r+1}) &\in \rho_r \qquad (2 \leq r \leq m-2), \\ (a_{m-1}^{-1} x_{m-1}^{-1} b_{m-1}, b_m a_m^{-1}) &\in \rho_{m-1}. \end{aligned}$$

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Theorem (IgD, Gould, Yang, 2019)

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holds in $\mathrm{IG}(\mathcal{E})$ if and only if $i_1 = j_1$, $\lambda_m = \mu_m$, and the problem

$$\mathbf{P}(G_1, \ldots, G_m; \rho_1(\lambda_1, i_2; \mu_1, j_2), \ldots, \rho_{m-1}(\lambda_{m-1}, i_m; \mu_{m-1}, j_m))$$

returns a positive answer on input $g_k, h_k \in G_k$, $1 \leq k \leq m$.

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(i) $m = 1$: We have $(i, g, \lambda) = (j, h, \mu)$ if and only if $i = j$, $\lambda = \mu$, and $g = h$.

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(ii) $m = 2$: $\mathbf{P}(G_1, G_2, \rho)$ is essentially the membership problem for $\rho \subseteq G_1 \times G_2^{\partial}$. The construction in [DGR17] was set up so that a certain segment of the word problem is equivalent to $\mathbf{P}(G, G, \rho_H)$ where

$$\rho_H = \{(h, h^{-1}) : \ h \in H\},$$

which is just the membership problem for H in G.

The principal applied result

Theorem (DGY, 2019)

Let $\mathcal{E}$ be a finite biordered set with the property that the maximal subgroups in all non-maximal $\mathscr{D}$-classes of* $\mathsf{IG}(\mathcal{E})$ *are finite. Then* $\mathsf{IG}(\mathcal{E})$ *has decidable word problem.*

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The maximal $\mathscr{D}$-classes necessarily yield free maximal subgroups, as there are no singular squares.

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- P.Silva (2002) $\Rightarrow$ effective version of Grunschlag's result

Applications

Corollary

For any $n \geq 1$, the free idempotent generated semigroups $\mathsf{IG}(\mathcal{E}_{\mathcal{T}_n})$ *and* $\mathsf{IG}(\mathcal{E}_{\mathcal{PT}_n})$ *have decidable word problems.*

Applications

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Question

Let Q be a finite field. Is the maximal subgroup of $\mathrm{IG}(\mathcal{E}_{M_n(Q)})$ contained in its $\mathscr{D}$-class $\overline{D_r}$ (corresponding to matrices of rank r) finite whenever $r \leq n-2$?

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Theorem

If $\mathcal{E}$ is finite, then $\mathrm{IG}(\mathcal{E})$ is always a Fountain (aka weakly abundant) semigroup satisfying the congruence condition.

The end-product

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A group-theoretical interpretation of the word problem for free idempotent generated semigroups

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A B S T R A C T

The set of idempotents of any semigroup carries the structure of a biordered set, which contains a great deal of information concerning the idempotent generated subsemigroup of the semigroup in question. This leads to the construction of a free idempotent generated semigroup $\mathsf{IG}(\mathcal{E})$ – the 'free-est' semigroup with a given biordered set $\mathcal{E}$ of idempotents. We show that when $\mathcal{E}$ is finite, the word problem for $\mathsf{IG}(\mathcal{E})$ is equivalent to a family of constraint satisfaction problems involving rational subsets of direct products of pairs of maximal subgroups of $\mathsf{IG}(\mathcal{E})$. As an application, we obtain decidability of the word problem for an important class of examples. Also, we prove that for finite $\mathcal{E}$, $\mathsf{IG}(\mathcal{E})$ is always a weakly abundant semigroup satisfying the congruence condition.

GRAZIE MILLE! THANK YOU!

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