The prefix membership problem for one-relator groups, and its semigroup-theoretical cousins

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Department of Mathematics and Informatics, University of Novi Sad

The eNBSAN Online Meeting 2020
CyberSpace, 24 June 2020
Starring

Robert D. Gray
(Uni of East Anglia, Norwich)

Lt. Col. Frank Slade
(US Army, retired)
Also starring

UEA campus bunnies
(providing the much-required positivity...)
The word problem (in groups, monoids,...)

Assume we have given a (finitely generated) group $G = \langle X \rangle$.

For starters, we'd very much like to know if two words represent the same element of $G$, and, in addition, is there an algorithm (think: computer program) which decides this.

The word problem for $G$:

**INPUT:** A word $w \in X^*$.  
**QUESTION:** Does $w$ represent the identity element 1 in $G$?

Similarly, one can ask about the word problem for monoids / inverse monoids / ..., with the difference being that the input requires two words $u, v$, and then we're keen to decide if $u = v$ holds in the corresponding monoid.
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Theorem (Shirshov, 1962)

Every one-relator Lie algebra has decidable word problem.
The one-relator monoid Riddle

Open Problem (still! – as of 2020)

Is the word problem decidable for all one-relator monoids
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- both \( u, v \) are non-empty, and have different initial letters and different terminal letters.

Lallement (1977) and L. Zhang (1992) provided alternative proofs for the result about special monoids.

The proof of Zhang is particularly compact and elegant.

NB. RIP S. I. Adyan (1 January 1931 – 5 May 2020).
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The connection to the inverse realm

Adyan & Oganessyan (1987): The word problem for one-relator monoids can be reduced to the special case of

$$\text{Mon}\langle X \mid asb = atc \rangle$$

where $a, b, c \in X$, $b \neq c$ and $s, t \in X^*$ (and their duals).
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Theorem (Ivanov, Margolis & Meakin, 2001)

If the word problem is decidable for all special inverse monoids

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where $w$ is a reduced word over $\overline{X}$ —
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If the word problem is decidable for all special inverse monoids 
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Theorem (Ivanov, Margolis & Meakin, 2001)

*If the word problem is decidable for all special inverse monoids \(\text{Inv}\langle X \mid w = 1 \rangle\) — where \(w\) is a reduced word over \(\overline{X}\) — then the word problem is decidable for every one-relator monoid.*

This holds basically because \(M = \text{Mon}\langle X \mid asb = atc \rangle\) embeds into \(I = \text{Inv}\langle X \mid asbc^{-1}t^{-1}a^{-1} = 1 \rangle\).
The plot thickens

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Conjecture (Margolis, Meakin, Stephen, 1987)
Every inverse monoid of the form $\text{Inv}\langle X \mid w = 1 \rangle$ has decidable word problem.
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Every inverse monoid of the form Inv\langle X \mid w = 1 \rangle has decidable word problem.

*There exists a one-relator inverse monoid Inv\langle X \mid w = 1 \rangle with undecidable word problem.*
Inverse monoid basics (1): Definitions & FIM

**Inverse monoid** = a monoid $M$ such that for every $a \in M$ there is a unique $a^{-1} \in M$ such that $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$. 
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Inverse monoids form a class of unary monoids defined by the laws

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Free inverse monoid $\text{FIM}(X)$: Munn, Scheiblich (1973/4)

Elements of $\text{FIM}(X)$ are represented as Munn trees = birooted finite subtrees of the Cayley graph of $\text{FG}(X)$. 
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Free inverse monoid $FIM(X)$: Munn, Scheiblich (1973/4)

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$$aa^{-1}bb^{-1}ba^{-1}abb^{-1} = bbb^{-1}a^{-1}ab^{-1}aa^{-1}b.$$
Inverse monoid basics (2): The $E$-unitary property

$E$-unitary inverse semigroups $=$ the well-behaved, “nice guys”.
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$E$-unitary inverse semigroups $=$ the well-behaved, “nice guys”.
For example, here are several (equivalent) definitions:

- For any $e \in E(S)$ and $x \in S$, $e \leq x$ (in the natural inverse semigroup order) $\Rightarrow x \in E(S)$.
- The minimum group congruence $\sigma$ on $S$ is idempotent-pure, which means that $E(S)$ constitutes a single $\sigma$-class.
- $\sigma = \sim$, where $\sim$ is the compatibility relation defined by $a \sim b \iff a^{-1}b, ab^{-1} \in E(S)$.

Theorem (Ivanov, Margolis & Meakin, 2001)
If $w$ is cyclically reduced, then $M = \text{Inv} \langle X \mid w = 1 \rangle$ is $E$-unitary.
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The key role of the prefix monoid

Consider a one-relator group $G$ given by $G \langle X \mid w = 1 \rangle$.

Prefix membership problem for $G = G \langle X \mid w = 1 \rangle$ = membership problem for $P_w$ within $G$.

Theorem (Ivanov, Margolis & Meakin, 2001) If $M = \text{Inv} \langle X \mid w = 1 \rangle$ is $E$-unitary, then word problem for $M =$ prefix membership problem for $G = G \langle X \mid w = 1 \rangle$.

Remark $G = G \langle X \mid w = 1 \rangle$ is the maximum group image of $M = \text{Inv} \langle X \mid w = 1 \rangle$. 
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word problem for $M = \text{prefix membership problem for } G = \langle X \mid w = 1 \rangle$.

Remark

$G = \langle X \mid w = 1 \rangle$ is the maximum group image of $M = \text{Inv} \langle X \mid w = 1 \rangle$. 
A Glimpse into the Toolbox
Membership problem (for a submonoid $M$ of a group $G$)
Membership problem (for a submonoid $M$ of a group $G$)

Submonoid membership problem for $G$: Is there an algorithm which, given $u, w_1, w_2, \ldots \in \bar{X}^*$, decides if $u \in \text{Mon}\langle w_1, w_2, \ldots \rangle$?
Rational subsets in groups

\[(X \cup X^{-1})^*\]

\[G = \langle X \rangle\]

\[L = \mathcal{L}(\alpha)\]

\[\text{regular language}\]

\[A = L \mathcal{J}\mathcal{T}\]

\[\text{rational subset of } G\]

\[\text{natural homomorphism}\]
Rational subset membership problem for a group $G = \langle X \rangle$:
Rational subset membership problem for a group $G = \langle X \rangle$:

**INPUT:** A word $w \in \overline{X}^*$ and a regular expression $\alpha$ over $\overline{X}$. 
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**QUESTION:** $w \in A_\alpha$?

(Here $A_\alpha \subseteq G$ is the image of $\mathcal{L}(\alpha)$, as in the previous pic.)
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**Theorem (Benois, 1969)**

*Every finitely generated free group has decidable RSMP.*
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(Here $A_\alpha \subseteq G$ is the image of $L(\alpha)$, as in the previous pic.)

**Theorem (Benois, 1969)**

Every finitely generated free group has decidable RSMP. Consequently, rational subsets of f.g. free groups are closed for intersection and complement.
Factorisations

In this slide we consider factorisations $w \equiv w_1 \ldots w_m$. 
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It is unital w.r.t. $M = \text{Inv} \langle X \mid w = 1 \rangle$ if each piece $w_i$ represents an invertible element (i.e. unit, $aa^{-1} = a^{-1}a = 1$) of $M$. 
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**Lemma**

*Unital fact.* $P_w \leq G = Gp\langle X \mid w = 1 \rangle$ is generated by $\bigcup_{i=1}^{m} \text{pref}(w_i)$.
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In this slide we consider factorisations \( w \equiv w_1 \ldots w_m \).

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**Lemma**

Unital fact. \( \implies P_w \subseteq G = \text{Gp} \langle X \mid w = 1 \rangle \) is generated by \( \bigcup_{i=1}^{m} \text{pref}(w_i) \).

In fact, for any factorisation of \( w \) we can consider the submonoid \( M(w_1, \ldots, w_m) \) of \( G \) generated by \( \bigcup_{i=1}^{m} \text{pref}(w_i) \).
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If $=$ holds, the considered factorisation is called conservative.
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**Theorem**

(i) *Any unital factorisation is conservative.* (aka previous Lemma)
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**Lemma**

*Unital fact.* $\implies P_w \subseteq G = \text{Gp} \langle X \mid w = 1 \rangle$ is generated by $\bigcup_{i=1}^m \text{pref}(w_i)$.

In fact, for any factorisation of $w$ we can consider the submonoid $M(w_1, \ldots, w_m)$ of $G$ generated by $\bigcup_{i=1}^m \text{pref}(w_i)$. In $G$, we have

$$P_w \subseteq M(w_1, \ldots, w_m).$$

If $=$ holds, the considered factorisation is called **conservative**.

**Theorem**

(i) *Any unital factorisation is conservative.* (aka previous Lemma)

(ii) *If $M = \text{Inv} \langle X \mid w = 1 \rangle$ is E-unitary then every conservative factorisation if unital.*
Amalgamated free product of groups $B \star_A C$
HNN extension of a group $G^*_{t,\phi}:A \rightarrow B$
The Results
Theorem A

\[ G = B \ast_A C \ (A, B, C \text{ finitely generated}) : \]
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- \( B, C \) have decidable word problems;
- the membership problem for \( A \) is decidable in both \( B \) and \( C \).
Theorem A

\( G = B \ast_A C \) (\( A, B, C \) finitely generated):

- \( B, C \) have decidable word problems;
- the membership problem for \( A \) is decidable in both \( B \) and \( C \).

Let \( M \) be a submonoid of \( G \) with the following properties:

(i) \( A \subseteq M \);  
(ii) \( M \cap B \) and \( M \cap C \) are f.g. and \( M = \text{Mon} \langle (M \cap B) \cup (M \cap C) \rangle \);  
(iii) the membership problem for \( M \cap B \) in \( B \) is decidable;  
(iv) the membership problem for \( M \cap C \) in \( C \) is decidable.

Then the membership problem for \( M \) in \( G \) is decidable.
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\begin{itemize}
  \item \(B, C\) have decidable word problems;
  \item the membership problem for \(A\) is decidable in both \(B\) and \(C\).
\end{itemize}

Let \(M\) be a submonoid of \(G\) with the following properties:

\begin{enumerate}
  \item \(A \subseteq M;\)
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\(M \cap B\) in \(B\) is decidable;
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Then the membership problem for \( M \) in \( G \) is decidable.
Rational intersections

\[ H \leq G \text{ closed for rational intersections:} \]

\[ R \in \text{Rat}(G) \implies R \cap H \in \text{Rat}(G) \]
Rational intersections

\( H \leq G \) closed for rational intersections:

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\( H \leq G \) effectively closed for rational intersections:

there is an algorithm which does the following
Rational intersections

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**INPUT**: A regular expression for \( R \in \text{Rat}(G) \).
Rational intersections

$H \leq G$ closed for rational intersections:

\[ R \in \text{Rat}(G) \implies R \cap H \in \text{Rat}(G) \]

$H \leq G$ effectively closed for rational intersections:

there is an algorithm which does the following

INPUT: A regular expression for $R \in \text{Rat}(G)$.
OUTPUT: Computes a regular expression for $R \cap H$. 
Theorem B

\[ G = B \ast_A C \] (\(A, B, C\) finitely generated):
Theorem B

\[ G = B \ast_A C \] (\( A, B, C \) finitely generated):

- \( B, C \) have decidable rational subset membership problems;
Theorem B

\[ G = B \star_A C \ (A, B, C \text{ finitely generated}): \]

- \( B, C \) have decidable rational subset membership problems;
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Let \( M \) be a submonoid of \( G \) such that \( M \cap B \) and \( M \cap C \) are f.g. and

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Then the membership problem for \( M \) in \( G \) is decidable.
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\( G = B \ast_A C \) (A, B, C finitely generated):

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Application #1: Unique marker letters

Theorem

\[ G = Gp \langle X \mid w = 1 \rangle \]
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Theorem

- $G = \langle X \mid w = 1 \rangle$
- $w \equiv u(w_1, \ldots, w_k)$ – a conservative factorisation of $w$
Application #1: Unique marker letters

Theorem

- $G = \text{Gp}\langle X \mid w = 1 \rangle$
- $w \equiv u(w_1, \ldots, w_k)$ – a conservative factorisation of $w$
- $\forall i \in [1, k]:$ there is a letter $x_i$ appearing exactly once in $w_i$ and not appearing in any $w_j$, $j \neq i$
Application #1: Unique marker letters

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Example

The group

$$G = \langle a, b, x, y | axbaybaybaxbaybaxb = 1 \rangle$$
Application #1: Unique marker letters

Theorem

- $G = Gp\langle X \mid w = 1 \rangle$
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Example

The group

$$G = Gp\langle a, b, x, y \mid (axb)(ayb)(ayb)(axb)(ayb)(axb) = 1 \rangle$$
Application #1: Unique marker letters

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1. \( G = \langle X \mid w = 1 \rangle \)
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\[ G = \text{Gp}\langle a, b, x, y \mid (axb)(ayb)(ayb)(axb)(ayb)(axb) = 1 \rangle \]

has decidable prefix membership problem \( \implies \) the inverse monoid

\[ M = \text{Inv}\langle a, b, x, y \mid axbaybaybaxbaybaxb = 1 \rangle \]

has decidable WP.
While waiting for a connecting flight at ORD sometime in the 1980s, Stuart Margolis and John Meakin came up with the following example, the (in)famous O'Hare (inverse) monoid:

$$\langle a, b, c, d \mid (abcd)(acd)(ad)(abbcd)(acd) = 1 \rangle$$
While waiting for a connecting flight at ORD sometime in the 1980s, Stuart Margolis and John Meakin came up with the following example, the (in)famous O’Hare (inverse) monoid:

\[ \text{Inv}\langle a, b, c, d \mid (abcd)(acd)(ad)(abbc)(acd) = 1 \rangle \]
Application #2: O’Hare-type examples

Proposition

Let $M = \text{Inv} \langle Y, a, d \mid (au_1 d) \ldots (au_m d) = 1 \rangle$, where $a, d$ do not appear in $u_{ij}$'s.
Application #2: O’Hare-type examples

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Let \( M = \text{Inv}\langle Y, a, d \mid (au_{i_1}d) \ldots (au_{i_m}d) = 1 \rangle \), where \( a, d \) do not appear in \( u_{ij} \)'s. Assume further that:

- some of the \( u_{ij} \)'s is the empty word;
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Let $M = \text{Inv} \langle Y, a, d \mid (au_1 d) \ldots (au_m d) = 1 \rangle$, where $a, d$ do not appear in $u_{ij}$’s. Assume further that:

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- for each $x \in Y$ we have $x \equiv \text{red}(u_r u_i^{-1})$ for some $r, s$;
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- each $au_{ij}d$ represents a unit of $M$.

Then $G = Gp \langle Y, a, d \mid (au_{i_1}d) \cdots (au_{i_m}d) = 1 \rangle$ has decidable prefix membership problem, and so $M$ as decidable WP. Consequently, the WP for the O’Hare monoid is decidable – just as announced at the WOW workshop in January 2018 by this fine gentleman:
Application #2: O’Hare-type examples

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Let $M = \text{Inv} \langle Y, a, d \mid (au_{i_1}d) \ldots (au_{i_m}d) = 1 \rangle$, where $a, d$ do not appear in $u_{i_j}$’s. Assume further that:

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Application #3: Disjoint alphabets

Theorem

\[ G = G \langle X \mid w = 1 \rangle, \ w \text{ is cyclically reduced} \]
Application #3: Disjoint alphabets

Theorem

- $G = \text{Gp}\langle X \mid w = 1 \rangle$, $w$ is cyclically reduced
- $w \equiv u(w_1, \ldots, w_k)$ – a conservative factorisation of $w$
Application #3: Disjoint alphabets

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Theorem

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- $i \neq j \Rightarrow w_i$ and $w_j$ have no letters in common

$\implies G$ has decidable prefix membership problem, and thus $M = \text{Inv} \langle X \mid w = 1 \rangle$ has decidable WP.
Application #3: Disjoint alphabets

Theorem

▶ $G = \text{Gp}\langle X \mid w = 1 \rangle$, \textit{w is cyclically reduced}
▶ $w \equiv u(w_1, \ldots, w_k)$ – a conservative factorisation of \textit{w}
▶ $i \neq j \Rightarrow w_i \text{ and } w_j \text{ have no letters in common}$

$\implies G \text{ has decidable prefix membership problem, and thus } M = \text{Inv}\langle X \mid w = 1 \rangle \text{ has decidable WP}$.

Example

The group

$G = \text{Gp}\langle a, b, c, d \mid (abab)(cdcd)(abab)(cdcd)(cdcd)(abab) = 1 \rangle$

has decidable prefix membership problem
Application #3: Disjoint alphabets

Theorem

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Example

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$G = \langle a, b, c, d \mid (abab)(cdcd)(abab)(cdcd)(cdcd)(abab) = 1 \rangle$

has decidable prefix membership problem $\implies$ the inverse monoid

$M = \text{Inv} \langle a, b, x, y \mid ababcdcdcdababc = 1 \rangle$

has decidable WP.
Theorem

The prefix membership problem is decidable for one-relator groups defined by cyclically pinched presentations:

\[ G = \langle X \cup Y \mid uv^{-1} = 1 \rangle \]

where \( u, v \) are reduced words over disjoint \( X, Y \), respectively.
Application #4: Cyclically pinched presentations

Theorem
The prefix membership problem is decidable for one-relator groups defined by cyclically pinched presentations:

\[ G = \langle X \cup Y \mid uv^{-1} = 1 \rangle \]

where \( u, v \) are reduced words over disjoint \( X, Y \), respectively.

Example
This implies decidability of the prefix membership problem for surface groups:

orientable (known) \[ Gp\langle a_1, \ldots, a_n, b_1, \ldots, b_n \mid [a_1, b_1] \ldots [a_n, b_n] = 1 \rangle \]
non-orientable (new) \[ Gp\langle a_1, \ldots, a_n \mid a_2 \ldots a_n = 1 \rangle \]
Application #4: Cyclically pinched presentations

Theorem
The prefix membership problem is decidable for one-relator groups defined by cyclically pinched presentations:

\[ G = \text{Gp}\langle X \cup Y \mid uv^{-1} = 1 \rangle \]

where \( u, v \) are reduced words over disjoint \( X, Y \), respectively.

Example
This implies decidability of the prefix membership problem for surface groups:

- orientable (known)
  \[ \text{Gp}\langle a_1, \ldots, a_n, b_1, \ldots, b_n \mid [a_1, b_1] \ldots [a_n, b_n] = 1 \rangle, \]

- non-orientable (new)
**Application #4: Cyclically pinched presentations**

**Theorem**

*The prefix membership problem is decidable for one-relator groups defined by cyclically pinched presentations:*

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*where* \( u, v \) *are reduced words over disjoint* \( X, Y \), *respectively.*

**Example**

This implies decidability of the prefix membership problem for surface groups:

- orientable (known)
  \[ \langle a_1, \ldots, a_n, b_1, \ldots, b_n \mid [a_1, b_1] \cdots [a_n, b_n] = 1 \rangle, \]

- non-orientable (new)
  \[ \langle a_1, \ldots, a_n \mid a_1^2 \cdots a_n^2 = 1 \rangle. \]
Theorem C

\[ G^* = G^*_{t, \phi : A \rightarrow B} \ (G, A, B \text{ finitely generated}) : \]

\[ \text{Let } M \text{ be a submonoid of } G^* \text{ with the following properties:} \]

(i) \[ A \cup B \subseteq M \];

(ii) \[ M \cap G \text{ is f.g. and } M = \text{Mon} \langle (M \cap G) \cup \{t, t^{-1}\} \rangle \];

(iii) the membership problem for \( M \cap G \) in \( G \) is decidable.

Then the membership problem for \( M \) in \( G^* \) is decidable.
Theorem C

\[ G^* = G^*_{t, \phi : A \to B} \ (G, A, B \text{ finitely generated}): \]

- \( G \) has decidable word problem;
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\[ G^* = G_{t,\phi:A\rightarrow B}^* \] (G, A, B finitely generated):

- G has decidable word problem;
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Let \( M \) be a submonoid of \( G^* \) with the following properties:

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Then the membership problem for \( M \) in \( G^* \) is decidable.
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\[ G^* = G^*_{t, \phi: A \to B} \quad (G, A, B \text{ finitely generated}): \]

- \( G \) has decidable rational subset membership problem;
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- \( A \leq G \) is effectively closed for rational intersections.
Theorem D

\[ G^* = G^*_{t, \phi: A \rightarrow B} \ (G, A, B \text{ finitely generated}): \]

- G has decidable rational subset membership problem;
- \( A \leq G \) is effectively closed for rational intersections.

For some finite \( W_0, W_1, \ldots, W_d, W'_1, \ldots, W'_d \subseteq G \) let

\[ M = \text{Mon}\langle W_0 \cup W_1 t \cup W_2 t^2 \cup \cdots \cup W_d t^d \cup tW'_1 \cup \cdots \cup t^d W'_d \rangle \]

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Then the membership problem for \( M \) in \( G^* \) is decidable.
Application #5: Exponent sum zero result

\[ G = G_p\langle X \mid w = 1 \rangle: \text{some } t \in X \text{ has exponent sum zero in } w. \]
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\[ G = \mathrm{Gp}\langle X \mid w = 1 \rangle: \text{some } t \in X \text{ has exponent sum zero in } w. \]

By general theory ("Magnus' method", also Lyndon & McCool), \( G \) is \( \cong \) an HNN extension of

\[ H = \mathrm{Gp}\langle X' \mid \rho_t(w) = 1 \rangle \]

where \( |\rho_t(w)| < |w| \), w.r.t. to free associated subgroups \( A, B \) (will show this in a minute on a concrete example).
Application #5: Exponent sum zero result

\[ G = \langle X \mid w = 1 \rangle : \text{some } t \in X \text{ has exponent sum zero in } w. \]

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**Theorem**

*Suppose that:*

1. \( \rho_t(w) \) is cyclically reduced;
2. \( H \) has decidable rational subset membership problem;
3. \( A \leq H \) is effectively closed for rational intersections;
4. \( w \) is either prefix \( t \)-positive or prefix \( t \)-negative.

\( \Rightarrow \) \( G \) has decidable prefix membership problem.
Application #5: Exponent sum zero result

\[ G = \langle \mathbb{G}p \langle X \mid w = 1 \rangle \rangle : \text{some } t \in X \text{ has exponent sum zero in } w. \]

By general theory ("Magnus' method", also Lyndon & McCool), \( G \) is isomorphic to an HNN extension of

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*Suppose that:*

- \( \rho_t(w) \) is cyclically reduced;
- \( H \) has decidable rational subset membership problem;
- \( A \leq H \) is effectively closed for rational intersections;
- \( w \) is either prefix \( t \)-positive or prefix \( t \)-negative.

\[ \implies G \text{ has decidable prefix membership problem.} \]
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\[ G = Gp\langle X \mid w = 1 \rangle: \] some \( t \in X \) has exponent sum zero in \( w \).

By general theory ("Magnus’ method", also Lyndon & McCool), \( G \) is \( \cong \) an HNN extension of

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Application #5: Exponent sum zero result

\[ G = G_p\langle X \mid w = 1 \rangle: \text{ some } t \in X \text{ has exponent sum zero in } w. \]

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\[ \implies G \text{ has decidable prefix membership problem.} \]
Application #5: Exponent sum zero result (example)

\[ w \equiv t^{-1}bcbt^{-8}bbct^6ct^3at^{-3}bt^3at^{-3}ct^2cta \]
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\[ w \equiv t^{-1} bcbt^{-8} bbct^6 ct^3 at^{-3} bt^3 at^{-3} ct^2 ct a \]

\[ \downarrow \]

\[ \rho_t(w) \equiv \]
Application #5: Exponent sum zero result (example)

\[ w \equiv t^{-1} b c b t^{-8} b b c t^{6} c t^{3} a t^{-3} b t^{3} a t^{-3} c t^{2} c t^{a} \]

\[ \downarrow \]

\[ \rho_t(w) \equiv b_1 \]
Application #5: Exponent sum zero result (example)

\[ w \equiv t^{-1} bcb t^{-8} b c t^6 c t^{-3} b t^3 a t^{-3} c t^2 c t a \]
\[ \downarrow \]
\[ \rho_t(w) \equiv b_1 c_1 \]
Application #5: Exponent sum zero result (example)

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\[ \downarrow \]
\[ \rho_t(w) \equiv b_1c_1b_1 \]
Application #5: Exponent sum zero result (example)

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\[ \downarrow \]

\[ \rho_t(w) \equiv b_1 c_1 b_1 b_9 \]
Application #5: Exponent sum zero result (example)

\[ w \equiv t^{-1} bcb t^{-8} bbct^6 c t^3 a t^{-3} b t^3 a t^{-3} c t^2 c t a \]

\[ \rho_t(w) \equiv b_1 c_1 b_1 b_9 b_9 \]
Application #5: Exponent sum zero result (example)

\[ w \equiv t^{-1} bcbt^{-8} bbct^6 ct^3 at^{-3} bt^3 at^{-3} ct^2 ct a \]

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\[ w \equiv t^{-1} bcbt^{-8} bbct^6 ct^3 a t^{-3} b t^3 a t^{-3} c t^2 c t a \]

\[ \downarrow \]

\[ \rho_t(w) \equiv b_1 c_1 b_1 b_9 b_9 c_9 c_3 a_0 \]
Application #5: Exponent sum zero result (example)

\[ w \equiv t^{-1} b c b t^{-8} b b c t^6 c t^3 a t^{-3} b t^3 a t^{-3} c t^2 c t a \]

\[ \Downarrow \]

\[ \rho_t(w) \equiv b_1 c_1 b_1 b_9 b_9 c_9 c_3 a_0 b_3 \]
Application #5: Exponent sum zero result (example)

\[ w \equiv t^{-1} b c b t^{-8} b b c t^6 c t^3 a t^{-3} b t^3 a t^{-3} c t^2 c t a \]

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\[ \downarrow \]

\[ \rho_t(w) \equiv b_1 c_1 b_1 b_9 b_9 c_9 c_3 a_0 b_3 a_0 c_3 \]
Application #5: Exponent sum zero result (example)

\[ w = t^{-1} b c b t^{-8} b b c t^6 c t^3 a t^{-3} b t^3 a t^{-3} c t^2 c t a \]

\[ \downarrow \]

\[ \rho_t(w) = b_1 c_1 b_1 b_9 b_9 c_9 c_3 a_0 b_3 a_0 c_3 c_1 \]
Application #5: Exponent sum zero result (example)

\[ w \equiv t^{-1} bcb t^{-8} b b c t^6 c t^3 a t^{-3} b t^3 a t^{-3} c t^2 c t a \]

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Application #5: Exponent sum zero result (example)

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\[ G = \langle X \mid w = 1 \rangle \] is \( \cong \) an HNN extension of

\[ H = \langle a_0, b_1, \ldots, b_9, c_1, \ldots, c_9 \mid \rho_t(w) = 1 \rangle \] (free of rank 18)

w.r.t. \( A = \langle b_1, \ldots, b_8, c_1, \ldots, c_8 \rangle \) and \( B = \langle b_2, \ldots, b_9, c_2, \ldots, c_9 \rangle \)

(which are free by Freiheitssatz);
Application #5: Exponent sum zero result (example)

\[ w \equiv t^{-1} b c b t^{-8} b b c t^{6} c t^{3} a t^{-3} b t^{3} a t^{-3} c t^{2} c t a \]

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\[ \longrightarrow G \] has decidable prefix membership problem.
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\[ w \equiv t^{-1} b c b t^{-8} b b c t^6 c t^3 a t^{-3} b t^3 a t^{-3} c t^2 c t a \]

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\[ \rho_t(w) \equiv b_1 c_1 b_1 b_9 b_9 c_9 c_3 a_0 b_3 a_0 c_3 c_1 a_0 \]

\( G = \text{Gp}\langle X \mid w = 1 \rangle \) is \( \cong \) an HNN extension of

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\[ \implies G \text{ has decidable prefix membership problem.} \]

+ \( w \) is cyclically reduced \( \implies M = \text{Inv}\langle X \mid w = 1 \rangle \) has decidable WP.
Application #5: Exponent sum zero result (example)

\[ w \equiv t^{-1} b c b t^{-8} b b c t^{6} c t^{3} a t^{-3} b t^{3} a t^{-3} c t^{2} c t a \]

\[ \rho_t(w) \equiv b_1 c_1 b_1 b_9 b_9 c_9 c_3 a_0 b_3 a_0 c_3 c_1 a_0 \]

\[ G = \langle X \mid w = 1 \rangle \] is \( \cong \) an HNN extension of \[ H = \langle a_0, b_1, \ldots, b_9, c_1, \ldots, c_9 \mid \rho_t(w) = 1 \rangle \] (free of rank 18)

w.r.t. \( A = \langle b_1, \ldots, b_8, c_1, \ldots, c_8 \rangle \) and \( B = \langle b_2, \ldots, b_9, c_2, \ldots, c_9 \rangle \) (which are free by Freiheitssatz);

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Further examples:

▶ large classes of Adyan-type presentations;
Application #5: Exponent sum zero result (example)

\[ w \equiv t^{-1} bcbt^{-8} bbct^6 ct^3 at^{-3} bt^3 at^{-3} ct^2 cta \]

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\[ \rho_t(w) \equiv b_1 c_1 b_1 b_9 b_9 c_9 c_3 a_0 b_3 a_0 c_3 c_1 a_0 \]

\[ G = \text{Gp}\langle X \mid w = 1 \rangle \text{ is } \cong \text{ an HNN extension of } \]
\[ H = \text{Gp}\langle a_0, b_1, \ldots, b_9, c_1, \ldots, c_9 \mid \rho_t(w) = 1 \rangle \text{ (free of rank 18)} \]

w.r.t. \[ A = \text{Gp}\langle b_1, \ldots, b_8, c_1, \ldots, c_8 \rangle \text{ and } B = \text{Gp}\langle b_2, \ldots, b_9, c_2, \ldots, c_9 \rangle \]

(\text{which are free by Freiheitssatz});

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Further examples:

\[ \blacktriangleright \text{ large classes of Adyan-type presentations;} \]
\[ \blacktriangleright \text{ conjugacy pinched presentations } \text{Gp}\langle X, t \mid t^{-1} utv^{-1} = 1 \rangle \]
\[ \quad (u, v \in X^* \text{ reduced}), \]
Application #5: Exponent sum zero result (example)

\[ w \equiv t^{-1} bcbt^{-8} bbct^6 ct^3 at^{-3} bt^3 at^{-3} ct^2 cta \]

\[
\downarrow
\]

\[ \rho_t(w) \equiv b_1 c_1 b_1 b_9 b_9 c_9 c_3 a_0 b_3 a_0 c_3 c_1 a_0 \]

\[ G = \langle X | w = 1 \rangle \text{ is } \cong \text{ an HNN extension of } H = \langle a_0, b_1, \ldots, b_9, c_1, \ldots, c_9 | \rho_t(w) = 1 \rangle \text{ (free of rank 18) w.r.t. } A = \langle b_1, \ldots, b_8, c_1, \ldots, c_8 \rangle \text{ and } B = \langle b_2, \ldots, b_9, c_2, \ldots, c_9 \rangle \text{ (which are free by Freiheitssatz)}; \]

\[ \implies G \text{ has decidable prefix membership problem.} \]
\[ + \ w \text{ is cyclically reduced } \implies M = \langle X | w = 1 \rangle \text{ has decidable WP.} \]

Further examples:

\[ \begin{align*}
\text{large classes of Adyan-type presentations;} \\
\text{conjugacy pinched presentations } \langle X, t | t^{-1} utv^{-1} = 1 \rangle \\
(u, v \in X^* \text{ reduced}), \text{ including Baumslag-Solitar groups:} \\
B(m, n) = \langle a, b | b^{-1} a^m ba^{-n} = 1 \rangle.
\end{align*} \]
The grand finale & an open problem

By modifying slightly the ideas from Bob’s Inventiones paper, we obtain

**Theorem**

*There exists a reduced word* $w$ *over a 3-letter alphabet* $X$ *such that* $G = Gp\langle X \mid w = 1 \rangle$ *has undecidable prefix membership problem.*
The grand finale & an open problem

By modifying slightly the ideas from Bob’s *Inventiones* paper, we obtain

**Theorem**

*There exists a reduced word* \( w \) *over a 3-letter alphabet* \( X \) *such that* \( G = \langle X \mid w = 1 \rangle \) *has undecidable prefix membership problem.*

**Open Problem**

Characterise the words \( w \in \langle X^* \rangle \) such that the prefix membership problem for \( \langle X \mid w = 1 \rangle \) is decidable.
The grand finale & an open problem

By modifying slightly the ideas from Bob’s *Inventiones* paper, we obtain

**Theorem**

There exists a reduced word $w$ over a 3-letter alphabet $X$ such that $G = \langle X \mid w = 1 \rangle$ has undecidable prefix membership problem.

**Open Problem**

Characterise the words $w \in \overline{X}^*$ such that the prefix membership problem for $\langle X \mid w = 1 \rangle$ is decidable. In particular, what about cyclically reduced words?
Thank you!

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