Facets of the Finite Basis Problem for Finite Involution Semigroups

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Glossary of terms

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- algebras generating congruence $\wedge$-semidistributive varieties with a finite residual bound (Willard, 2000)

Negative results

Examples of finite NFB algebras:

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2	0	2	2

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Tarski's Finite Basis Problem: Is there any algorithmic way to distinguish between finite FB and NFB algebras?

McKenzie's solution of the Tarski problem

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M. V. Volkov: *The finite basis problem for finite semigroups*, Sci. Math. Jpn. **53** (2001), 171–199.
http://csseminar.kadm.usu.ru/MATHJAP_revisited.pdf

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A_2 is representable by matrices (over any field).

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- *matrix semigroups* $\mathcal{M}_n(\mathbb{F})$ *for any* $n \geq 2$ *and any finite field* $\mathbb{F}$

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Examples

- groups
- inverse semigroups
- regular *-semigroups $(xx^*x \approx x)$
- matrix semigroups with transposition $\mathcal{M}_n(\mathbb{F}) = (\mathrm{M}_n(\mathbb{F}), \cdot, ^{\mathrm{T}})$

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Furthermore, let K_3 be the 10-element unary Rees matrix semigroup over a trivial group $E = \{e\}$ with the sandwich matrix

$$\begin{pmatrix} e & e & e \\ e & e & 0 \\ e & 0 & e \end{pmatrix},$$

while $(i, e, j)^* = (j, e, i)$ and $0^* = 0$.

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Fact

K_3 generates the variety of all strict combinatorial regular *-semigroups (studied by K. Auinger in 1992).

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Theorem (K. Auinger, M. V. Volkov, cca. 1991/92)

Let S be a unary semigroup such that $\mathbf{V} = \mathrm{var}\, S$ contains K_3. If there exist a group G which belongs to $\mathbf{V}$ but not to $\mathrm{H}(\mathbf{V})$, then S is NFB.

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- *matrix semigroups $(\mathrm{M}_2(\mathbb{F}), \cdot, ^\dagger)$, where $\mathbb{F}$ is either a finite field such that $|\mathbb{F}| \equiv 3 \pmod 4$, or a subfield of $\mathbb{C}$ closed under complex conjugation, and † is the unary operation of taking the Moore-Penrose inverse.*

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Also, the following open problem was both intriguing and inviting.

Problem

Do finite INFB involution semigroups exist at all?

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INFB algebras are a powerful tool for proving the NFB property; namely, the INFB property is "contagious":

> if $\operatorname{var} A$ is locally finite and contains an INFB algebra B, then A is NFB.

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In particular, B is NFB.

Finite INFB semigroups: a success story

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Theorem (Sapir, 1987)

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$$S \text{ is INFB} \Longleftrightarrow S \not\models Z_n \approx W$$

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Sapir also found an effective structural description of finite INFB semigroups, thus proving

Theorem (Sapir, 1987)

It is decidable whether a finite semigroup is INFB or not.

Examples of finite INFB semigroups

The example: the 6-element Brandt inverse monoid

$$B_2^1 = \langle a, b : \ a^2 = b^2 = 0, \ aba = a, \ bab = b \rangle \cup \{1\}.$$

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Corollary

For any $n \geq 2$ and any (semi)ring R, the matrix semigroup $\mathcal{M}_n(R)$ is (I)NFB.

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B_2^1 fails to satisfy a nontrivial identity of the form $Z_n \approx W$. Hence, it is INFB.

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For any $n \geq 2$ and any (semi)ring R, the matrix semigroup $\mathcal{M}_n(R)$ is (I)NFB.

Since $B_2^1 \in \operatorname{var} A_2^1$, where A_2 is the 5-element semigroup from Volkov's theorem, we have that A_2^1 is (I)NFB as well.

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The same argument applies to $\mathcal{T}_n$ ($n \geq 3$), $\mathcal{R}_n$ ($n \geq 2$), $\mathcal{PT}_n$ ($n \geq 2$),...

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So, once again:

Problem

Do finite INFB involution semigroups exist at all?

An INFB criterion for involution semigroups

Yes!

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Theorem (ID, cca. 2007/08)

Let S be an involution semigroup such that $\operatorname{var} S$ *is locally finite. If S fails to satisfy any nontrivial identity of the form*

$$Z_n \approx W,$$

where W is an involutorial word (a word over the 'doubled' alphabet $X \cup X^*$), then S is INFB.

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How about a (finite) example?

‘C’mon baby, let’s do the twist...!’

Rescue: Luckily, B_2^1 admits one more involution aside from the inverse one: define the nilpotents a, b (and, of course, $0, 1$) to be fixed by *, which results in $(ab)^* = ba$ and $(ba)^* = ab$.

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Remark

Analogously, one can also define TA_2^1, the “involutorial version” of A_2^1, which is also INFB.

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So, what about $\mathcal{M}_2(\mathbb{F})$ if $|\mathbb{F}| \equiv 3 \pmod 4$?

(We already know it is NFB.)

Non-INFB results

Theorem (ID, 2010)

Let S be a finite involution semigroup satisfying a nontrivial identity of the form $Z_n \approx W$ such that $B_2^1 \notin \operatorname{var} S$. Then S is not INFB.

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Proof idea: Stretching the approach of Margolis & Sapir (1995) developed for finitely generated quasivarieties of semigroups to what seems to be the final limits of that method: certain semigroup quasiidentities can be "encoded" into unary semigroup identities.

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No finite regular *-semigroup is INFB.
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Remark

The ordinary power semigroup $\mathcal{P}_G = (\mathcal{P}(G), \cdot)$ is INFB if and only if G is not Dedekind.

Non-INFB results

Proposition (Crvenković, 1982)

If a finite involution semigroup S admits a Moore-Penrose inverse †, then the inverse is term-definable in S.

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Proposition

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This completes our classification! $\heartsuit$

Solution to the (I)NFB problem for matrix involution semigroups

Theorem (Auinger, ID, Volkov, 2008-10)

Let $n \geq 2$ and $\mathbb{F}$ be a finite field. Then

(1) $\mathcal{M}_n(\mathbb{F})$ *is not finitely based;*

(2) $\mathcal{M}_n(\mathbb{F})$ *is INFB if and only if either $n \geq 3$, or $n = 2$ and $|\mathbb{F}| \not\equiv 3 \pmod 4$.*

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Test-Example

Is $xyxzxyx \approx xyxx^*xzxyx$ implying the non-INFB property?

THANK YOU!

Questions and comments to:

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http://sites.dmi.rs/personal/dolinkai