Canone Inverso

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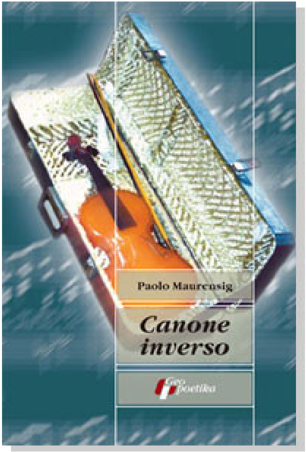
Paolo Maurensig
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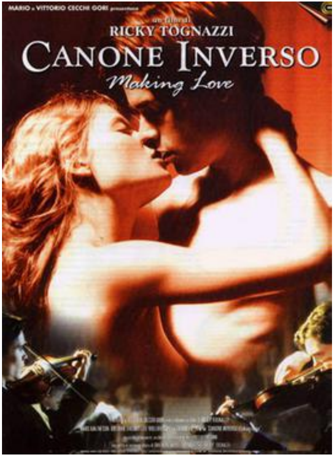
RICKY TOGNAZZI
CANONE INVERSO
Making Love

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Few basic facts:

- Every $a \in M$ has a unique inverse ($axa = a$, $xax = x$): a^{-1}.
- All idempotents are of the form aa^{-1} $(a \in M)$.
- Idempotents form a semilattice.

Additional *fun facts*

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- right units = injections $X \to X$;
- the maximal subgroup "around" $\mathrm{id}_A \cong \mathbb{S}_A$.

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- The order is compatible with the operations.
- The order extends the semilattice order on idempotents.

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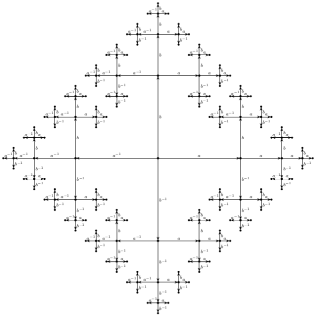
Fact: *F-inverse $\Longrightarrow$ E-unitary*

Free inverse monoids (I)

Start from the Cayley graph of the free group $FG(X)$:



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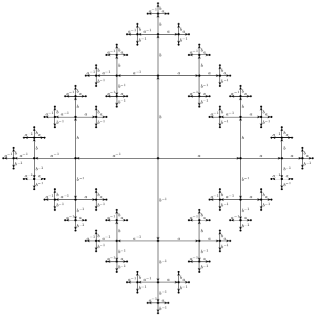

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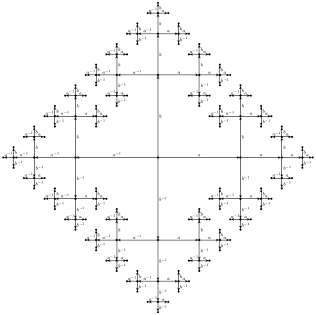
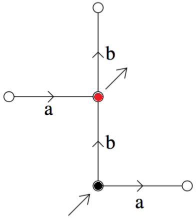


Munn / Scheiblich (1973/74): $FIM(X)$ = pairs (Γ, g) where:

- $g \in FG(X)$,
- Γ is a finite connected subgraph of the Cayley graph of $FG(X)$ containing the vertices 1 and g.

Free inverse monoids (II)

Munn trees:



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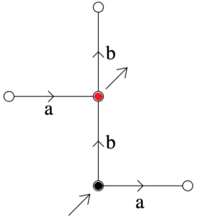

Btw: This solves the WP for $FIM(X)$ – e.g. the tree above verifies

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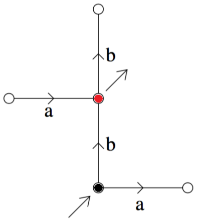

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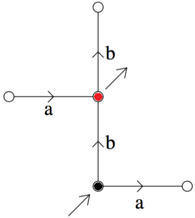

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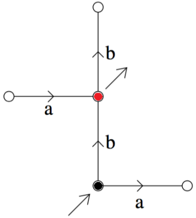

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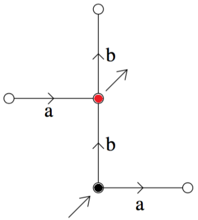

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Margolis & Meakin (1989): Wait! Why only $FG(X)$? We can do this for the Cayley graph of any X-generated group G...!

Thus we get $M(G, X)$ – the Margolis-Meakin expansion = the universal X-generated E-unitary inverse monoid with max. group image G.

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$$\mathsf{Inv}\langle A \mid u_i = v_i \ (i \in I)\rangle$$

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The inverse monoid M is special if it admits a presentation where all words v_i are empty (i.e. all relations are of the form $u_i = 1$).

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Three "cute" results for $M = \mathsf{Inv}\langle A \mid u_i = 1 \ (i \in I)\rangle$:

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(Ivanov, Margolis, Meakin, 2001) 

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Theorem (S.I.Adian, 1966)

Yes, if:

- *the monoid is special, i.e. v is an empty word, or*
- *the words u, v are non-empty and they have both distinct first letters and distinct last letters.*

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Theorem (Adian & Oganessian, 1987)

The general 1-relator monoid problem reduces to the cases:

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Hence, solving WP for $\mathrm{Inv}\langle A \mid w = 1 \rangle \Longrightarrow$ (That's a) bingo!

Mr Party Spoiler

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There is a 1-relator special inverse monoid (SIM)

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Reason: $\mathrm{RU}(M)$ has undecidable membership in M. Namely, there is a 1-relator group $G = \mathrm{Gp}\langle a, b \mid \dots = 1 \rangle$ and a f.g. submonoid T with undecidable membership in G s.t. $\forall\, u \in \{a, b, a^{-1}, b^{-1}\}^*$ we have

$$u \in T \iff tut^{-1} \in \mathrm{RU}(M).$$

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Reason: $\mathrm{RU}(M)$ has undecidable membership in M. Namely, there is a 1-relator group $G = \mathsf{Gp}\langle a, b \mid \ldots = 1\rangle$ and a f.g. submonoid T with undecidable membership in G s.t. $\forall\, u \in \{a, b, a^{-1}, b^{-1}\}^*$ we have

$$u \in T \iff tut^{-1} \in \mathrm{RU}(M).$$

However, not all is lost, as the relators from the IMM theorem are of a quite particular form.

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...and some of them are explicitly unsettling (and disturbing). :) :)

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Absolute beauty, it simply couldn't be better. :)

The Higman Embedding Theorem

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A group H is weakly recursively presented if there exists a f.p. group $G = \mathrm{Gp}\langle A \mid \mathfrak{R}\rangle$ and a r.e. language $L = \{u_i : \ i \in \mathbb{N}\}$ (over $A \cup A^{-1}$) such that the set of elements of G represented by words from L is precisely H.

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This is basically: recursively presented "minus" finite generation.

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Bonus: There are f.p. SIMs containing infinitely many pairwise non-isomorphic maximal subgroups.

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Theorem (IgD, Gray, 2023)

The class of all prefix monoids consists precisely of the following ones:

$$M * \Sigma_k^*$$

where M is recursively presented group embeddable monoid, $|\Sigma_k| = k$, and $k \geq \mu_M$ where $\mu_M \geq 0$ is a constant depending (only) on M.

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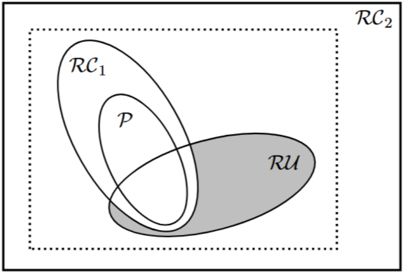

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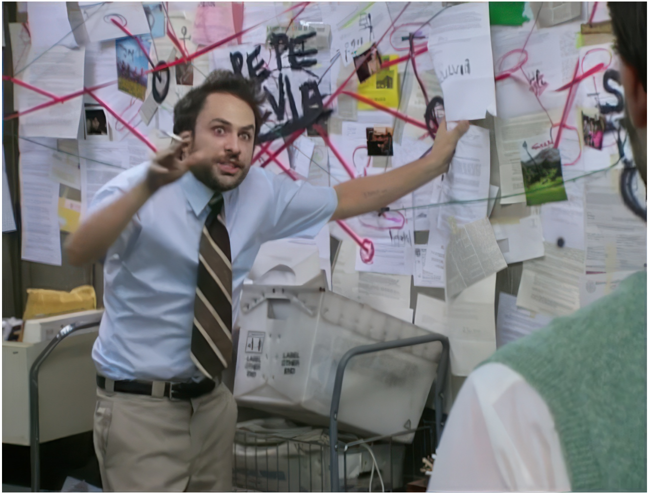
- There is a finitely RC-presented right cancellative monoid S such that the group U_S is not finitely presented (even though $S \setminus U_S$ is an ideal).

The visual of the construction



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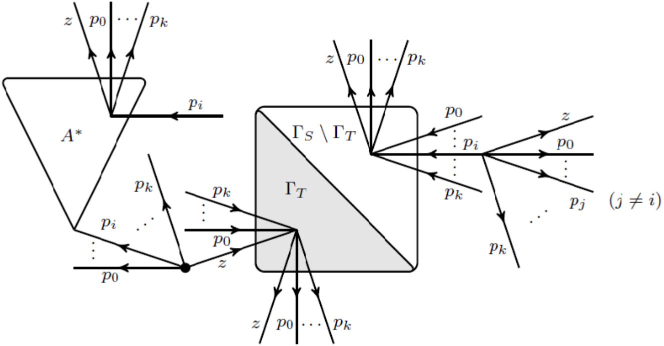

Outro: An ongoing project here in Ljubljana (1)

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Theorem (IgD, Kudryavtseva, 2025/6)

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- $\mathrm{Inv}\langle a, b, c \mid abc = 1\rangle$ is F-inverse but not strongly F-inverse.
- Every inverse monoid with a free canonical group image is strongly F-inverse, and so in particular $\mathrm{Inv}\langle A \mid e = 1\rangle$ for any Dyck word over A.
- So, there exist strongly F-inverse monoids $\mathrm{Inv}\langle A \mid w = 1\rangle$ for cyclically non-reduced words w as well.

Hvala za vašo pozornost!
Hvala Ljubljana! 😊 ❤️