Canone Inverso

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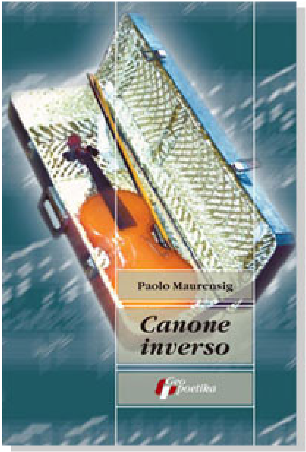
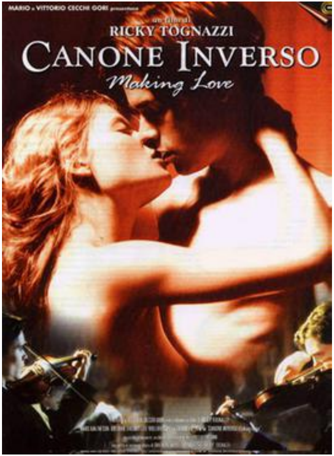

Paolo Maurensig
Canone inverso
poetika


RICKY TOGNAZZI
CANONE INVERSO
Making Love

Inverse monoids

$(M, \cdot, 1, ^{-1})$ – where $(M, \cdot, 1)$ is a monoid, $^{-1}$ is an involution, so:

$$(xy)^{-1} = y^{-1}x^{-1}, \qquad \left(x^{-1}\right)^{-1} = x,$$

x^{-1} is an inverse of x:

$$xx^{-1}x = x,$$

and the idempotents commute:

$$xx^{-1}yy^{-1} = yy^{-1}xx^{-1}.$$

Few basic facts:

- Every $a \in M$ has a unique inverse ($axa = a$, $xax = x$): a^{-1}.
- All idempotents are of the form aa^{-1} ($a \in M$).
- Idempotents form a semilattice.

Additional *fun facts*

- Each $\mathscr{R}$-class / $\mathscr{L}$-class contains a unique idempotent ($a \mathscr{R} aa^{-1}$ / $a \mathscr{L} a^{-1}a$).
- Hence, $a \mathscr{R} b \iff aa^{-1} = bb^{-1}$ / $a \mathscr{L} b \iff a^{-1}a = b^{-1}b$.
- $a \mathscr{R} 1$ C $aa^{-1} = 1$ – right units
- $a \mathscr{H} 1 \iff aa^{-1} = a^{-1}a = 1$ – units

Main example: the symmetric inverse monoid $\mathcal{I}_X$: its elements are the partial bijections $f : A \to B$ $(A, B \subseteq X)$

- $f \mathscr{R} g$ / $f \mathscr{L} g \iff \mathrm{dom}(f) = \mathrm{dom}(g)$ / $\mathrm{im}(f) = \mathrm{im}(g)$;
- the idempotents = identity maps on subsets of X;
- right units = injections $X \to X$;
- the maximal subgroup "around" $\mathrm{id}_A \cong \mathbb{S}_A$.

The natural order

$$a \leq b \quad \Longleftrightarrow \quad a = aa^{-1}b$$

Equivalent to:

- $a = ba^{-1}a$
- $a = eb$ for some $e \in E(M)$
- $a = bf$ for some$f \in E(M)$
- ...

- The order is compatible with the operations.
- The order extends the semilattice order on idempotents.

The maximum group image of an inverse monoid

ρ is a group congruence of M if M/ρ is a group.

Fact: *An intersection of a family of group congruences is again a group congruence.*

$\Longrightarrow(\exists)$: σ – the minimum group congruence of M.

$\Longrightarrow$ M/σ – maximum group image of the inverse monoid M.

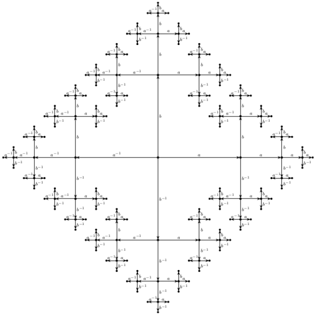
M is E-unitary if $1/\sigma = E(M)$.

M is F-inverse if every σ-class has a greatest element w.r.t. $\leq$.

Fact: *F-inverse $\Longrightarrow$ E-unitary*

Free inverse monoids (I)

Start from the Cayley graph of the free group $FG(X)$:

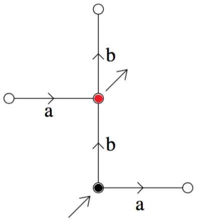


Munn / Scheiblich (1973/74): $FIM(X)$ = pairs (Γ, g) where:

- $g \in FG(X)$,
- Γ is a finite connected subgraph of the Cayley graph of $FG(X)$ containing the vertices 1 and g.

Free inverse monoids (II)

Munn trees:



Btw: This solves the WP for $FIM(X)$ – e.g. the tree above verifies

$$aa^{-1}bb^{-1}ba^{-1}abb^{-1} = bbb^{-1}a^{-1}ab^{-1}aa^{-1}b.$$

Margolis & Meakin (1989): Wait! Why only $FG(X)$? We can do this for the Cayley graph of any X-generated group G...!

Thus we get $M(G, X)$ – the Margolis-Meakin expansion = the universal X-generated E-unitary inverse monoid with max. group image G.

Presentations (I)

$$\mathrm{Inv}\langle A \mid u_i = v_i \ (i \in I)\rangle$$

where $u_i, v_i \in (A \cup A^{-1})^*$. This defines $FIM(A)/\rho_{\mathfrak{R}}$ (where $\rho_{\mathfrak{R}}$ is the congruence on $FIM(A)$ generated by $\mathfrak{R} = \{(u_i, v_i) : \ i \in I\}$).

The maximum group image of $\mathrm{Inv}\langle A \mid u_i = v_i \ (i \in I)\rangle$ is:

$$\mathrm{Gp}\langle A \mid u_i v_i^{-1} = 1 \ (i \in I)\rangle.$$

The inverse monoid M is special if it admits a presentation where all words v_i are empty (i.e. all relations are of the form $u_i = 1$).

Presentations (II)

Three "cute" results for $M = \mathrm{Inv}\langle A \mid u_i = 1\ (i \in I)\rangle$:

- The elements of M represented by the prefixes of the words u_i generate the monoid of right units $\mathrm{RU}(M)$.
- There is a factorisation of the words u_i such that the *pieces* represent units, and which is the finest one in the sense that the *pieces* cannot be factorised further into words representing units. If so, the *pieces* generate the group of units U_M.
- If $w \in (A \cup A^{-1})^*$ is a cyclically reduced word, then

$$\mathrm{Inv}\langle A \mid w = 1\ (i \in I)\rangle$$

is E-unitary.

(Ivanov, Margolis, Meakin, 2001) 

The WP for 1-relator structures (I)

Theorem (W.Magnus, 1932)

The word problem is decidable for all 1-relator groups $\mathrm{Gp}\langle A \mid w = 1\rangle$.

Still open: Is the word problem decidable for all 1-relator monoids $\mathrm{Mon}\langle A \mid u = v\rangle$?

Theorem (S.I.Adian, 1966)

Yes, if:

- *the monoid is special, i.e. v is an empty word, or*
- *the words u, v are non-empty and they have both distinct first letters and distinct last letters.*

The WP for 1-relator structures (II)

Theorem (Adian & Oganessian, 1987)

The general 1-relator monoid problem reduces to the cases:

- $\mathrm{Mon}\langle a, b \mid aUb = aVa \rangle$,
- $\mathrm{Mon}\langle a, b \mid aUb = a \rangle$.

(Very relevant) Fact: Both these presentations define right cancellative monoids.

Theorem (I+M+M, 2001)

$\mathrm{Mon}\langle A \mid u = v \rangle$ *embeds into* $\mathrm{Inv}\langle A \mid uv^{-1} = 1 \rangle$.

- $\mathrm{Mon}\langle a, b \mid aUb = aVa \rangle$ embeds into $\mathrm{Inv}\langle a, b \mid aUba^{-1}V^{-1}a^{-1} = 1 \rangle$,
- $\mathrm{Mon}\langle a, b \mid aUb = a \rangle$ embeds into $\mathrm{Inv}\langle a, b \mid aUba^{-1} = 1 \rangle$.

Hence, solving WP for $\mathrm{Inv}\langle A \mid w = 1 \rangle \Longrightarrow$ (That's a) bingo!

Mr Party Spoiler

Theorem (R.D.Gray, Invent. Math. 2020)

There is a 1-relator special inverse monoid (SIM)

$$\mathsf{Inv}\langle a, b, t \mid \ldots = 1\rangle$$

with an undecidable word problem.

Reason: $\mathrm{RU}(M)$ has undecidable membership in M. Namely, there is a 1-relator group $G = \mathsf{Gp}\langle a, b \mid \ldots = 1\rangle$ and a f.g. submonoid T with undecidable membership in G s.t. $\forall\, u \in \{a, b, a^{-1}, b^{-1}\}^*$ we have

$$u \in T \iff tut^{-1} \in \mathrm{RU}(M).$$

However, not all is lost, as the relators from the IMM theorem are of a quite particular form.

In order to gain awe and a full grasp...

...of the complexity of the 1-relator problem for (inverse) monoids, we are keen to answer the following questions:

- What can the groups of units of f.p. SIMs be?
- What can the maximal subgroups of f.p. SIMs be?
- What can the prefix monoids of f.p. groups be?
- What can the monoids of right units of f.p. SIMs be?

Some of the answers (sadly, not all) are contained in:

- R.D.Gray, M.Kambites, Maximal subgroups of finitely presented special inverse monoids, *J. Eur. Math. Soc.* (to appear), 25pp.
- IgD, R.D.Gray, Prefix monoids of groups and right units of special inverse monoids, *Forum Math. Sigma* **11** (2023), e97:1–19.
- IgD, R.D.Gray, On right units of special inverse monoids, (submitted), arXiv:2512.15591, 45pp.

...and some of them are explicitly unsettling (and disturbing). :) :)

Let us take a look at the case of ordinary monoids

$M = \mathrm{Mon}\langle A \mid w = 1 \rangle$

- The group U_M is always 1-relator.
  (After all, this is a key ingredient in the modern proofs of the decidability of the WP for 1-relator special monoids.)
- All maximal subgroups of M are the same, i.e. isomorphic to U_M. (Malheiro, 2005)
- We always have $\mathrm{RU}(M) \cong U_M * X^*$ for some finite set X.
- All Schützenberger graphs (restrictions of the Cayley graph to the $\mathscr{R}$-classes) are isomorphic.

Absolute beauty, it simply couldn't be better. :)

The Higman Embedding Theorem

We say a group $\mathrm{Gp}\langle A \mid w_i = 1 \ (i \in I)\rangle$ is recursively presented if A is finite and $\{w_i : \ i \in I\} \subseteq (A \cup A^{-1})^*$ is a r.e. language.

Theorem (G.Higman, 1961)

A finitely generated group G embeds into a finitely presented one if and only if G is recursively presented.

Remark: There are analogous results for monoids (Murskiĭ, 1967) and inverse monoids (Belyaev, 1984).

A group H is weakly recursively presented if there exists a f.p. group $G = \mathrm{Gp}\langle A \mid \mathfrak{R}\rangle$ and a r.e. language $L = \{u_i : \ i \in \mathbb{N}\}$ (over $A \cup A^{-1}$) such that the set of elements of G represented by words from L is precisely H.
This is basically: recursively presented "minus" finite generation.

Subgroups of finitely presented SIMs

Theorem (Gray, Kambites)

Let H be a group.

(i) *There is a f.p. SIM M such that $U_M \cong H \iff H$ is recursively presented.*

(ii) *There is a f.p. SIM M such that $H \cong$ to a maximal subgroup of $M \iff H$ is weakly recursively presented.*

Bonus: There are f.p. SIMs containing infinitely many pairwise non-isomorphic maximal subgroups.

Prefix monoids

= f.g. monoids $\cong$ prefix monoid of a f.p. group.

Preliminary facts:

- Every prefix monoid is group embeddable and recursively presented.
- A group is a prefix monoid $\iff$ it is finitely presented.
- Every f.p. group embeddable monoid is prefix.

Theorem (IgD, Gray, 2023)

The class of all prefix monoids consists precisely of the following ones:

$$M * \Sigma_k^*$$

where M is recursively presented group embeddable monoid, $|\Sigma_k| = k$, and $k \geq \mu_M$ where $\mu_M \geq 0$ is a constant depending (only) on M.

RU-monoids

$=$ f.g. monoids $\cong \mathrm{RU}(M)$ for some f.p. SIM M.

Preliminary facts:

- Every prefix monoid is right cancellative and recursively presented.
- A group is a RU-monoid $\iff$ it is finitely presented.

$$\mathsf{MonRC}\langle A \mid \mathfrak{R} \rangle$$

$= A^*/\mathfrak{R}^{\mathsf{RC}}$ where $\mathfrak{R}^{\mathsf{RC}}$ is the least right cancellative congruence on A^* containing $\mathfrak{R}$.

☞ *The intersection of any family of right cancellative congruences is again a right cancellative congruence.*

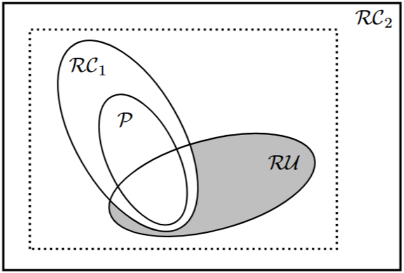
Theorem (IgD, Gray, 2023)

Every finitely RC-presented monoid is an RU-monoid.

Classes of right cancellative monoids

The problem is: we (currently) don't have a Higman Theorem for RC-presentations, and thus:

- $\mathcal{RC}_1$ = f.g. submonoid of f.RC-p. right cancellative monoids
- $\mathcal{RC}_2$ = recursively RC-presented right cancellative monoids



More on RU-monoids (1)

Theorem (IgD, Gray, 2025)
Let M be a f.p. SIM. If

$$\mathrm{RU}(M) \cong U_M * T$$

for a f.g. monoid T with a trivial group of units $\Longrightarrow$ U_M is f.p.

Consequences:

- G a f.g. group that is not f.p. $\Longrightarrow$ $G * X^* \notin \mathcal{RU}$ ($\forall$ finite X).
- $\mathcal{P} \not\subseteq \mathcal{RU}$ (and $\mathcal{RU} \not\subseteq \mathcal{P}$).
- $\mathcal{RC}_1 \not\subseteq \mathcal{RU}$.
- $\mathcal{RU} \neq \mathcal{RC}_2$.

More on RU-monoids (2)

Theorem (IgD, Gray, 2025)

Every f.g. submonoid of a f.RC-p. monoid (aka $\in \mathcal{RC}_1$) is isomorphic to a submonoid N of a f.p. SIM M such that:

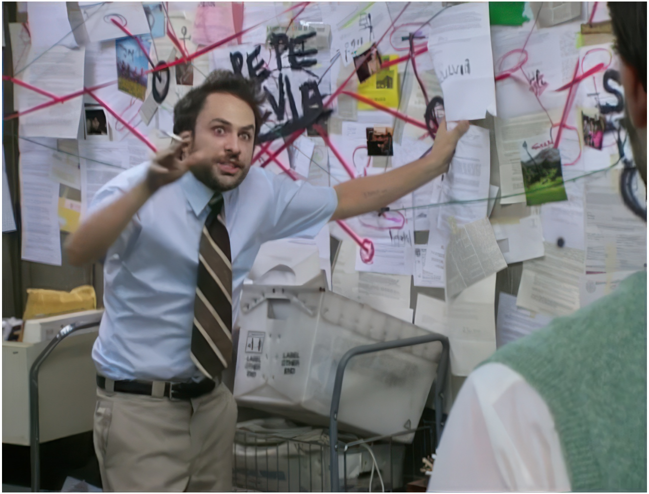
(a) $N \subseteq \mathrm{RU}(M)$,

(b) $U_M \subseteq N$.

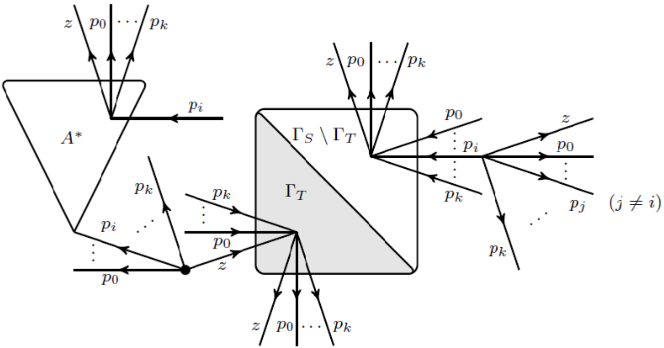
Bonuses:

- There is a f.p. SIM M such that $\mathrm{RU}(M)$is not finitely RC-presented, while the group U_M is trivial.
- There is a(n E-unitary) f.p. SIM M such that $\mathrm{RU}(M)$ is finitely RC-presented, but not finitely presented as a monoid.
- There is a finitely RC-presented right cancellative monoid S such that the group U_S is not finitely presented (even though $S \setminus U_S$ is an ideal).

The visual of the construction



The visual of the construction



Outro: An ongoing project here in Ljubljana (1)

Goal

Describe the 1-relator inverse monoids $\mathrm{Inv}\langle A \mid w = 1\rangle$ with the F-inverse property.

Remember: F-inverse $\Longrightarrow$ E-unitary, so it is a good idea to start with the case when w is cyclically reduced.

Example (Kambites, Szakács, 2026)

$\mathrm{Inv}\langle a, b, c, d \mid bcb^{-1}ad^{-1}a^{-1} = 1\rangle$ is not F-inverse. [!!!]

An X-generated F-inverse monoid S with maximum group image G is strongly F-inverse if the canonical map $M(G, X) \to S$ maps all maximal elements in each σ-class of $M(G, X)$ (which are the simple paths $1 \rightsquigarrow g$) to the greatest element of the corresponding σ-class of S.

Outro: An ongoing project here in Ljubljana (2)

Theorem (IgD, Kudryavtseva, 2025/6)

$\mathrm{Inv}\langle A \mid w = 1\rangle$, *$w$ cyclically reduced, is strongly F-inverse* $\iff$ *every piece w_i in the decomposition $w = w_1 \cdots w_k$ into minimal invertible pieces has length at most 2.*

Examples and non-examples:

- $\mathrm{Inv}\langle a, b, c \mid abc = 1\rangle$ is F-inverse but not strongly F-inverse.
- Every inverse monoid with a free canonical group image is strongly F-inverse, and so in particular $\mathrm{Inv}\langle A \mid e = 1\rangle$ for any Dyck word over A.
- So, there exist strongly F-inverse monoids $\mathrm{Inv}\langle A \mid w = 1\rangle$ for cyclically non-reduced words w as well.

Hvala za vašo pozornost!
Hvala Ljubljana! 😊 ❤️