A Nonfinitely Based Finite Semiring

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The finite basis problem

A – a finite algebra

Eq(***A***) – the set of all identities true in ***A***

Is Eq(***A***) finitely axiomatizable
(finitely based)?

McKenzie (1996): in general, undecidable

Finitely based finite algebras

- groups: Oates & Powell (1966)
- commutative semigroups: Perkins (1968)
- lattices (& other lattice-based algebras): McKenzie (1970)
- rings: Львов, Kruse (1973)

Some NFB finite algebras

- Мурский (1965): a 3-element groupoid
  - this is a special case of NFB graph algebras – Baker, McNulty, Werner (1987)
- Perkins (1968): a 6-element semigroup = the Brandt monoid B_2^1 of order 2

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- the Perkins' semigroup is INFB = each l.f. variety containing it is NFB (Sapir, 1987)

Semirings

Semiring = an algebra $(\Sigma,+,\cdot,0)$ such that

- $(\Sigma,+,0)$ is a commutative monoid,
- $(\Sigma,\cdot)$ is a semigroup,
- the multiplication distributes over addition.

If + is an idempotent operation ($x+x=x$), then we have ai-semirings.

Σ_7

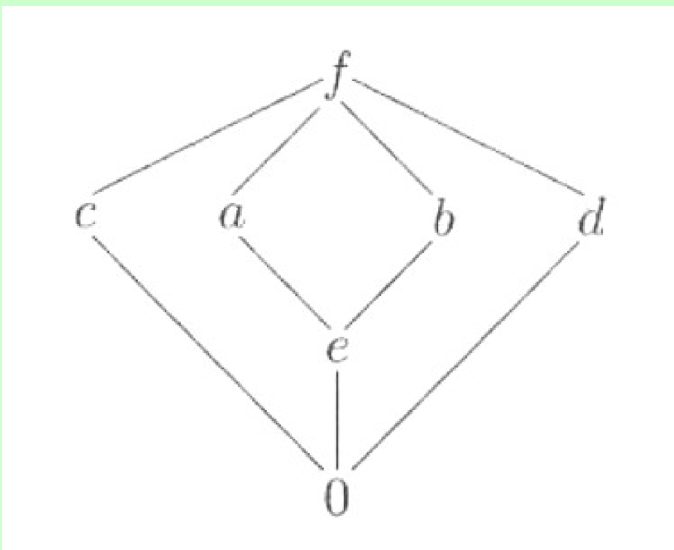
- a subsemiring of Rel(2), the semiring of binary relations on a two element set, formed by:
  - the four relations with 3 pairs,
  - the empty, the diagonal, and the full relation
- alternatively, the ai-semiring formed by 7 Boolean matrices

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

(remember that we have $\underline{1+1=1}$ in the 2-element Boolean semiring)

Σ_7(continued)



.	0	a	b	c	d	e	f
0	0	0	0	0	0	0	0
a	0	a	f	c	f	a	f
b	0	f	b	f	d	b	f
c	0	f	c	f	a	c	f
d	0	d	f	b	f	d	f
e	0	a	b	c	d	e	f
f	0	f	f	f	f	f	f

equations of B_2^1 = semigroup equations of Σ_7

Is there such a thing as a NFB finite semiring?

Theorem A. Σ_7 is NFB.

According to *MathSciNet*, this is a first example of such kind.

What follows is a (hopefully) VERY short outline of the proof idea.

IMAGIGAM words

- a word of the form

$$yLyL^R$$

where L is a linear word not containing y, and L^R is the reverse of L

- for all n, $B_2^{\,1}$ (and so Σ_7) satisfies the imagigam equations

$$yx_1x_2\ldots x_nyx_n\ldots x_2x_1 = yx_n\ldots x_2x_1yx_1x_2\ldots x_n$$

Isoterms #1

A word *u* is an <u>isoterm</u> for an ai-semiring identity

$$\Sigma_i u_i = \Sigma_j v_j$$

if for each semigroup substitution φ such that φ(u_i) is (for some *i*) a subword of *u* we have that

- either not all φ-values of u_i's are equal, or
- all φ-values of both u_i's and v_j's are equal

Isoterms #2

- for a fixed ai-semiring Σ and words u, v we write $u \leq v$ if Σ satisfies $u + v = v$
- a word w is minimal if $u \leq w$ implies that u is either 0, or w
- a minimal word = an isoterm for all identities of Σ (an isoterm of Σ)

Isoterms #3

Let *n* be a natural number and ***Σ*** an ai-semiring.

A word *u* in at least *n* letters is an *n*-isoterm of ***Σ*** if it is an isoterm for all equations of ***Σ*** in less than *n* letters.

Why isoterms?

An easy proposition. Let Σ be an ai-semiring. Suppose that for arbit-rary large n we manage to find a word w_n which **is** an n-isoterm, but **not** an isoterm of Σ.

Then Σ is NFB.

Why isoterms?

If one translates all notions to <u>semigroups</u> this is <u>exactly</u> the tool used by Perkins!

Namely, the <u>imagigam words</u> turn out to be suitable: Perkins proves that

$$yx_1x_2 \ldots x_n y x_n \ldots x_2 x_1$$

is always a (semigroup) *n*-isoterm, while the imagigam equations show that it is <u>not</u> an isoterm of the Perkins' monoid.

René, I've got a plan…

Can we do the same for Σ_7?

I.e., is the nth imagigam word an n-isoterm (in the ai-semiring sense) of Σ_7? (It is obviously not an isoterm of Σ_7.)

How to find n-isoterms at all?

A good lemma always saves the day!

Lemma. Let w be a word, with precisely n letters occurring in it, let Σ be an ai-semiring, and let $k<n$ be such that

(1) each word u in less than n letters, such that w contains a value of u (under some substitution), is minimal with respect to Σ,

(2) w satisfies a certain combinatorial (and technical – but not too much) condition called the k-joint substitution property.

Then w is a (k+1)-isoterm of Σ.

In Σ_7, the imagigam words satisfy both conditions!

1) Each word in at most *n* variables that has a value in the *n*th imagigam word is minimal in Σ_7.
2) Each imagigam word containing at least $4k+2$ letters has the *k*-joint substitution property.

1) is a classical combinatorics-on-words issue; for the proof of 2) the key thing is to use a fact from elementary geometry (!)

1) + 2) + Easy Prop. => Theorem A.

To tell the truth, we do not need the \`full strength' of Eq(Σ_7), only 7 its particular features so that we obtain a slightly more general result…

Theorem B.

Theorem B. *Let Σ be an ai-semiring. Call Σ* special *if it satisfies the following conditions:*

(a) *the inequalities of Σ are* closed under deletion, *i.e. for any words u, v such that $u \preceq v$ we have $c(u) = c(v)$, and if u', v' are obtained respectively from u, v by deleting all occurrences of a given variable (provided u, v contain at least two variables), then $u' \preceq v'$,*

(b) $yx \not\preceq xy$,

(c) *x and xyx are minimal with respect to Σ,*

(d) *x^2y, xyx, yx^2 are mutually $\preceq$-incomparable,*

(e) *$w \not\preceq (xy)^2$ whenever $w \in \{xyx, yxyxy\}$ or w contains one of x^2, y^2 as a subword,*

(f) *$xyzxy \not\preceq xyzyx$, $yxzyx \not\preceq xyzyx$ and $xzyxy \not\preceq xzy^2x$,*

(g) *$w \not\preceq xyztxtz$ for $w \in \{xytzxtz, xyztxzt, xytzxzt\}$.*

If Σ is special and satisfies all the imagigam identities, then it is nonfinitely based.

Open questions

- Q1: Are the semirings *Rel(n)* of binary relations on an *n*-element set, *n*>1, finitely based or not?
- Q2: Is Σ_7 INFB?

Clearly enough, A2:Yes=>A1:They're not.

- Q3: If A2 is Yes, is the same conclusion true for each finite ai-semiring in which all Zimin words are minimal (a feature easily proved in Σ_7 by induction)?

Thank you!