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Generalized Stochastic Processes in Infinite Dimensional Spaces with Applications to Singular Stochastic Partial Differential Equations

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Introduction

White noise theory, as a discipline of infinite dimensional analysis, had a fast development due to its broad spectrum of applications in the modeling of stochastic dynamical phenomena arising in physics, economy, biology etc. White noise is an excellent model for prompt, extremely large fluctuations, but since it does not exists in classical sense, it requires the notion of a generalized random process. In [HKPS], [HØUZ], [Ku] generalized random processes are considered as functions $X(t,\omega)$ generalized by the argument ω (ω is an element of a probability space Ω) and continuous by the time variable t, while in [SM], [Ul], [Wa] generalized random processes are considered as generalized functions in the t argument. Various conditions of continuity lead to differences in the structure of these classes of processes, which demand different methods in order to solve differential equations involving these types of processes. In [AHR1], [AHR2], [OR1], [OR2], [RO], stochastic differential equations are solved by regularization methods in the framework of Colombeau theory. On the other hand, in [HØUZ], [LØUZ], [Øks] equations are solved using the Wick product and the Hermite–transformation or the S–transformation. The authors in [MFA] make use of operator semigroups, and their solutions are generalized random processes with values in a separable Hilbert space.

The investigation of the structure of various types of generalized random processes is a prerequisite to solving various classes of stochastic differential equations with singularities. Many physical, economical, biological or even social phenomena can be represented by a mathematical model in form of a stochastic differential equation involving singular coefficients, singular data and singular initial values. In order to solve these equations, one needs the concept of a generalized random process which can deal with these types of singularities, i.e. one needs processes which are generalized by both arguments (ω and t). Processes of this type were introduced in [Se], [PS1] and will be subject also of this dissertation. A further step is to consider processes taking values in an infinite dimensional space; for this purpose we will consider a separable Hilbert space as the state space of the process.

The dissertation is organized in three chapters. Chapter 1 is expository and it represents a short overview of generalized functions theory. Spaces of deterministic and stochastic generalized functions are introduced, which will be used in the sequel. All theorems are stated without proof since they are parts of well-developed mathematical theories.

Chapter 2 and 3 contain the original parts of the dissertation. The subject of Chapter 2 are generalized random processes and their structure representation. We will use the acronym GRP for "generalized random process" and the already established acronyms SDE, SPDE, CONS for "stochastic differential equation", "stochastic partial differential equation" and "complete orthonormal system", respectively. Generalized random processes by various types of continuity are considered and classified as GRPs of type (O), (I) and (II). In particular, structure theorems for Hilbert space valued generalized random processes are obtained: Expansion theorems for GRPs (I) considered as elements of the spaces $\mathcal{L}(\mathcal{A}, S(H)_{-1})$ are derived, and structure representation theorems for GRPs (II) on $\mathcal{K}\{M_p\}$ spaces on a set with arbitrary large probability are given. Especially, Gaussian GRPs (II) are proven to be representable as a sum of derivatives of classical Gaussian processes with appropriate growth rate at infinity. For GRPs (I) a Colombeau type extension is introduced, and Wick products are defined. Also, in Chapter 2 applications to some classes of stochastic ordinary differential equations are presented. In order to emphasize the differences in the concept of generalized random processes defined by various conditions of continuity, the stochastic differential equation $y'(\omega, t) = f(\omega, t)$ is considered, where y is a generalized random process having a point value at $t = 0$ in sense of Lojasiewicz. Other SDEs considered in Chapter 2 are $P(\mathcal{R})X = g$ and $P(\mathcal{R})X = X\blacklozenge\tilde{W} + g$, where g is a GRP (I) and $\tilde{W}$ is white noise.

Applications to the eponymous singular stochastic partial differential equations are the subject of Chapter 3. The main part of it is devoted to elliptic equations; elliptic SPDEs in economics are typical of steady–state problems such as perpetual options in multi–factor models, in physics they describe diffusion processes in anisotropic media, while in medicine they are a good model for brain functions.

We treat the linear elliptic stochastic Dirichlet problem in the framework of GRPs (I) combined with Sobolev space and Colombeau algebra methods. The equation $Lu = h + \nabla f$ with given stochastic boundary condition is uniquely solved, where the boundary condition, as well as the data h and f are GRPs regarded as linear continuous mappings from the Sobolev space into the Kondratiev space, or as Colombeau extended GRPs. The operator L is assumed to be strictly elliptic in divergence form $Lu = \nabla\cdot(A\cdot\nabla u + bu) + c\cdot\nabla u + du$. Its coefficients: the elements of the matrix A and of the vectors b, c

and d are (in the most general setting) assumed to be Colombeau generalized random processes. Stability and regularity properties of the solutions are also obtained. Another SDE considered in Chapter 3 is the stochastic Helmholtz equation, which is solved by means of the Fourier transformation.

I would like to take this opportunity to express my huge thank to my parents, Erika and György for their love, encouragement, kindness and patience. Special thanks go to my friends and colleagues for providing constant support during my work on this dissertation. A complete list of their names would fill pages. I wish to thank Professor Zagorka Lozanov–Crvenković and Assistant Professor Danijela Rajter–Ćirić for many fruitful discussions about Probability and Stochastics. To Professor Marko Nedeljkov I owe a debt of gratitude for a careful reading of this text and providing a sequence of improvements. I would like to mention here his excellent lectures on PDEs; some of the examples in this dissertation were inspired by things I learned at the time I was conducting exercises for his lectures. A great thank goes to Professor Michael Oberguggenberger for his hospitality and support during my stay at the Insitute of Engineering Mathematics, Geometry and Computer Science in Innsbruck, Austria. In particular, I wish to thank him for his interesting Seminar on Stochastic Analysis and for our discussions during my stay in Innsbruck. Last but not least, I want to express my deep appreciation for the constant support, help and encouragement in many areas given to me by my mentor, Professor Stevan Pilipović. He represents an inexhaustible source of knowledge, who keeps the doors of his office always open for his students. I wish to thank him for his comprehensive guidance through the mathematical universe, showing me some of its most interesting parts like Functional Analysis, Measure Theory, Generalized Functions (just to mention some of them), which finally converged to this dissertation.

Contents

There is no branch of mathematics, however abstract, which may not some day be applied to phenomena of the real world.

(Nikolai Lobatchevsky)

Chapter 1

Generalized Function Spaces

In this introductory chapter we give a brief overview of some classes of deterministic (e.g. Schwartz, Zemanian, Colombeau) and stochastic (Hida, Kondratiev etc.) generalized function spaces. Definitions of some basic concepts, their most important properties and relations are given, which are necessary to understand the methods used in the sequent chapters of the dissertation. Most of the material presented here is familiar and therefore given without proof but with references for further reading.

Some basic notation we will use throughout the dissertation is the following: Let V be a topological vector space, V' its dual space, and $\mathcal{L}(V, U)$ be the space of all linear continuous mappings from V into a topological vector space U. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $Z^p = L^p(\Omega)$, $p \geq 1$, be the space of random variables X such that $\int_\Omega |X|^p dP < \infty$. By $L^r(\mathbb{R}^n)$, $r \geq 1$, we denote the space of r–integrable functions with respect to the Lebesgue measure m, by $C^k(\mathbb{R}^n)$ denote the space of k–times continuously differentiable functions, and by $C_0(\mathbb{R}^n)$ the space of continuous functions with compact support.

1.1 Nuclear Spaces

For convenience of the reader not so familiar with notions in functional analysis, we provide a basic overview of some concepts such as nuclear spaces or Gel'fand triples, which are fundamental in infinite dimensional analysis and white noise theory. It is outside the scope of this chapter to give a comprehensive discussion of all concepts and details related to nuclear spaces. The reader is referred to [GV], [HKPS] and [Tr] for proofs and further details.

Projective and inductive topologies

Let E be a vector space, $\{E_\alpha\}_{\alpha\in\Lambda}$ a family of locally convex topological vector spaces and $\{h_\alpha\}_{\alpha\in\Lambda}$ a family of linear mappings such that $h_\alpha : E \to E_\alpha$ and $\bigcap_{\alpha\in\Lambda} h_\alpha^{-1}(0) = 0$. The *projective topology* is the strongest locally convex topology on E with respect to which all mappings h_α are continuous. Let $\{g_\alpha\}_{\alpha\in\Lambda}$ be a family of linear mappings such that $g_\alpha : E_\alpha \to E$ and $\bigcup_{\alpha\in\Lambda} g_\alpha(E_\alpha) = E$. The *inductive topology* is the finest locally convex topology on E with respect to which all mappings g_α are continuous.

Countably Hilbert spaces

Let V and W be separable Hilbert spaces. An operator $A : V \to W$ is of *Hilbert-Schmidt type* if there exist an orthonormal basis $\{e_n\}$ in V and $\{f_n\}$ in W, and if there exist $\lambda_n > 0$, $n \in \mathbb{N}$, such that $\sum_{n\in\mathbb{N}} \lambda_n^2 < \infty$ and $A(x) = \sum_{n\in\mathbb{N}} \lambda_n(e_n|x)f_n$. An operator $A : V \to W$ is *nuclear*, if there exist an orthonormal basis $\{e_n\}$ in V and $\{f_n\}$ in W, and if there exist $\lambda_n > 0$, $n \in \mathbb{N}$, such that $\sum_{n\in\mathbb{N}} \lambda_n < \infty$ and $A(x) = \sum_{n\in\mathbb{N}} \lambda_n(e_n|x)f_n$.

Let E_0 be a separable Hilbert space endowed with the inner product $(\cdot|\cdot)_0$ and let $(\cdot|\cdot)_p$ be a countable family of inner products. The space E_0 endowed with this family of inner products is called a *countably Hilbert space* if it is complete with respect to the metric $|x| = \sum_{p=0}^{\infty} 2^{-p}\frac{|x|_p}{1+|x|_p}$.

Without loss of generality we may assume the inner products are ordered in an ascending way, i.e. $|\cdot|_p \leq |\cdot|_{p+1}$, $p \in \mathbb{N}_0$. Denote by E_p the completition of E_0 with respect to $|\cdot|_p$. Clearly, E_p is a separable Hilbert space, $\ldots E_{p+1} \subseteq E_p \ldots \subseteq E_0$ and $E = projlim_{p\to\infty} E_p = \bigcap_{p\in\mathbb{N}_0} E_p$.

Topologies on the dual

On the dual of a countably Hilbert space E' the *strong topology* is defined by the basis of neighborhoods of zero $B(A;\varepsilon) = \{L \in E' : sup_{x\in A}|\langle L, x\rangle| < \varepsilon\}$, where A varies through all bounded sets and $\varepsilon > 0$. The strongest topology on E' such that for each fixed $x \in E$, all mappings $E' \to \mathbb{C}$ given by $L \mapsto \langle L, x\rangle, \quad L \in E'$ are continuous, is called the *weak topology.* The dual of a countably Hilbert space is complete with respect to the strong topology.

Denote now by E_{-p} the dual of E_p. Thus, we obtain an increasing chain of Hilbert spaces $E_0 \subseteq E_{-1} \subseteq \ldots E_{-p} \subseteq E_{-(p+1)} \ldots$, and for all $p \in \mathbb{N}_0$ the inclusion $E_p \subseteq E_0 \subseteq E_{-p}$ is continuous. Let $E' = \bigcup_{p\in\mathbb{N}_0} E_{-p}$ and endow it with the inductive topology. Since E is dense in each E_p, the dual of E is (in algebraic sense) E' and the inductive topology on E' is equivalent to the strong topology on $\left(\bigcap_{p\in\mathbb{N}_0}^{\infty} E_p\right)'$.

Nuclear spaces

A countably Hilbert space E is called *nuclear*, if for every $p \in \mathbb{N}$ there exists $q \geq p$ such that the inclusion mapping $E_q \to E_p$ is nulcear. Recall that the inclusion $E_q \to E_p$ is nuclear if there exists a CONS $\{e_k\}$ in E_q such that $\sum_{k\in\mathbb{N}} |e_k|_p^2 < \infty$. All finite dimensional spaces are nuclear. Moreover, nuclear spaces have similar properties as finite dimensional spaces, e.g. if E is nuclear, then $A \subset E$ is compact if and only if it is bounded and closed. Infinite dimensional Banach spaces are not nuclear. The notion of a nuclear space is not reduced only to countably Hilbert spaces; one can define also *nuclear topological vector spaces* (for this we refer to [Tr]). For example, the dual of a nuclear countably Hilbert space is a nuclear topological vector space. Some examples of nuclear spaces we will use in this dissertation are the Schwartz spaces $\mathcal{S}$, $\mathcal{S}'$, $\mathcal{D}$, $\mathcal{D}'$, but also C_0, C_0^∞. Also, the Zemanian spaces $\mathcal{A}$, $\mathcal{A}'$ which are yet to be introduced, are nuclear under some additional conditions.

Gel'fand triples

Let V be a nuclear space and V' its dual. If there exists a Hilbert space H such that V is dense in H, then the triple $V \subseteq H \subseteq V'$ is called a *Gel'fand triple*. In fact, here we identified H with its dual H'. In case of nuclear countably Hilbert spaces we have the Gel'fand triple $E \subseteq E_0 \subseteq E'$.

Tensor products

In order to define a topological structure on the tensor product of two topological vector spaces $X \otimes Y$, there are two possibilities at our disposal (see [Tr]). One possibility is to construct a seminorm topology relying directly on the seminorm topologies of X and Y (this will be the so called π-topology). The other possibility is to use an embedding of $X \otimes Y$ in some space related to X and Y, over which a topology already exists (this construction will lead to the so called ε-topology). In general, the π-topology is finer than the ε-topology.

The π-topology on $X \otimes Y$ is the strongest locally convex topology such that the canonical bilinear mapping $X \times Y \to X \otimes Y$, $(x, y) \mapsto x \otimes y$

is continuous. In terms of seminorms, the π-topology can be described as follows: Let p, q be seminorms on X, Y, respectively and let $\theta \in X \otimes Y$. Define $(p \otimes q)(\theta) = \inf \sum_{j=1}^{n} p(x_j)q(y_j)$, where the infimum is taken over all representations of θ of the form $\theta = \sum_{j=1}^{n} x_j \otimes y_j$ for some $x_j \in X$, $y_j \in Y$, $j = 1, 2, \ldots n$ and $n \in \mathbb{N}$. Especially, for $x \in X$, $y \in Y$, $(p \otimes q)(x \otimes y) = p(x)q(y)$. The completition of $X \otimes Y$ with respect to the π-topology is in [Tr] denoted by $X \hat{\otimes}_{\pi} Y$.

In order to define the ε-topology, we observe that there exists a canonical bilinear mapping $X \otimes Y \to B(X', Y')$, defined by $(x, y) \mapsto [(x', y') \mapsto \langle x', x\rangle\langle y', y\rangle]$. Here $B(X', Y')$ denotes the space of continuous bilinear forms on $X' \times Y'$. Thus, $X \otimes Y \cong B(X', Y')$. Denote by $\mathcal{B}_{\varepsilon}(X', Y')$ the space of separately continuous bilinear forms on $X \otimes Y$ equipped with the topology of uniform convergence on the products $A \otimes B$, where A is an equicontinuous subset of X' and B is an equicontinuous subset of Y'. Since $B(X', Y')$ is a subspace of $\mathcal{B}_{\varepsilon}(X', Y')$, it inherits this topology. Due to the above result, $X \otimes Y$ also can be provided with this subspace topology. We call this subspace topology on $X \otimes Y$ the ε-topology. The completition of $X \otimes Y$ with respect to the ε-topology is in [Tr] denoted by $X \hat{\otimes}_{\varepsilon} Y$.

It is known that if X is a nuclear topological vector space, and Y is a Fréchet space, then $X \hat{\otimes}_{\varepsilon} Y = X \hat{\otimes}_{\pi} Y$. Since we will deal only with tensor products of spaces, where at least one of them is nuclear, we will suppress the index and simply denote by $X \otimes Y$ the completition of the tensor product space with respect to either of the topologies.

1.2 The Space of Tempered Distributions

We use notation $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ for multiindeces, $D^{\alpha} = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$ for the differential operator, and $x^{\alpha} = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. The length of a multiindex α is defined as $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$.

1.2.1 Hermite functions

The *Hermite polynomial of order n*, $n \in \mathbb{N}_0$, is defined by $h_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}})$, $x \in \mathbb{R}$. It is well known that the family $\{\frac{1}{\sqrt{n!}} h_n : n \in \mathbb{N}_0\}$ constitutes an orthonormal basis of the space $L^2(\mathbb{R}, d\mu)$, where $d\mu = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ is the Gaussian measure. The *Hermite function of order $n+1$*, $n \in \mathbb{N}_0$, is defined as

$$\xi_{n+1}(x) = \frac{1}{\sqrt[4]{\pi}\sqrt{n!}} e^{-\frac{x^2}{2}} h_n(\sqrt{2}x), \quad x \in \mathbb{R}.$$

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}^d$ be a multi-index. Define $\xi_\alpha = \xi_{\alpha_1} \otimes \xi_{\alpha_2} \otimes \cdots \otimes \xi_{\alpha_d}$. The set of multi-indeces α can be ordered in an ascending sequence as it is described in [HØUZ]. Denote by $\alpha^{(j)}$ the jth multi-index in this ordering. Hence, the family of vectors ξ_α can also be enumerated into a countable family $\eta_j = \xi_{\alpha^{(j)}}$, $j \in \mathbb{N}$. The family of functions $\{\eta_j : j \in \mathbb{N}\}$ is an orthonormal basis of the space $L^2(\mathbb{R}^d)$.

1.2.2 Schwartz spaces

The *Schwartz space of rapidly decreasing functions* is defined as

$$\mathcal{S}(\mathbb{R}^d) = \{f \in \mathcal{C}^\infty(\mathbb{R}^d) : \forall \alpha, \beta \in \mathbb{N}_0^d,\ \|f\|_{\alpha,\beta} < \infty\},$$

and the topology on $\mathcal{S}(\mathbb{R}^d)$ is given by the family of seminorms $\|f\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^d} |x^\alpha D^\beta f(x)|$, $\alpha, \beta \in \mathbb{N}_0^d$. It is well known that $\mathcal{S}(\mathbb{R}^d)$ is a nuclear space. The *Schwartz space of tempered distributions* is the dual space $\mathcal{S}'(\mathbb{R}^d)$.

An equivalent construction of the topology on $\mathcal{S}(\mathbb{R}^d)$ can be obtained using the self-adjoint operator $A = -\triangle + |x|^2 + 1$, where $\triangle$ is the Laplace operator. The operator A is densely defined on $L^2(\mathbb{R}^d)$; more precisely the domain of A contains $\mathcal{S}(\mathbb{R}^d)$. The Hermite functions $\{\eta_n\}$, $n \in \mathbb{N}$, which form an orthonormal basis of $L^2(\mathbb{R}^d)$ are the eigenvectors of A, i.e. $A\eta_n = \lambda_n \eta_n$, where $\{\lambda_n = 2(n_1 + \cdots + n_d) - d + 1 :\ (n_1, \ldots, n_d) \in \mathbb{N}^d\}$ is the spectar of the operator A. Since zero does not belong to the spectrum, there exists the inverse operator A^{-1}, and moreover $\|A^{-1}\| = \frac{1}{\lambda_1}$.

For $p \in \mathbb{N}$ define the norm $|\cdot|_p$ on $L^2(\mathbb{R}^d)$ by $|f|_p = |A^p f|_{L^2(\mathbb{R}^d)}$. Let $S_p(\mathbb{R}^d)$ be the closure of $\{f \in \mathcal{C}^\infty(\mathbb{R}^d) : |f|_p < \infty\}$ in $L^2(\mathbb{R}^d)$. Then $S_p(\mathbb{R}^d)$ is a Hilbert space with scalar product $(f|g)_p = (A^p f | A^p g)_0$. For each $p \in \mathbb{N}$, $\mathcal{S}(\mathbb{R}^d)$ is dense in $S_p(\mathbb{R}^d)$ and the inclusion $S_{p+1}(\mathbb{R}^d) \subseteq S_p(\mathbb{R}^d)$ is of Hilbert-Schmidt type. The family of seminorms $\{|\cdot|_{\alpha,\beta} :\ \alpha, \beta \in \mathbb{N}_0^d\}$ and the family of norms $\{|\cdot|_p :\ p \in \mathbb{N}\}$ are equivalent on $\mathcal{S}(\mathbb{R}^d)$. Thus, the projective limit of spaces $S_p(\mathbb{R}^d)$ is isomorphic to $\mathcal{S}(\mathbb{R}^d)$, i.e.

$$\mathcal{S}(\mathbb{R}^d) = \bigcap_{p \in \mathbb{N}} S_p(\mathbb{R}^d) = \{f \in \mathcal{C}^\infty(\mathbb{R}^d) : \forall p \in \mathbb{N},\ |f|_p < \infty\}.$$

For $p \in \mathbb{N}$ define $|f|_{-p} = |A^{-p} f|_{L^2(\mathbb{R}^d)}$ and let $S_{-p}(\mathbb{R}^d)$ be the completition of $\mathcal{S}(\mathbb{R}^d)$ with respect to $|\cdot|_{-p}$. The $|\cdot|_{-p}$ norm topology is equivalent to the strong topology on the dual space $S_p'(\mathbb{R}^d)$ defined by $|\cdot|_{-p} = \sup_{\psi \in \mathcal{S}(\mathbb{R}^d), |\psi|_p \leq 1} |\langle \cdot, \psi \rangle|$. Thus, $S_{-p}(\mathbb{R}^d)$ is isomorphic to $S_p'(\mathbb{R}^d)$.

In set–theoretical sense we have $\mathcal{S}'(\mathbb{R}^d) = \bigcup_{p \in \mathbb{N}} S_{-p}(\mathbb{R}^d)$, and moreover the locally convex inductive limit of the spaces $S_{-p}(\mathbb{R}^d)$ is isomorphic to

$\mathcal{S}'(\mathbb{R}^d)$ equipped with the strong topology. Thus, $\mathcal{S}(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d) \subseteq \mathcal{S}'(\mathbb{R}^d)$ form a Gel'fand triple. Moreover, we have continuous inclusions

$$\mathcal{S}(\mathbb{R}^d) \subseteq S_p(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d) \subseteq S_p'(\mathbb{R}^d) \subseteq \mathcal{S}'(\mathbb{R}^d).$$

For each $p \in \mathbb{N}$ the operator A^{-p} is of Hilbert-Schmidt type on $L^2(\mathbb{R}^d)$, with Hilbert-Schmidt norm $\|A^{-p}\|_{HS}^2 = \sum_{n=1}^{\infty} \lambda_n^{-2p}$. The vectors $\{\lambda_n^{-p}\eta_n\}$ form an orthonormal basis of $S_p(\mathbb{R}^d)$.

One can easily characterize the Schwartz spaces by Hermite functions (recall, λ_k are the eigenvalues of the operator A, and η_k is the Hermite CONS of $L^2(\mathbb{R}^d)$): A function f belongs to $\mathcal{S}(\mathbb{R}^d)$ if and only if it is of the form $f = \sum_{k\in\mathbb{N}} a_k\eta_k$, where $\sum_{k\in\mathbb{N}} \lambda_k^p |a_k|^2 < \infty$ for each $p \in \mathbb{N}$ and the coefficients are given by $a_k = \langle f, \eta_k\rangle \in \mathbb{C}$. A function f belongs to $\mathcal{S}'(\mathbb{R}^d)$ if and only if it is of the form $f = \sum_{k\in\mathbb{N}} a_k\eta_k$, where $\sum_{k\in\mathbb{N}} \lambda_k^{-p}|a_k|^2 < \infty$ for some $p \in \mathbb{N}$ and the coefficients are given by $a_k = \langle f, \eta_k\rangle \in \mathbb{C}$.

Note that the space of tempered distributions can be constructed also on a bounded open set $I \subset \mathbb{R}^d$. In this case, $\mathcal{S}'(I)$ coincides with $\mathcal{E}'(I)$, where $\mathcal{E}'(I)$ denotes the space of distributions having compact support.

1.2.3 Hilbert space valued tempered distributions

Let H be a separable Hilbert space with orthonormal basis $\{e_i : i \in \mathbb{N}\}$. By $\mathcal{S}(\mathbb{R}^d; H)$ and $\mathcal{S}'(\mathbb{R}^d; H)$ we will denote the H–valued Schwartz test function space, and the H–valued Schwartz generalized function space, respectively. Since $\mathcal{S}(\mathbb{R}^d)$ is nuclear, we have (see [Tr, p.533]) that

$$\mathcal{S}(\mathbb{R}^d; H) \cong \mathcal{S}(\mathbb{R}^d) \otimes H, \qquad \mathcal{S}'(\mathbb{R}^d; H) \cong (\mathcal{S}(\mathbb{R}^d) \otimes H)' \cong \mathcal{S}'(\mathbb{R}^d) \otimes H,$$

where the symbol $\otimes$ is understood as the tensor product on the test space, and the π–completition (or in this case the equivalent ϵ–completition) of the tensor product of dual spaces in the second case. Also, since H is a Hilbert space, we may identify it with its dual and thus consider $\mathcal{S}'(\mathbb{R}^d; H)$ as $\mathcal{B}(\mathcal{S}(\mathbb{R}^d), H)$ i.e. as the space of bilinear continuous mappings $\mathcal{S}(\mathbb{R}^d) \times H \to \mathbb{R}$. We will denote by $\langle \cdot, \cdot\rangle$ this dual bilinear pairing.

Since the family $\{e_i\eta_k : i, k \in \mathbb{N}\}$ is an orthogonal basis of $L^2(\mathbb{R}^d) \otimes H$, we get a Hermite basis characterization also in the H–valued case: A function f belongs to $\mathcal{S}(\mathbb{R}^d; H)$ if and only if it is of the form $f = \sum_{i\in\mathbb{N}}\sum_{k\in\mathbb{N}} a_{i,k}\eta_k e_i$, where $\sum_{i\in\mathbb{N}}\sum_{k\in\mathbb{N}} \lambda_k^p |a_{i,k}|^2 < \infty$ for each $p \in \mathbb{N}$ and the coefficients are given by $a_{i,k} = \langle f, \eta_k e_i\rangle \in \mathbb{C}$. A function f belongs to $\mathcal{S}'(\mathbb{R}^d; H)$ if and only if it is of the form $f = \sum_{i\in\mathbb{N}}\sum_{k\in\mathbb{N}} a_{i,k}\eta_k e_i$, where $\sum_{i\in\mathbb{N}}\sum_{k\in\mathbb{N}} \lambda_k^{-p}|a_{i,k}|^2 < \infty$ for some $p \in \mathbb{N}$ and the coefficients are given by $a_{i,k} = \langle f, \eta_k e_i\rangle \in \mathbb{C}$.

1.3 Zemanian Spaces

Let I be an open interval in $\mathbb{R}$, and let $\mathcal{R}$ be a formally self-adjoint linear differential operator of the form

$$\mathcal{R} = \theta_0 D^{n_1}\theta_1 \cdots D^{n_\nu}\theta_\nu = \bar{\theta}_\nu(-D)^{n_\nu}\cdots(-D)^{n_2}\bar{\theta}_1(-D)^{n_1}\bar{\theta}_0 \tag{1.1}$$

where $D = d/dx$, θ_k are smooth complex functions without zero-points in I, and n_k are integers $k = 1, 2, \ldots, \nu$. Suppose that there exist a sequence of real numbers $\langle\lambda_n\rangle_{n\in\mathbb{N}}$, $\lim_{n\to\infty}|\lambda_n| = \infty$, and a sequence of smooth functions $\langle\psi_n\rangle_{n\in\mathbb{N}}$ in $L^2(I)$ which are the eigenvalues and eigenfunctions, respectively, of the operator $\mathcal{R}$, i.e. $\mathcal{R}\psi_n = \lambda_n\psi_n$, $n \in \mathbb{N}$. We can enumerate them in an ascending order: $|\lambda_1| \le |\lambda_2| \le |\lambda_3| \le \cdots \to \infty$. This re-ordering is made just for technical reasons, but it is neither unique nor unavoidable. In Example 2.3.1 we will use a Zemanian space without this ordering. Suppose that $\{\psi_n : n \in \mathbb{N}\}$ forms a CONS in $L^2(I)$ with respect to the usual inner product denoted by $(\cdot|\cdot)$. Each function $f \in L^2(I)$ can be represented as an infinite sum $f = \sum_{n=1}^{\infty}(f|\psi_n)\psi_n$ converging in $L^2(I)$. Define inductively: $\mathcal{R}^0 = \mathcal{I}$, $\mathcal{R}^{k+1} = \mathcal{R}(\mathcal{R}^k)$, $k \in \mathbb{N}$. Note that $\lambda_n = 0$ for some $n \in \mathbb{N}$ implies $\lambda_k = 0$ for every $k < n$. From now on, if $\lambda_n = 0$, we replace it with $\widetilde{\lambda_n} = 1$, else we put $\widetilde{\lambda_n} = \lambda_n$, $n \in \mathbb{N}$.

1.3.1 The spaces $\mathcal{A}_k$, $\mathcal{A}'_k$, $\mathcal{A}$, $\mathcal{A}'$

Define:

$$\mathcal{A}_k = \{f = \sum_{n=1}^{\infty} a_n\psi_n : \sum_{n=1}^{\infty}|a_n|^2\widetilde{\lambda_n}^{2k} < \infty\}, \quad k \in \mathbb{Z}.$$

If $k \in \mathbb{N}_0$, then $\mathcal{A}_k \subseteq L^2(I)$. For each $k \in \mathbb{N}_0$, $\mathcal{A}_k$ is a Hilbert space when provided with the inner product $(f|g)_k = \sum_{n=1}^{\infty} a_n\bar{b}_n\widetilde{\lambda_n}^{2k}$, where $f = \sum_{n=1}^{\infty} a_n\psi_n$, $g = \sum_{n=1}^{\infty} b_n\psi_n \in \mathcal{A}_k$. Denote by $\|\cdot\|_k$ the norm induced by this inner product. The dual space $\mathcal{A}'_k$, equipped with the usual dual norm, is isomorphic with $\mathcal{A}_{-k}$. Thus, we have a sequence of linear continuous canonical inclusions

$$\cdots \subseteq \mathcal{A}_{k+1} \subseteq \mathcal{A}_k \subseteq \cdots \mathcal{A}_0 = L^2(I) \subseteq \mathcal{A}_{-1} \subseteq \mathcal{A}_{-2} \subseteq \cdots$$

The set

$$S = \{f \in L^2(I) : f = \sum_{n=1}^{m} a_n\psi_n,\ a_n \in \mathbb{C}, m \in \mathbb{N}\}, \tag{1.2}$$

i.e. the linear span of the set $\{\psi_n : n \in \mathbb{N}\}$, is dense in each $\mathcal{A}_k$, $k \in \mathbb{Z}$. Define:

$$\mathcal{A} = \bigcap_{k\in\mathbb{N}_0} \mathcal{A}_k = \left\{ f \in L^2(I) : \ f = \sum_{n=1}^{\infty} a_n\psi_n, \, \forall k, \sum_{n=1}^{\infty} |a_n|^2 \widetilde{\lambda_n}^{2k} < \infty \right\},$$

$$\mathcal{A}' = \bigcup_{k\in\mathbb{N}_0} \mathcal{A}_{-k} = \left\{ f = \sum_{n=1}^{\infty} b_n\psi_n : \exists k, \sum_{n=1}^{\infty} |b_n|^2 \widetilde{\lambda_n}^{-2k} < \infty \right\}.$$

The *Zemanian space of test functions* $\mathcal{A}$ is equipped with the projective topology, and its dual $\mathcal{A}'$, the *Zemanian space of generalized functions*, is equipped with the inductive topology which is equivalent to the strong dual topology. The action of a generalized function $f = \sum_{n=1}^{\infty} a_n\psi_n \in \mathcal{A}'$ onto a test function $\varphi = \sum_{n=1}^{\infty} b_n\psi_n$ is given by the dual pairing $\langle f, \varphi\rangle = \sum_{n=1}^{\infty} a_n b_n$. The orthonormal basis of $\mathcal{A}_k$, $k \in \mathbb{N}_0$, is the family of functions $\{\widetilde{\lambda_n}^{-k}\psi_n : n \in \mathbb{N}\}$. Note that $\mathcal{A}$ is nuclear if for some $p \geq 0$ the condition $\sum_{n\in\mathbb{N}} \widetilde{\lambda_n}^{-2p} < \infty$ holds.

Example 1.3.1 *In particular, for the choice $\mathcal{R} = -\frac{d^2}{dx^2} + x^2 + 1$, defined on a maximal domain in $L^2(\mathbb{R})$, $\mathcal{A}'$ is the space of tempered distributions $\mathcal{S}'(\mathbb{R})$.*

1.3.2 The spaces $\exp_p \mathcal{A}$, $\exp_p \mathcal{A}'$, $\mathrm{Exp}\mathcal{A}$, $\mathrm{Exp}\mathcal{A}'$

Let $p \in \mathbb{N}$. Denote $\exp_p x = \underbrace{\exp(\exp(\cdots(\exp x))\cdots)}_{p}$ and define (see [Pi]) $\exp_p \mathcal{A}$ as the projective limit of the family

$$\exp_{p,k} \mathcal{A} = \{\varphi = \sum_{n=1}^{\infty} a_n\psi_n : \sum_{n=1}^{\infty} |a_n|^2 (\exp_p \widetilde{\lambda_n})^{2k} < \infty\}, \quad k \in \mathbb{N}_0,$$

equipped with the norm $\|\varphi\|_{p,k} = \sum_{n=1}^{\infty} |a_n|^2 (\exp_p \widetilde{\lambda_n})^{2k}$, $k \in \mathbb{N}_0$. Thus,

$$\exp_p \mathcal{A} = \bigcap_{k\in\mathbb{N}_0} \exp_{p,k} \mathcal{A}, \quad \exp_p \mathcal{A}' = \bigcup_{k\in\mathbb{N}_0} \exp_{p,-k} \mathcal{A}.$$

Clearly, S is dense in each $\exp_{p,k} \mathcal{A}$. The canonical inclusions $\exp_{p,k+1} \mathcal{A} \subseteq \exp_{p,k} \mathcal{A}$ are compact. Moreover, $\exp_p \mathcal{A}$ is nuclear if for some $c \in \mathbb{N}_0$ the series $\sum_{n=1}^{\infty} (\exp_p \widetilde{\lambda_n})^{-2c}$ converges. Define the pair of test and generalized function spaces $\mathrm{Exp}\mathcal{A}$ and $\mathrm{Exp}\mathcal{A}'$ as

$$\mathrm{Exp}\mathcal{A} = \operatorname*{projlim}_{p\to\infty} \exp_p \mathcal{A}, \quad \mathrm{Exp}\mathcal{A}' = \operatorname*{indlim}_{p\to\infty} \exp_p \mathcal{A}'.$$

The canonical inclusions $\exp_{p+1}\mathcal{A} \subseteq \exp_p\mathcal{A}$ are continuous and compact. The set S is dense in $\exp_p\mathcal{A}$ for each $p \in \mathbb{N}$. Hence, $\mathrm{Exp}\mathcal{A}$ is dense in each $\exp_p\mathcal{A}$ as well as in $\mathcal{A}$.

1.3.3 Hilbert space valued Zemanian functions

Similarly as we considered H–valued Schwartz functions for a separable Hilbert space H with orthonormal basis $\{e_i : i \in \mathbb{N}\}$, we generalize this concept also to Zemanian functions. We will assume that $\mathcal{A}$ is nuclear, i.e. that there exists some $p \geq 0$, such that $\sum_{n=1}^{\infty} \widetilde{\lambda_n}^{-2p} < \infty$. This is necessary in order to have an isomorphism of $\mathcal{A}'(I;H)$ with the tensor product space $\mathcal{A}' \otimes H$ (refer to [Tr, Prop.50.7]). In general case, $\mathcal{A}' \otimes H$ would be isomorphic to a subspace of $\mathcal{A}'(I;H)$.

Denote by $\mathcal{A}_k(I;H)$ the space of functions $f : I \to H$ of the form $f = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} a_{i,n}\psi_n e_i$ such that $\|f\|_{k;H} = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} |a_{i,n}|^2 \widetilde{\lambda_n}^{2k} < \infty$. Let $\mathcal{A}(I;H) = \mathrm{projlim}_{k\to\infty}\mathcal{A}_k(I;H)$. Clearly, $\mathcal{A}'_k(I;H)$ is isomorphic to $\mathcal{A}_{-k}(I;H)$ and we may define $\mathcal{A}'(I;H) = \mathrm{indlim}_{k\to\infty}\mathcal{A}_{-k}(I;H)$.

A similar construction can be carried out also for the spaces $\mathrm{Exp}\mathcal{A}$ and $\mathrm{Exp}\mathcal{A}'$. We denote their Hilbert space valued versions by $\mathrm{Exp}\mathcal{A}(I;H)$ and $\mathrm{Exp}\mathcal{A}'(I;H)$.

1.4 Sobolev Spaces

Let I be an open subset of $\mathbb{R}^d$. The αth weak derivative of f, denoted by $D^\alpha f$ is given by the action

$$\int_I D^\alpha f(x)\varphi(x)dx = -\int_I f(x)D^\alpha\varphi(x)dx,$$

for all $\varphi \in C_0^\infty(\mathbb{R}^d)$.

Denote by $W^{k,p}(I)$ the space of weakly differentiable functions f such that $D^\alpha f \in L^p(I)$ for all $|\alpha| \leq k$. We endow $W^{k,p}(I)$ with the norm

$$\|f\|_{W^{k,p}} = \sum_{|\alpha|\leq k} \|D^\alpha f\|_{L^p(I)}.$$

Clearly, $W^{k,2}(I)$ is a Hilbert space.

Another important space we will consider is $W_0^{k,p}(I)$ defined as the closure of $C_0^\infty(I)$ in $W^{k,p}(I)$. The dual space of $W_0^{k,p}(I)$ will be denoted by $W^{-k,p}(I)$. An isomorphism between $W_0^{k,p}(I)$ and $W^{-k,p}(I)$ can be established via the

Laplace operator. By its Hilbert structure, we also may identify $W_0^{k,2}(I)$ with $W^{-k,2}(I)$. Thus, we obtain a Gel'fand triple

$$W_0^{k,2}(I) \subseteq L^2(I) \subseteq W^{-k,2}(I).$$

For further notions and properties of Sobolev spaces we refer to the monograph [Ad].

1.5 $\mathcal{K}\{M_p\}$ Spaces

Now we give a brief overview of some basic notions from the theory of $\mathcal{K}\{M_p\}$ spaces, which are constructed similarly as the tempered distributions, but are more general. For further details refer to [GS] or [PS].

1.5.1 The weight functions M_p

Let $(M_p)_{p\in\mathbb{N}_0}$ be a sequence of continuous functions on $\mathbb{R}$ such that the following conditions are satisfied:

$$1 \leq M_p(x) \leq M_{p'}(x),\ x \in \mathbb{R},\ \ p < p'. \tag{1.3}$$

(P) For every $p \in \mathbb{N}_0$ there is $p' \in \mathbb{N}_0$ such that

$$\lim_{|x|\to\infty} M_p(x)M_{p'}^{-1}(x) = 0.$$

(N) For every $p \in \mathbb{N}_0$ there is $p' \in \mathbb{N}_0$ such that $M_pM_{p'}^{-1} \in L^1(\mathbb{R})$.

$\mathcal{K}\{M_p\}$ is defined as a space of smooth functions $\varphi \in C^\infty(\mathbb{R})$ endowed with the family of norms

$$\|\varphi\|_p = \sup\{M_p(x)|\varphi^{(i)}(x)| :\ x \in \mathbb{R},\ i \leq p\}\ ,\ p \in \mathbb{N}_0.$$

We refer the reader to [GS] for the properties of $\mathcal{K}\{M_p\}$ and its strong dual $\mathcal{K}'\{M_p\}$. In this paper we will consider a subclass of such spaces. Namely, as in [GV, p.82], we will assume that $\{M_p,\ p \in \mathbb{N}_0\}$ are smooth functions such that

(I) for every $k, p \in \mathbb{N}_0$ there exist $p' \in \mathbb{N}_0$ and $C > 0$ such that

$$|M_p^{(k)}(x)| \leq CM_{p'}(x), \quad x \in \mathbb{R}.$$

With the quoted conditions on $M_p,\ p \in \mathbb{N}_0$, the sequence of norms $\|\cdot\|_p$, $p \in \mathbb{N}_0$, is equivalent to the sequence of norms

$$\|\varphi\|_{p,2} = \sup\{(\int_{\mathbb{R}} |M_p(x)\varphi^{(i)}(x)|^2 dx)^{1/2} : \ \ i \leq p\},\ p \in \mathbb{N}_0.$$

For example, if we choose $M_p(x) = (1+|x|^2)^{\frac{p}{2}}$, we obtain the space of rapidly decreasing functions $\mathcal{S}(\mathbb{R})$ as $\mathcal{K}\{M_p\}$.

Further on we will also assume that the weight functions M_p satisfy condition

(T) for every $p \in \mathbb{N}_0$ there exist $\tilde{p} \in \mathbb{N}_0$ such that

$$M_p(x-u)M_p(u) \leq M_{\tilde{p}}(x), \quad 0 \leq u \leq |x|,\ x \in \mathbb{R}.$$

Note that the functions $M_p(x) = (1+|x|^2)^{\frac{p}{2}}$, and the functions defined to be $M_p(x) = e^{p|x|^r}$, for $|x| > x_0 > 0$ and smooth around zero, $r \in [1,\infty)$ satisfy condition (T).

1.5.2 The functions f_s^{x-}, f_s^{x+}

We now give a brief overview of the techniques we will use in the sequel.

Let $d \in \mathbb{N}_0$ and $\mathcal{K}_d$ be the closure of $\mathcal{K}\{M_p\}$ with respect to $\|\cdot\|_{d,2}$, i.e. $\mathcal{K}_d = \{\phi \in C^\infty(\mathbb{R}) :\ M_d|\phi^{(i)}| \in L^2(\mathbb{R}), i = 1,2,\ldots,d\}$. Denote:

$$\begin{aligned}
\mathcal{K}_d^- &= \{\phi \in \mathcal{K}_d :\ \text{supp}\ \phi \subset (-\infty, b) \text{ for some } b \in \mathbb{R}\},\\
\mathcal{K}_d^0 &= \mathcal{K}_d \cap C_0,\\
\mathcal{K}_d^+ &= \{\phi \in \mathcal{K}_d :\ \text{supp}\ \phi \subset (b, \infty) \text{ for some } b \in \mathbb{R}\}.
\end{aligned}$$

Let for $t \in \mathbb{R}$,

$$f_s(t) = \begin{cases} \dfrac{H(t)t^{s-1}}{\Gamma(s)}, & s > 0 \\[2ex] f_{s+n}^{(n)}(t), & s \leq 0,\ s+n > 0,\ n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where H denotes the Heaviside function. Note that for $s < 0$ we have $f_s^{(s)} = \delta$, where δ denotes the Dirac delta distribution . For $s, r \in \mathbb{R}$ we have $f_s * f_r = f_{s+r}$.

For fixed $x \in \mathbb{R}$ denote

$$\begin{aligned}
f_s^{x-}(t) &= f_s(x-t), \quad t \in \mathbb{R},\\
f_s^{x+}(t) &= f_s(t-x), \quad t \in \mathbb{R}.
\end{aligned}$$

Clearly, $f_s^{x-}, f_s^{x+} \in C^{s-2}(\mathbb{R})$ and supp $f_s^{x-} \subset (-\infty, x]$, supp $f_s^{x+} \subset [x, \infty)$.

Note that for $\phi \in \mathcal{K}\{M_p\}$ following holds:

$$\left(\int_{\mathbb{R}} f_{d+2}^{x-}(t)\phi(t)dt\right)^{(d+2)} = \left(\int_{-\infty}^{x} f_{d+2}(x-t)\phi(t)dt\right)^{(d+2)} = \phi(x),$$

$$\left(\int_{\mathbb{R}} f_{d+2}^{x+}(t)\phi(t)dt\right)^{(d+2)} = \left(\int_{x}^{\infty} f_{d+2}(t-x)\phi(t)dt\right)^{(d+2)} = \phi(x).$$

Now, let $g \in \mathcal{K}'\{M_p\}$ such that supp $g \subset (a, \infty)$ for some $a \in \mathbb{R}$. There exist $C > 0$ and $d \in \mathbb{N}$ such that

$$|\langle g, \phi\rangle| \leq C\|\phi\|_d, \quad \phi \in \mathcal{K}\{M_p\}.$$

Then, g can be extended onto every $\phi \in \mathcal{K}_d$. Moreover, g can be extended onto every $\varphi \in C^\infty(\mathbb{R})$ such that supp $\varphi \subset (-\infty, b)$, for some $b \in \mathbb{R}$. Thus, g can be extended to every f_{d+2}^{x-}, $x \in \mathbb{R}$. Note that the mapping $x \mapsto \langle g, f_{d+2}^{x-}\rangle$ is continuous. Thus, for $G(x) = \langle g, f_{d+2}^{x-}\rangle$, $x \in \mathbb{R}$, we have

$$G^{(d+2)} = g, \tag{1.5}$$

where the derivative is understood in distributional sense. A similar argument holds for $G(x) = \langle g, f_{d+2}^{x+}\rangle$, $x \in \mathbb{R}$, where $g \in \mathcal{K}'\{M_p\}$ such that supp $g \subset (-\infty, a)$ for some $a \in \mathbb{R}$.

1.5.3 Point value of a distribution

Recall the definition from [Lo] that a distribution $g \in \mathcal{K}'\{M_p\}$ has at $t = t_0$ the *point value* in sense of Lojasiewicz, denoted as $g(t_0) = C$, where $C \in \mathbb{R}$, if

$$\lim_{\varepsilon\to 0}\langle g(t_0 + \varepsilon x), \varphi(x)\rangle = C \int_{\mathbb{R}} \varphi(x)dx, \quad \varphi \in \mathcal{K}\{M_p\}.$$

Let $h \in \mathcal{K}'\{M_p\}$, supp $h \subset (a, \infty)$, $a \in \mathbb{R}$. We say that H is a *primitive* of h in a neighborhood of t_0, $t_0 \in (a, \infty)$, if $H' = f$ in a neighborhood of t_0.

1.5.4 $H-$valued $\mathcal{K}\{M_p\}$ spaces

Conditions (P) and (N) imply (see[PS]) that $\mathcal{K}\{M_p\}$ is a nuclear space. Thus, if H is a Hilbert space with orthonormal basis $\{e_i : \ i \in \mathbb{N}\}$, we can consider the $H-$valued $\mathcal{K}\{M_p\}$ spaces, denoted by $\mathcal{K}\{M_p\}(H)$, as the tensor product $\mathcal{K}\{M_p\}(H) \cong \mathcal{K}\{M_p\} \otimes H$. Thus, a function ϕ belongs to $\mathcal{K}\{M_p\}(H)$ if and only if it is of the form $\phi = \sum_{i=1}^\infty \phi_i e_i$, $\phi_i \in \mathcal{K}\{M_p\}$, $i \in \mathbb{N}$, and $\|\phi\|_{p,2;H}^2 = \sum_{i=1}^\infty \|\phi_i\|_{p,2}^2 < \infty$ for all $p \in \mathbb{N}_0$.

1.6 Colombeau Generalized Functions

Main references for the so called simplified and full Colombeau algebras are [Co1], [Co2], [Mo], [GKOS]. However, we want to extend this concept and construct other Colombeau–type algebras of generalized function spaces. For this approach we refer to [NP] and [DHPV].

Let V be a topological vector space on $\mathbb{C}$ with an an increasing sequence of seminorms $\{p_k : \ k \in \mathbb{N}\}$ defining the topology on V. The vector space $\mathcal{E}_M(V)$, called *the space of moderate nets*, is constituted of functions $R : (0,1) \to V$, $\epsilon \mapsto R(\epsilon) = R_\epsilon$, such that for every $k \in \mathbb{N}$ there exists $a > 0$ with the property $p_k(R_\epsilon) = \mathcal{O}(\epsilon^{-a})$. The space of nets $R_\epsilon \in \mathcal{E}_M(V)$ with the property $p_k(R_\epsilon) = \mathcal{O}(\epsilon^{a})$ for all $k \in \mathbb{N}$ and all $a > 0$, is called *the space of negligible nets* and will be denoted by $\mathcal{N}(V)$. The quotient space

$$\mathcal{G}(V) = \mathcal{E}_M(V)/\ \mathcal{N}(V)$$

is called the *polynomial Colombeau extension of* V. Its elements (equivalence classes) are denoted by $[F_\epsilon]$, $[F_\epsilon]$, ... etc. If V is an algebra whose products are continuous for all seminorms, i.e. $p_k(ab) \leq C p_k(a) p_k(b)$, for all $a, b \in V$, $k \in \mathbb{N}$, then $\mathcal{N}(V)$ is an ideal of the algebra $\mathcal{E}_M(V)$. In this case $\mathcal{G}(V)$ is an algebra.

The following two cases were originally constructed by Colombeau:

If $V = \mathbb{C}$, then $\mathcal{G}(V)$ is called the *Colombeau algebra of generalized constants.*

If $V = C^\infty(I)$, I is an open subset of $\mathbb{R}^d$, and $p_k(f) = \sup\{|D^\alpha f(x)| : x \in U_k,\ |\alpha| \leq k\}$, where U_k, $k \in \mathbb{N}$ is an increasing sequence of compact sets exhausting I, then $\mathcal{G}(V)$ is called the *simplified Colombeau algebra of generalized functions* and is usually denoted by $\mathcal{G}(I)$. Here one can easily deal with differentiation and multiplication, which are defined by $D^\alpha[F_\epsilon] = [D^\alpha F_\epsilon]$ and $[f_\epsilon][g_\epsilon] = [f_\epsilon g_\epsilon]$. The space of Schwartz generalized functions $\mathcal{S}'(I)$ can be embedded into $\mathcal{G}(I)$ by regularization: If $f \in \mathcal{S}'(I)$ and has compact support, then for a fixed mollifier function ρ, we have $f * \rho_\epsilon - f \in \mathcal{N}(I)$ and $[f * \rho_\epsilon] \in \mathcal{G}(I)$.

In the following chapters we will apply this type of construction also to Sobolev spaces, and spaces of stochastic processes.

One can also consider H–valued Colombeau functions; for the case of a Banach space H and the simplified Colombeau algebra see [GKOS, p.70]. The H–valued simplified Colombeau algebra is denoted by $\mathcal{G}(I;H)$. Note that H can be identified with a subspace of $\mathcal{G}(I;H)$ via the constant embedding $H \ni f \mapsto [(x,\epsilon) \mapsto f] \in \mathcal{G}(I;H)$. If H is finite dimensional with a basis $\{h_j\ j = 1,2,\ldots,m\}$, then $\mathcal{G}(I;H) \cong \bigoplus_{j=1}^m \mathcal{G}(I;\mathbb{R}) h_j$. This idea will be used in Chapter 2 to construct Colombeau algebras of generalized random processes (we will put some space of generalized random variables in place of H).

1.7 Spaces of Generalized Random Variables

The spaces of generalized random variables are stochastic analogues of deterministic generalized functions: They have no point value for $\omega \in \Omega$, only an average value with respect to a test random variable. For details refer to the sources that have been drawn upon: [HKPS], [HØUZ] or [Ku].

1.7.1 White noise space

Consider the Gel'fand triple $\mathcal{S}(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d) \subseteq \mathcal{S}'(\mathbb{R}^d)$, the Borel σ-algebra $\mathcal{B}$ generated by the weak topology on $\mathcal{S}'(\mathbb{R}^d)$ and the characteristic function $C(\phi) = \exp\{-\frac{1}{2}|\phi|^2_{L^2(\mathbb{R}^d)}\}$. According to the Bochner-Minlos theorem, there exists a unique measure μ on $(\mathcal{S}'(\mathbb{R}^d), \mathcal{B})$ such that for each $\phi \in \mathcal{S}(\mathbb{R}^d)$ the relation

$$\int_{\mathcal{S}'(\mathbb{R}^d)} e^{i\langle\omega,\phi\rangle} d\mu(\omega) = C(\phi)$$

holds. Here $\langle\omega, \phi\rangle$ is the dual pairing of $\mathcal{S}'(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$. The triplet $(\mathcal{S}'(\mathbb{R}^d), \mathcal{B}, \mu)$ is called *the white noise space* and μ is called the *white noise measure* or the *Gaussian measure* on $\mathcal{S}'(\mathbb{R}^d)$.

From now on we assume that the basic probability space $(\Omega, \mathcal{F}, P)$ is the space $(\mathcal{S}'(\mathbb{R}^d), \mathcal{B}, \mu)$. Put $(L)^2 = L^2(\mathcal{S}'(\mathbb{R}^d), \mathcal{B}, \mu)$. It is a Hilbert space equipped with the inner product $(F|G)_{(L)^2} = E_\mu(F\overline{G})$.

In a multi-dimensional case, for a given $m \in \mathbb{N}$, $m > 1$, define $\mathcal{S}_m = \prod_{i=1}^m \mathcal{S}_i(\mathbb{R}^d)$, where $\mathcal{S}_i(\mathbb{R}^d)$ is a copy of $\mathcal{S}(\mathbb{R}^d)$, and let $\mathcal{S}'_m = \prod_{i=1}^m \mathcal{S}'_i(\mathbb{R}^d)$. Equip the space $\mathcal{S}'_m$ with the product Borel σ-algebra and with the product measure $\mu_m = \mu \times \cdots \times \mu$. The triple $(\mathcal{S}'_m, \mathcal{B}, \mu_m)$ is called the *m-dimensional d-parameter white noise space*. Put $(L)^{2,m} = L^2(\mathcal{S}'_m, \mu_m)$, and for $N \in \mathbb{N}$, $N > 1$, let $(L)^{2,m,N} = \bigoplus_{k=1}^N (L)^{2,m}$ be the direct sum of N identical copies of the m-dimensional d-parameter white noise space. Here the finite dimension N is the state space dimension. Later, instead of the N–dimensional state space, we will also consider the infinite–dimensional case, when the state space is a separable Hilbert space H.

1.7.2 The Wiener – Itô chaos expansion

Denote by $\mathcal{J} = (\mathbb{N}_0^{\mathbb{N}})_c$ the set of sequences of integers which have only finitely many nonzero components. For a given $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathcal{J}$ define the *Fourier-Hermite polynomial* as $H_\alpha(\omega) = \prod_{i=1}^{\infty} h_{\alpha_i}(\langle \omega, \eta_i \rangle)$, $\omega \in \mathcal{S}'(\mathbb{R}^d)$. In the multi-dimensional valued case ($m > 1$) the orthonormal basis of the space $K = \bigoplus_{k=1}^{m} L^2(\mathbb{R}^d)$ is constituted of vectors of length m of the form $e^{(k)} = (0, \ldots, \eta_j, 0, \ldots)$, where η_j takes the ith place in the sequence, and i, j are integers such that $k = i + (j-1)m$, $i \in \{1, 2, \ldots, m\}$, $j \in \mathbb{N}$. In this case we define $H_\alpha^{(m)}(\omega) = \prod_{k=1}^{\infty} h_{\alpha_k}(\langle \omega, e^{(k)} \rangle)$, $\omega \in \mathcal{S}'_m$. The Fourier-Hermite polynomials $H_\alpha^{(m)}$ form an orthogonal basis of $(L)^{2,m}$. It is also known that $\|H_\alpha^{(m)}\|_{(L)^{2,m}} = \sqrt{\alpha_1! \alpha_2! \cdots}$, $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathcal{J}$.

The Wiener-Itô expansion theorem (see [HØUZ]) states that each element $F \in (L)^{2,m,N}$ has a unique representation of the form

$$F(\omega) = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha^{(m)}(\omega), \quad \omega \in \mathcal{S}'_m, \quad c_\alpha \in \mathbb{R}^N, \, \alpha \in \mathcal{J},$$

such that $\|F\|^2_{(L)^{2,m,N}} = \sum_{\alpha \in \mathcal{J}} \alpha! c_\alpha^2$, where $c_\alpha^2 = (c_\alpha | c_\alpha)$ is the standard inner product in $\mathbb{R}^N$.

Example 1.7.1 *Let $\varepsilon_j = (0, 0, \ldots, 1, 0, \ldots)$ be a sequence of zeros with the number 1 as the jth component. The one-dimensional d-parameter Brownian motion $B(t, \omega) = \langle \omega, \kappa_{[0,t]} \rangle$ has expansion*

$$B(t, \omega) = \sum_{j=1}^{\infty} \left(\int_0^t \eta_j(u) du \right) H_{\varepsilon_j}(\omega), \quad t \in \mathbb{R}^d, \, \omega \in \mathcal{S}'(\mathbb{R}^d).$$

For every $t \in \mathbb{R}^d$ fixed, $B(t, \cdot)$ is an element of $(L)^2$. Brownian motion is a Gaussian process whose almost all trajectories are continuous, but nowhere differentiable functions.

1.7.3 The Kondratiev spaces $(S)_\rho^{m,N}$ and $(S)_{-\rho}^{m,N}$

We will use notation $\alpha^\beta = \alpha_1^{\beta_1} \alpha_2^{\beta_2} \cdots$ for given multi-indeces $\alpha, \beta \in \mathcal{J}$, and

$$(2\mathbb{N})^\gamma = \prod_{j=1}^{\infty} (2j)^{\gamma_j},$$

where $\gamma = (\gamma_1, \gamma_2, \ldots) \in \mathcal{J}$. Then, $\sum_{\alpha \in \mathcal{J}} (2\mathbb{N})^{-p\alpha} < \infty$ if and only if $p > 1$, and $\sum_{\alpha \in \mathcal{J}} e^{-p(2\mathbb{N})^\alpha} < \infty$ if and only if $p > 0$.

We will use the definition of the Kondratiev spaces given in [HØUZ] where the authors provide an equivalent construction of the original one introduced by Kondratiev. The *space of the Kondratiev stochastic test functions (space of Kondratiev test random variables)* $(S)^{m,N}_{\rho}$ consists of elements $f = \sum_{\alpha\in\mathcal{I}} c_\alpha H^{(m)}_\alpha \in (L)^{2,m,N}$, $c_\alpha \in \mathbb{R}^N$, $\alpha \in \mathcal{I}$, such that

$$\|f\|^2_{\rho,p} = \sum_{\alpha\in\mathcal{I}} c^2_\alpha (\alpha!)^{1+\rho}(2\mathbb{N})^{p\alpha} < \infty \quad \text{for all } p \in \mathbb{N}_0.$$

The *space of the Kondratiev stochastic generalized functions (space of Kondratiev generalized random variables)* $(S)^{m,N}_{-\rho}$ consists of formal expansions of the form $F = \sum_{\alpha\in\mathcal{I}} b_\alpha H^{(m)}_\alpha$, $b_\alpha \in \mathbb{R}^N$, $\alpha \in \mathcal{I}$, such that

$$\|F\|^2_{-\rho,-p} = \sum_{\alpha\in\mathcal{I}} b^2_\alpha (\alpha!)^{1-\rho}(2\mathbb{N})^{-p\alpha} < \infty \quad \text{for some } p \in \mathbb{N}_0.$$

The action of F onto a test function f is given by $\langle F, f\rangle = \sum_{\alpha\in\mathcal{I}} (b_\alpha|c_\alpha)\alpha!$ where $(\cdot|\cdot)$ is the standard inner product in $\mathbb{R}^N$.

The *generalized expectation* of F is defined as $E(F) = \langle F, 1\rangle = b_0$.

Note that the space $(S)^{m,N}_{\rho}$ can also be constructed as the projective limit of the family $(S_p)^{m,N}_{\rho} = \{f = \sum_{\alpha\in\mathcal{I}} c_\alpha H_\alpha : \|f\|_{\rho,p} < \infty\}$, $p \in \mathbb{N}_0$. Also, space $(S)^{m,N}_{-\rho}$ can also be constructed as the inductive limit of the family $(S_{-p})^{m,N}_{-\rho} = \{f = \sum_{\alpha\in\mathcal{I}} c_\alpha H_\alpha : \|f\|_{-\rho,-p} < \infty\}$, $p \in \mathbb{N}_0$.

In particular, for $\rho = 0$ the Kondratiev spaces are called the *Hida spaces* of test and generalized stochastic functions.

Example 1.7.2 *One-dimensional d-parameter singular white noise is defined via the formal expansion*

$$W(t,\omega) = \sum_{k=1}^{\infty} \eta_k(t) H_{\varepsilon_k}(\omega)$$

where $t \in \mathbb{R}^d$, *and* ε_k, η_k *are as in the previous example. For every* $t \in \mathbb{R}^d$ *fixed,* $W(t,\cdot)$ *belongs to the Hida space of generalized stochastic functions. Singular white noise is the distributional derivative of Brownian motion.*

1.7.4 The Spaces $\exp(S)^{m,N}_{\rho}$ and $\exp(S)^{m,N}_{-\rho}$

In [Se], [PS1] following space of generalized random variables was introduced: The space of stochastic test functions $\exp(S)^{m,N}_{\rho}$ consists of elements $f = \sum_{\alpha\in\mathcal{I}} c_\alpha H^{(m)}_\alpha \in (L)^{2,m,N}$, $c_\alpha \in \mathbb{R}^N$, $\alpha \in \mathcal{I}$, such that

$$\|f\|^2_{\rho,p,exp} = \sum_{\alpha\in\mathcal{I}} c^2_\alpha (\alpha!)^{1+\rho} e^{p(2\mathbb{N})^{\alpha}} < \infty, \quad \text{for all } p \in \mathbb{N}_0.$$

The space of stochastic generalized functions $\exp(S)_{-\rho}^{m,N}$ consists of formal expansions of the form $F = \sum_{\alpha\in\mathcal{I}} b_\alpha H_\alpha^{(m)}$, $b_\alpha \in \mathbb{R}^N$, $\alpha \in \mathcal{I}$, such that

$$\|F\|^2_{-\rho,-p,exp} = \sum_{\alpha\in\mathcal{I}} b_\alpha^2 (\alpha!)^{1-\rho} e^{-p(2\mathbb{N})^\alpha} < \infty, \qquad \text{for some } p \in \mathbb{N}_0.$$

Note that for each $\rho \in [0,1]$, $\exp(S)_{\rho}^{m,N}$ is nuclear and $\exp(S)_{-\rho}^{m,N} \subseteq \exp(S)_{-1}^{m,N}$ (the canonical inclusion $\exp(S)_{1}^{m,N} \subseteq \exp(S)_{\rho}^{m,N}$ is compact). Moreover, following relationship to the Kondratiev spaces holds:

$$\exp(S)_{\rho}^{m,N} \subseteq (S)_{\rho}^{m,N} \subseteq (L)^{2,m,N} \subseteq (S)_{-\rho}^{m,N} \subseteq \exp(S)_{-\rho}^{m,N},$$

i.e. the generalized random variable space $\exp(S)_{-\rho}^{m,N}$ is wider than the corresponding Kondratiev one. The canonical inclusion $\exp(S)_{\rho}^{m,N} \subseteq (S)_{\rho}^{m,N}$ is compact. From the construction it follows that $\exp(S)_{\rho}^{m,N}$ is dense in $(L)^{2,m,N}$, i.e. $\exp(S)_{\rho}^{m,N} \subseteq (L)^{2,m,N} \subseteq \exp(S)_{-\rho}^{m,N}$ is a Gel'fand triple.

1.7.5 Hilbert space valued generalized random variables

Let H be a separable Hilbert space with orthonormal basis $\{e_i : i \in \mathbb{N}\}$. As already suggested in Section 1.7.1, we will treat H as the state space, i.e. we replace the N–dimensional case with an infinitedimensional state space (see [MFA] and the references therein). While in [MFA] the case $m = 1$ is considered, we keep our white noise space dimension to be $m \geq 1$.

Recall that the basic probability space is $(\mathcal{S}'_m, \mathcal{B}, \mu_m)$. Denote by $L^{2,m}(\Omega; H)$ the space of functions on Ω with values in H, which are square integrable with respect to μ_m. The family of functions $\{H_\alpha^{(m)} e_i : i \in \mathbb{N}, \alpha \in \mathcal{I}\}$ is an orthogonal basis of the Hilbert space $L^{2,m}(\Omega; H)$. Each element of $L^{2,m}(\Omega; H)$ can be represented in either of the following forms:

$$f(\omega) = \sum_{i=1}^{\infty} a_i(\omega) e_i, \qquad a_i = \langle f, e_i \rangle_H \in (L)^{2,m}, \qquad \sum_{i=1}^{\infty} \|a_i\|^2_{(L)^{2,m}} < \infty,$$

$$f(\omega) = \sum_{i=1}^{\infty} \sum_{\alpha\in\mathcal{I}} a_{i,\alpha} H_\alpha^{(m)}(\omega) e_i, \; a_{i,\alpha} = \langle f, H_\alpha^{(m)} e_i \rangle \in \mathbb{R}, \; \sum_{i=1}^{\infty} \sum_{\alpha\in\mathcal{I}} \alpha! |a_{i,\alpha}|^2 < \infty,$$

$$f(\omega) = \sum_{\alpha\in\mathcal{I}} a_\alpha H_\alpha^{(m)}(\omega), \qquad a_\alpha = \langle f, H_\alpha^{(m)} \rangle_{(L)^{2,m}} \in H, \qquad \sum_{\alpha\in\mathcal{I}} \alpha! \|a_\alpha\|^2_H < \infty.$$

Now, one can build up spaces of H–valued generalized random variables (H–valued Kondratiev spaces and others) over $L^{2,m}(\Omega;H)$ following the same pattern as in the $\mathbb{R}^N$–valued case. Let $\rho\in[0,1]$.

Define $S(H)^m_\rho$ as the space of functions $f\in L^{2,m}(\Omega;H)$, $f(\omega)=\sum_{i=1}^\infty\sum_{\alpha\in\mathcal{I}}a_{i,\alpha}H^{(m)}_\alpha(\omega)e_i$, $a_{i,\alpha}\in\mathbb{R}$, such that for all $p\in\mathbb{N}_0$,

$$\|f\|^2_{\rho,p;H}=\sum_{i=1}^\infty\sum_{\alpha\in\mathcal{I}}\alpha!^{1+\rho}|a_{i,\alpha}|^2(2\mathbb{N})^{p\alpha}=\sum_{\alpha\in\mathcal{I}}\sum_{i=1}^\infty\alpha!^{1+\rho}|a_{i,\alpha}|^2(2\mathbb{N})^{p\alpha}<\infty.$$

Define $S(H)^m_{-\rho}$ as the space of formal expansions $F(\omega)=\sum_{i=1}^\infty\sum_{\alpha\in\mathcal{I}}b_{i,\alpha}H^{(m)}_\alpha(\omega)e_i$, $b_{i,\alpha}\in\mathbb{R}$, such that for some $q\in\mathbb{N}_0$,

$$\|F\|^2_{-\rho,-q;H}=\sum_{i=1}^\infty\sum_{\alpha\in\mathcal{I}}\alpha!^{1-\rho}|b_{i,\alpha}|^2(2\mathbb{N})^{-q\alpha}=\sum_{\alpha\in\mathcal{I}}\sum_{i=1}^\infty\alpha!^{1-\rho}|b_{i,\alpha}|^2(2\mathbb{N})^{-q\alpha}<\infty.$$

Note, we can also write

$$f(\omega)=\sum_{\alpha\in\mathcal{I}}\sum_{i=1}^\infty a_{i,\alpha}H^{(m)}_\alpha(\omega)e_i=\sum_{\alpha\in\mathcal{I}}a_\alpha H^{(m)}_\alpha(\omega)=\sum_{i=1}^\infty a_i(\omega)e_i,$$

where $a_\alpha=\sum_{i=1}^\infty a_{i,\alpha}e_i\in H$ and $a_i=\sum_{\alpha\in\mathcal{I}}a_{i,\alpha}H^{(m)}_\alpha(\omega)\in(S)^m_\rho$. Also,

$$\|f\|^2_{\rho,p;H}=\sum_{\alpha\in\mathcal{I}}\alpha!^{1+\rho}\|a_\alpha\|^2_H(2\mathbb{N})^{p\alpha}=\sum_{i=1}^\infty\|a_i\|^2_{\rho,p}.$$

The same holds also for

$$F(\omega)=\sum_{\alpha\in\mathcal{I}}\sum_{i=1}^\infty b_{i,\alpha}H^{(m)}_\alpha(\omega)e_i=\sum_{\alpha\in\mathcal{I}}b_\alpha H^{(m)}_\alpha(\omega)=\sum_{i=1}^\infty b_i(\omega)e_i,$$

where $b_\alpha=\sum_{i=1}^\infty b_{i,\alpha}e_i\in H$ and $b_i=\sum_{\alpha\in\mathcal{I}}b_{i,\alpha}H^{(m)}_\alpha(\omega)\in(S)^m_{-\rho}$. Also,

$$\|F\|^2_{-\rho,-q;H}=\sum_{\alpha\in\mathcal{I}}\alpha!^{1-\rho}\|b_\alpha\|^2_H(2\mathbb{N})^{-q\alpha}=\sum_{i=1}^\infty\|b_i\|^2_{-\rho,-q}.$$

The action of F onto f is given by

$$\langle F,f\rangle=\sum_{\alpha\in\mathcal{I}}\alpha!\langle b_\alpha,a_\alpha\rangle_H.$$

Similarly as in the finite-dimensional case, we have

$$S(H)^{(m)}_1\subseteq S(H)^{(m)}_\rho\subseteq S(H)^{(m)}_0\subseteq L^{2,m}(\Omega;H)\subseteq S(H)^{(m)}_{-0}\subseteq S(H)^{(m)}_{-\rho}\subseteq S(H)^{(m)}_{-1}.$$

The same construction can be carried out for the exponential growth rate spaces. Let $\exp S(H)^m_\rho$ be the space of functions $f \in L^{2,m}(\Omega;H)$, $f(\omega) = \sum_{i=1}^\infty \sum_{\alpha\in\mathcal{J}} a_{i,\alpha} H^{(m)}_\alpha(\omega) e_i$, $a_{i,\alpha}\in\mathbb{R}$, such that for all $p\in\mathbb{N}_0$,

$$\|f\|^2_{\rho,p,exp;H} = \sum_{i=1}^\infty \sum_{\alpha\in\mathcal{J}} \alpha!^{1+\rho} |a_{i,\alpha}|^2 e^{p(2\mathbb{N})^\alpha} = \sum_{\alpha\in\mathcal{J}} \sum_{i=1}^\infty \alpha!^{1+\rho} |a_{i,\alpha}|^2 e^{p(2\mathbb{N})^\alpha} < \infty.$$

The corresponding space of stochastic generalized functions $\exp S(H)^m_{-\rho}$ consists of formal expansions of the form $F(\omega) = \sum_{i=1}^\infty \sum_{\alpha\in\mathcal{J}} b_{i,\alpha} H^{(m)}_\alpha(\omega) e_i$, $b_{i,\alpha}\in\mathbb{R}$, such that for some $q\in\mathbb{N}_0$,

$$\|F\|^2_{-\rho,-q,exp;H} = \sum_{i=1}^\infty \sum_{\alpha\in\mathcal{J}} \alpha!^{1-\rho} |b_{i,\alpha}|^2 e^{-q(2\mathbb{N})^\alpha} = \sum_{\alpha\in\mathcal{J}} \sum_{i=1}^\infty \alpha!^{1-\rho} |b_{i,\alpha}|^2 e^{-q(2\mathbb{N})^\alpha} < \infty.$$

Both $S(H)^m_\rho$ and $\exp S(H)^m_\rho$ are countably Hilbert spaces and

$$\exp S(H)^m_\rho \subseteq S(H)^m_\rho \subseteq L^{2,m}(\Omega;H) \subseteq S(H)^m_{-\rho} \subseteq \exp S(H)^m_{-\rho}.$$

In general, $S(H)^m_\rho$ and $\exp S(H)^m_\rho$ are not nuclear spaces. They would be nuclear e.g. if H were finite–dimensional (see [Tr, Prop.50.1,Cor.]).

Note that since $(S)^m_\rho$ and $\exp(S)^m_\rho$ are nuclear spaces, by [Tr, Prop.50.7] we have again (as for the Schwartz spaces in the deterministic case) an isomorphism with tensor product spaces:

$$S(H)^m_{-\rho} \cong (S)^m_{-\rho} \otimes H, \qquad \exp S(H)^m_{-\rho} \cong \exp(S)^m_{-\rho} \otimes H.$$

As in [MFA] one can define H–valued Brownian motion and singular white noise:

Example 1.7.3 *Let $n : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$, $(i,j) \mapsto n(i,j)$ be the usual bijection given in the following table:*

	1	2	3	4	5	6	7	$\cdots$	j	$\cdots$
1	1	3	6	10	15	21	28	$\cdots$		
2	2	5	9	14	20	27			$\vdots$	
3	4	8	13	19	26				$\vdots$	
4	7	12	18	25					$\vdots$	
5	11	17	24						$\vdots$	
6	16	23							$\vdots$	
7	22								$\vdots$	
$\vdots$	$\vdots$								$\vdots$	
i		$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$n(i,j)$	$\cdots$
$\vdots$									$\vdots$	

Let $\varepsilon_j = (0, 0, \ldots, 1, 0, \ldots)$ *be a sequence of zeros with the number 1 as the* j*th component and* ξ_j *the Hermite functions. Then,*

$$\beta_i(t, \omega) = \sum_{j=1}^{\infty} \int_0^t \xi_j(s) ds H_{\varepsilon_{n(i,j)}}(\omega), \quad i \in \mathbb{N}$$

is a sequence of independent (one-dimensional d-parameter $\mathbb{R}$*–valued) Brownian motions. Rewrite this as*

$$\beta_i(t, \omega) = \sum_{k=1}^{\infty} \theta_{ik}(t) H_{\varepsilon_k}(\omega), \quad \theta_{ik}(t) = \begin{cases} \int_0^t \xi_j(s) ds, & k = n(i,j) \\ 0, & k \neq n(i,j) \end{cases} .$$

The formal sum

$$\mathbf{B}(t, \omega) = \sum_{i=1}^{\infty} \beta_i(t, \omega) e_i = \sum_{k=1}^{\infty} \theta_k(t) H_{\varepsilon_k}(\omega), \quad \theta_k(t) = \delta_{n(i,j),k} \int_0^t \xi_j(s) ds \; e_i \tag{1.6}$$

is an H*–valued Brownian motion. Note that while* $\mathbb{R}$*–valued Brownian motion was an element of* $(L)^2$ *for fixed* $t \geq 0$*, now* $\mathbf{B}(t, \omega)$ *does not (!) belong to* $L^2(\Omega; H)$*. However, the sum in* (1.6) *does converge in* $S(H)_{-0}$ *for each* $t \geq 0$ *fixed.*

Example 1.7.4 *H–valued (one-dimensional, d-parameter) singular white noise is defined by the formal sum*

$$\mathbf{W}(t, \omega) = \sum_{k=1}^{\infty} \kappa_k(t) H_{\varepsilon_k}(\omega), \quad \kappa_k(t) = \delta_{n(i,j),k} \xi_j(t) \; e_i.$$

It is also an element of $S(H)_{-0}$*.*

The infinite! No other question has ever moved
so profoundly the spirit of man.

(David Hilbert)

Chapter 2

Generalized Stochastic Processes

Generalized random (stochastic) processes can be defined in various ways depending on whether the author regards them as a family of random variables or as a family of trajectories, but also depending on the type of continuity implied onto this family. It is well–known that a classical stochastic process $X(t, \omega)$, $t \in T \subseteq \mathbb{R}^d$, $\omega \in \Omega$, can be regarded either as a family of random variables $X(t, \cdot)$, $t \in T$, as a family of trajectories $X(\cdot, \omega)$, $\omega \in \Omega$, or as a family of functions $X : T \times \Omega \to \mathbb{R}^n$ ($\mathbb{R}^n$ is the state space) such that for each fixed $t \in T$, $X(t, \cdot)$ is an $\mathbb{R}^n$–valued random variable and for each fixed $\omega \in \Omega$, $X(\cdot, \omega)$ is an $\mathbb{R}^n$–valued deterministic function (called a trajectory). For classical stochastic processes these three concepts are equivalent, but if one replaces the space of trajectories with some space of deterministic generalized functions, or if one replaces the space of random variables with some space of generalized random variables, then different types of generalized stochastic processes are obtained. Each of these approaches is an acorn from which a great theory has grown. The classification of generalized stochastic processes by various conditions of continuity leads to structural theorems such as integral representations and series expansions, which will be subject of the following sections. Let us give now a historical overview of various definitions of the concept of generalized random processes (GRPs).

Processes generalized with respect to the t argument

One possible definition of a GRP is the one used by J.B. Walsh (see [Wa]) as a measurable mapping $X : \Omega \to \mathcal{D}'(T)$. For each $\phi \in \mathcal{D}(T)$, the mapping $\Omega \to \mathbb{R}$, $\omega \mapsto \langle X(\omega), \phi\rangle$ is a random variable. This definition is motivated by the fact that trajectories of Brownian motion are nowhere differentiable and can be considered as elements of $\mathcal{D}'(T)$.

K. Itô defined in [It] a GRP as a linear and continuous mapping from $L^2(\mathbb{R})$ to the space $L^2(\Omega)$ of random variables with finite second moments, while Inaba considered in [In] a GRP as a continuous mapping from a certain space of test functions to the space $L^2(\Omega)$. Also, Gel'fand and Vilenkin [GV] have considered GRPs in this sense. Further on, we will refer to GRPs defined in this sense as to **GRPs of type (I)**.

On the other hand, O. Hanš, M. Ullrich, L. Swartz and others (see [Ha], [LP], [LP1], [LP2], [SM], [Ul]) defined a GRP as a mapping $\xi : \Omega \times V \to \mathbb{C}$ such that for every $\varphi \in V$, $\xi(\cdot, \varphi)$ is a complex random variable, and for every $\omega \in \Omega$, $\xi(\omega, \cdot)$ is an element in V', where V denotes a topological vector space and V' its dual space. Further on, we will refer to GRPs defined in this sense as to **GRPs of type (II)**.

Note that if a GRP is of type (I), then we do not have continuity of sample paths for each fixed $\omega \in \Omega$, only continuity in distributional sense. If we assume for a GRP (I) the continuity for a.e. fixed $\omega \in \Omega$, then it is also of type (II). However, in [Wa] Walsh proved that, if the underlying test space V is nuclear, then for a GRP (I) there exists a version which is also a GRP (II). This result need not hold true if V is not nuclear, e.g. if it is a Hilbert space, as it will be shown in Example 2.5.1.

Vice versa, if a GRP is of type (II), this does not ensure its continuity as a mapping from the test space $\mathcal{K}\{M_p\}$ into $L^2(\Omega)$ or even $L^0(\Omega)$ equipped with convergence in probability. Nevertheless, we show in Theorem 2.5.2 under which conditions is a GRP of type (II) also a GRP (I) up to a set of arbitrarily small measure. This result can not be improved, i.e. one can not obtain a version up to a set of measure zero, as it is shown in Example 2.5.5. In Example 2.5.1 we show that the the result need not hold true if we consider GRPs (II) on a Hilbert space.

It remains as an enticing open question to investigate the general relationship of GRPs (I) and (II) and to find which conditions one should propose on not nuclear spaces in order to obtain an equivalence between the two types of continuity.

Another possibility to generalize the concept of a stochastic process is to take advantage of Colombeau generalized function spaces. This approach was recently used by M. Oberguggenberger, F. Russo, S. Albeverio, M. Nedeljkov,

D. Rajter and others (see [OR1], [OR2], [RO], [AHR1], [AHR2], [NR].) In this framework a GRP is defined as a mapping $X : \Omega \to \mathcal{G}(T)$, ($\mathcal{G}(T)$ is the simplified Colombeau algebra of generalized functions) having the property that there exists a net of functions $R_\epsilon : \Omega \times T \to \mathbb{C}$, $\epsilon \in (0,1)$, such that $R_\epsilon(\omega, \cdot)$ represents $X(\omega)$ for a.e. $\omega \in \Omega$, and for every $\epsilon \in (0,1)$, $(\omega, t) \to R_\epsilon(\omega, t)$ is a measurable (classical) stochastic process. Recall that $\mathcal{D}'(T)$ can be embedded into $\mathcal{G}(T)$, thus this is a generalization of the Walsh definition of a GRP.

Also, in [LPP], Colombeau GRPs of second order were considered as elements of the vector–valued Colombeau algebra $\mathcal{G}(T; L^2(\Omega))$, where $L^2(\Omega)$ is the Hilbert space of random variables with finite second moments.

Processes generalized with respect to the ω argument

T. Hida, Y. Kondratiev, B. Øksendal, H.-H. Kuo, and many others (refer to [HKPS], [HØUZ], [Ku]) have developed a very general concept of GRPs via chaos expansions. In [HØUZ] GRPs are defined as measurable mappings $T \to (S)_{-1}$, where $(S)_{-1}$ denotes the Kondratiev space, but one can consider also some other spaces of generalized random variables instead of it. Thus, they are pointwisely defined with respect to the parameter $t \in T$ and generalized with respect to $\omega \in \Omega$. Further on, we will refer to GRPs defined in this sense as to **GRPs of type (O)**.

Examples of GRPs (O) are Brownian motion and singular white noise defined via the chaos expansions in Example 1.7.1 and Example 1.7.2, respectively.

Processes generalized with respect to both arguments

GRPs of this type were introduced in [Se] and [PS1], where we in fact generalized and unified the concept of a GRP in Inaba's sence and the concept of a GRP (O), and considered GRPs as linear continuous mappings from the Zemanian test space into $(S)_{-1}$. There we gave structural properties of these GRPs by series expansions in spaces of Zemanian generalized functions and a simultaneous chaos expansion. Since these processes, as elements of $\mathcal{L}(\mathcal{A}, (S)_{-1})$, are very close to the concept of a GRP (I), we will also call them GRPs of type (I).

As in the deterministic theory of distributions, where we identify locally integrable functions and regular distributions, we can identify elements $f \in L^1_{loc}(\mathbb{R}^d, (S)_{-1})$ with the corresponding linear mappings $\tilde{f} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^d), (S)_{-1})$. In other words, the class of locally Pettis-integrable GRPs (O) can be embedded into the class of GRPs (I).

We will follow a similar idea in this dissertation and replace $L^2(\Omega)$ with the more general $(S)_{-1}$ and consider the Colombeau extension of a GRP (I) i.e. we will consider GRPs as elements of $\mathcal{G}(\mathbb{R}^d;(S)_{-1})$. Similarly as in the deterministic theory of distributions, where $\mathcal{S}'(\mathbb{R}^d)$ can be embedded into $\mathcal{G}(\mathbb{R}^d)$, we will have the vector–valued analogy of it: The space of GRPs (I) $\mathcal{L}(\mathcal{S}(\mathbb{R}^d),(S)_{-1})$ can be embedded into $\mathcal{G}(\mathbb{R}^d;(S)_{-1})$.

2.1 Generalized Random Processes of Type (O) and (I)

We give now an overview of the results obtained in [Se] and [PS1]. Elements of the spaces $\mathcal{L}(\mathcal{A},(S)_{-1})$ and $\mathcal{L}(\mathcal{A},\exp(S)_{-1})$, but also of $\mathcal{L}(\mathrm{Exp}\mathcal{A},(S)_{-1})$ and $\mathcal{L}(\mathrm{Exp}\mathcal{A},\exp(S)_{-1})$ are GRPs (I). As already mentioned, these processes are generalized both by the time-parameter t and by the random parameter ω.

Let $t \mapsto U(t,\cdot) \in (S)_{-1}$ be a GRP (O) which is locally integrable in the Pettis sense, $U \in L^1_{loc}(\mathbb{R}^d,(S)_{-1})$. Then there is an associated GRP (I), i.e. a linear, continuous mapping $\tilde{U} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^d),(S)_{-1})$ such that

$$[\tilde{U},\varphi](\omega) = \int_{\mathbb{R}^d} U(t,\omega)\varphi(t)dt, \quad \varphi \in \mathcal{S}(\mathbb{R}^d),\, \omega \in \mathcal{S}'(\mathbb{R}^d), \tag{2.1}$$

where $[\cdot,\cdot]$ denotes the action of $\tilde{U} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^d),(S)_{-1})$ onto a $\varphi \in \mathcal{S}(\mathbb{R}^d)$; so that $[\tilde{U},\varphi] \in (S)_{-1}$.

Further we will also denote by $[\cdot,\cdot]$ the action of an element from $\mathcal{L}(\mathcal{A},(S)_{-1})$ or $\mathcal{L}(\mathcal{A},\exp(S)_{-1})$ onto an element from $\mathcal{A}$, and with $\langle\cdot,\cdot\rangle$ the classical dual pairing in $\mathcal{A}'$ and $\mathcal{A}$.

Series expansion of GRPs (O)

Since GRPs (O) with values in $(L)^{2,m,N}$ are defined pointwisely with respect to the parameter $t \in \mathbb{R}^d$, their expansions follow directly from the Wiener–Itô theorem: Let $Z : \mathbb{R}^d \to (S)^{m,N}_{-\rho}$ be a GRP (O). Then it has an expansion

$$Z(t) = \sum_{\alpha\in\mathcal{J}} c_\alpha(t) H^{(m)}_\alpha, \qquad t \in \mathbb{R}^d, \tag{2.2}$$

where $c_\alpha : \mathbb{R}^d \to \mathbb{R}^N$, $\alpha \in \mathcal{J}$ are measurable functions. If $c_\alpha \in L^1(\mathbb{R}^d,\mathbb{R}^N)$, $\alpha \in \mathcal{J}$, and if there exists $p \in \mathbb{N}_0$ such that

$$\int_{\mathbb{R}^d} \|Z(t)\|_{-\rho,-p}dt = \sum_{\alpha\in\mathcal{J}} \alpha!^{1-\rho}|c_\alpha(t)|^2_{L^1(\mathbb{R}^d)}(2\mathbb{N})^{-p\alpha} < \infty,$$

then $Z \in L^1(\mathbb{R}^d, (S)_{-\rho}^{m,N})$, that is Z is a Pettis-integrable process.

The expansion theorems of GRPs (I) will give an extension of the expansion of the Pettis-integrable GRPs (O) in the sense that the coefficients $c_\alpha(t)$, $\alpha \in \mathcal{I}$, in (2.2) will be generalized functions from $\mathcal{A}'$.

Series expansion of GRPs (I)

Consider for example GRPs (I) as elements of the space $\mathcal{A}^* = \mathcal{L}(\mathcal{A}, (S)_{-1})$. Elements of $\mathcal{A}_k^* = \mathcal{L}(\mathcal{A}_k, (S)_{-1})$ are called GRPs (I) of $\mathcal{R}$-order k. We have a chain of continuous canonical inclusions

$$(L^2(I))^* = \mathcal{A}_0^* \subseteq \mathcal{A}_1^* \subseteq \cdots \subseteq \mathcal{A}_k^* \subseteq \mathcal{A}^* = \bigcup_{k \in \mathbb{N}_0} \mathcal{A}_k^*.$$

For technical reasons we assume that the set of multi-indeces $\mathcal{I}$ is ordered in a lexicographic order and denote by α^j, $j \in \mathbb{N}$, the jth element in this ordering.

Definition 2.1.1 *Let $f_j \in \mathcal{A}'$, $j = 1, 2, \ldots, m$ and let $\theta_{\alpha^j} \in (S)_{-1}$, $j = 1, 2, \ldots, m$. Then $\sum_{j=1}^m f_j \otimes \theta_{\alpha^j}$ is a GRP (I), more precisely an element of $\mathcal{A}^*$ defined by*

$$\left[\sum_{j=1}^m f_j \otimes \theta_{\alpha^j}, \varphi\right] = \sum_{j=1}^m \langle f_j, \varphi \rangle \theta_{\alpha^j}, \qquad \varphi \in \mathcal{A}. \tag{2.3}$$

Theorem 2.1.1 *Let $k \in \mathbb{N}_0$. The following conditions are equivalent:*

(i) $\Phi \in \mathcal{A}_k^*$.

(ii) Φ *can be represented in the form*

$$\Phi = \sum_{j=1}^\infty f_j \otimes H_{\alpha^j}, \qquad f_j \in \mathcal{A}_{-k}, \quad j \in \mathbb{N} \tag{2.4}$$

and there exists $k_0 \in \mathbb{N}_0$ such that for each bounded set $B \subseteq \mathcal{A}_k$

$$\sup_{\varphi \in B} \sum_{j=1}^\infty |\langle f_j, \varphi \rangle|^2 \, (2\mathbb{N})^{-k_0 \alpha^j} < \infty. \tag{2.5}$$

(iii) Φ *can be represented in the form (2.4) and there exists $k_1 \in \mathbb{N}_0$ such that*

$$\sum_{j=1}^\infty \|f_j\|_{-k}^2 (2\mathbb{N})^{-k_1 \alpha^j} < \infty. \tag{2.6}$$

Since $\mathcal{A}^*$ is constructed as the inductive limit of the family $\mathcal{A}_k^*$, $k \in \mathbb{N}_0$, we obtain the following expansion theorem for a GRP (I).

Theorem 2.1.2 $\Phi \in \mathcal{A}^*$ *if and only if there exist* k, $k_0 \in \mathbb{N}_0$ *such that series expansion (2.4) and condition (2.5) hold.*

Consistency of the expansions of GRPs (O) and GRPs (I)

Let U be a GRP (O) given by the expansion

$$U(t,\omega) = \sum_{i=1}^{\infty} a_i(t) H_{\alpha^i}(\omega), \qquad t \in \mathbb{R}, \quad \omega \in \mathcal{S}'(\mathbb{R}),$$

such that $a_i(t) \in L^1_{loc}(\mathbb{R})$, $i \in \mathbb{N}$. Then due to (2.1), there is a GRP (I), denoted by $\tilde{U}$ associated with U, such that

$$[\tilde{U}, \varphi](\omega) = \int_{\mathbb{R}} \sum_{i=1}^{\infty} a_i(t) H_{\alpha^i}(\omega)\varphi(t)dt = \sum_{i=1}^{\infty} \langle \tilde{a}_i, \varphi\rangle H_{\alpha^i}(\omega), \quad \omega \in \mathcal{S}'(\mathbb{R}),$$

where $\tilde{a}_i \in \mathcal{S}'(\mathbb{R})$ is the generalized function associated with the function $a_i(t) \in L^1_{loc}(\mathbb{R})$, $i \in \mathbb{N}$. Thus, $\tilde{U}$ has expansion

$$\tilde{U} = \sum_{i=1}^{\infty} \tilde{a}_i \otimes H_{\alpha^i}.$$

This means that the expansion theorems for GRPs (I) are consistent with the expansion theorems for GRPs (O).

Expansion theorems for ${}^{exp}\mathcal{A}^* = \mathcal{L}(\mathcal{A}, \exp(S)_{-1})$

Analogously to the previous theorems concerning GRPs (I) with values in $(S)_{-1}$, one can consider GRPs (I) taking values in other spaces of generalized stochastic functions. The expansion theorems can be carried over, *mutatis mutandis*, to these GRPs.

Consider for example the space $\exp(S)_{-1}$, which will provide a larger class of GRP (I). Let ${}^{exp}\mathcal{A}^* = \mathcal{L}(\mathcal{A}, \exp(S)_{-1})$ and ${}^{exp}\mathcal{A}_k^* = \mathcal{L}(\mathcal{A}_k, \exp(S)_{-1})$ be GRPs (I) and GRPs (I) of $\mathcal{R}$-order k, respectively. Further, let all the other terms be defined analogously as for $\mathcal{A}^*$; i.e. we replace $(S)_{-1}$ with $\exp(S)_{-1}$ in Definition 2.1.1 and else where necessary.

Theorem 2.1.3 *Let* $k \in \mathbb{N}_0$. *The following conditions are equivalent:*

(i) $\Phi \in {}^{exp}\mathcal{A}_k^*$.

(ii) Φ *can be represented in the form*

$$\Phi = \sum_{j=1}^{\infty} f_j \otimes H_{\alpha^j}, \qquad f_j \in \mathcal{A}_{-k}, \quad j \in \mathbb{N}, \tag{2.7}$$

and there exists $k_0 \in \mathbb{N}_0$*, such that for each bounded set* $B \subseteq \mathcal{A}_k$

$$\sup_{\varphi \in B} \sum_{j=1}^{\infty} |\langle f_j, \varphi \rangle|^2 e^{-k_0 (2\mathbb{N})^{\alpha^j}} < \infty. \tag{2.8}$$

(iii) Φ *can be represented in the form (2.7) and there exists* $k_1 \geq 0$*, such that*

$$\sum_{j=1}^{\infty} \|f_j\|^2_{-k} e^{-k_1 (2\mathbb{N})^{\alpha^j}} < \infty.$$

For examples of GRPs (I) and definition of the Wick product in spaces $\mathcal{A}^*$ refer to [PS1].

2.2 Hilbert Space Valued Generalized Random Processes of Type (I)

Now we expand the results of [PS1] and consider Hilbert space valued GRPs. Throughout this section, and also further on, H will denote a separable Hilbert space with orthonormal basis $\{e_i : \ i \in \mathbb{N}\}$.

We replace the Kondratiev space $(S)_{-1}$ with the H–valued Kondratiev space $S(H)_{-1}$ and define H–valued GRPs (I) as linear continuous mappings from the Zemanian test space $\mathcal{A}$ into $S(H)_{-1}$ i.e. as elements of

$$\mathcal{A}(H)^* = \mathcal{L}(\mathcal{A}, S(H)_{-1}).$$

Elements of $\mathcal{A}(H)^*_k = \mathcal{L}(\mathcal{A}_k, S(H)_{-1})$ are called H–valued GRPs (I) of $\mathcal{R}$-order k. Thus, $L \in \mathcal{A}(H)^*_k$ if and only if there exists $k_0 \in \mathbb{N}$ such that $L \in \mathcal{L}(\mathcal{A}_k, S(H)_{-1,-k_0})$. Note that $\mathcal{L}(\mathcal{A}_k, S(H)_{-1,-k_0})$ is a Banach space with the usual dual norm

$$\|L\|^*_{-k;H} = \sup \left\{ \|[L, g]\|_{-1,-k_0;H} : \ g \in \mathcal{A}_k, \|g\|_k \leq 1 \right\}.$$

Clearly, $\mathcal{A}(H)'_k \subseteq \mathcal{A}(H)^*_k$, and $\|f\|^*_{-k;H} = \|f\|_{-k;H}$ if $f \in \mathcal{A}(H)_{-k}$. We have a chain of continuous canonical inclusions

$$L^2(I; H) = \mathcal{A}(H)^*_0 \subseteq \mathcal{A}(H)^*_1 \subseteq \cdots \subseteq \mathcal{A}(H)^*_k \subseteq \mathcal{A}(H)^* = \bigcup_{k \in \mathbb{N}_0} \mathcal{A}(H)^*_k.$$

Definition 2.2.1 *Let $f_j \in \mathcal{A}'$, $j = 1, 2, \ldots, m$ and let $\theta_{\alpha^j} \in S(H)_{-1}$, $j = 1, 2, \ldots, m$. Then $\sum_{j=1}^{m} f_j \otimes \theta_{\alpha^j}$ is an element of $\mathcal{A}(H)^*$ defined by*

$$\left[\sum_{j=1}^{m} f_j \otimes \theta_{\alpha^j}, \varphi\right] = \sum_{j=1}^{m} \langle f_j, \varphi\rangle \theta_{\alpha^j}, \qquad \varphi \in \mathcal{A}. \tag{2.9}$$

Recall that each $\theta_{\alpha^j} \in S(H)_{-1}$ can be represented as

$$\theta_{\alpha^j}(\omega) = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \theta_{jik} H_{\alpha^k}(\omega) e_i, \quad \theta_{jik} \in \mathbb{R}. \tag{2.10}$$

Thus, (2.9) can be written in an equivalent form

$$\begin{aligned} \sum_{j=1}^{m} \langle f_j, \varphi\rangle \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \theta_{jik} H_{\alpha^k}(\omega) e_i &= \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \langle F_{ik}, \varphi\rangle H_{\alpha^k}(\omega) e_i \\ &= \left[\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} F_{ik} \otimes H_{\alpha^k}(\omega) e_i, \varphi\right], \end{aligned}$$

where $F_{ik} = \sum_{j=1}^{m} f_j \theta_{jik} \in \mathcal{A}'$.

Lemma 2.2.1 *Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}'$ and $\langle \theta_{\alpha^j}\rangle_{j\in\mathbb{N}}$ be a sequence in $S(H)_{-1}$. If there exists $k_0 \in \mathbb{N}_0$ such that for any bounded set $B \in \mathcal{A}$,*

$$\sup_{\varphi\in B} \sum_{j=1}^{\infty} |\langle f_j, \varphi\rangle| \cdot \|\theta_{\alpha^j}\|_{-1,-k_0;H} < \infty, \tag{2.11}$$

then $\sum_{j=1}^{\infty} f_j \otimes \theta_{\alpha^j}$ defined by

$$\sum_{j=1}^{\infty} f_j \otimes \theta_{\alpha^j} = \lim_{m\to\infty} \sum_{j=1}^{m} f_j \otimes \theta_{\alpha^j}$$

is an element of $\mathcal{A}(H)^*$.

Proof. Denote $\Upsilon = \sum_{j=1}^{\infty} f_j \otimes \theta_{\alpha^j}$ and $\Upsilon_m = \sum_{j=1}^{m} f_j \otimes \theta_{\alpha^j}$, $m \in \mathbb{N}$. Clearly, $\Upsilon_m \in \mathcal{A}^*$. If (2.11) holds, then $\Upsilon_m \in \mathcal{L}(\mathcal{A}, S(H)_{-1,-k_0})$.

The sequence of partial sums Υ_m, $m \in \mathbb{N}$, is a Cauchy sequence in $\mathcal{A}(H)^*$ because, for given $\varepsilon > 0$ and $m > n$,

$$\|[\Upsilon_m, \varphi] - [\Upsilon_n, \varphi]\|_{-1,-k_0;H} = \sum_{j=n+1}^{m} |\langle f_j, \varphi\rangle| \cdot \|\theta_{\alpha^j}\|_{-1,-k_0;H} < \varepsilon,$$

if n, m are large enough.

Since

$$\|\Upsilon_m\|^* = \sup\{\|[\Upsilon, \varphi]\|_{-1,-k_0;H} : \varphi \in \mathcal{A}, \|\varphi\| \leq 1\}$$
$$\leq \sup\{\sum_{j=1}^{m} |\langle f_j, \varphi\rangle| \cdot \|\theta_{\alpha^j}\|_{-1,-k_0;H} : \varphi \in \mathcal{A}, \|\varphi\| \leq 1\},$$

it follows that

$$\sup_{m\in\mathbb{N}} \|\Upsilon_m\|^* \leq \sup_{m\in\mathbb{N}} \left(\sup\{\sum_{j=1}^{m} |\langle f_j, \varphi\rangle| \cdot \|\theta_{\alpha^j}\|_{-1,-k_0;H} : \varphi \in \mathcal{A}, \|\varphi\| \leq 1\} \right)$$
$$\leq \sup\{\sum_{j=1}^{\infty} |\langle f_j, \varphi\rangle| \cdot \|\theta_{\alpha^j}\|_{-1,-k_0;H} : \varphi \in \mathcal{A}, \|\varphi\| \leq 1\} < \infty,$$

by condition (2.11). According to the Banach-Steinhaus theorem, $\Upsilon = lim_{m\to\infty}\Upsilon_m$ also belongs to $\mathcal{L}(\mathcal{A}, S(H)_{-1,-k_0})$. □

Theorem 2.2.1 *Let $k \in \mathbb{N}_0$. The following conditions are equivalent:*

(i) $\Phi \in \mathcal{A}(H)^*_k$.

(ii) Φ *can be represented in the form*

$$\Phi = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f_{ij} \otimes H_{\alpha^j}\, e_i, \qquad f_{ij} \in \mathcal{A}_{-k}, \quad i, j \in \mathbb{N} \tag{2.12}$$

and there exists $k_0 \in \mathbb{N}_0$ such that for each bounded set $B \subseteq \mathcal{A}_k$

$$\sup_{\varphi\in B} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |\langle f_{ij}, \varphi\rangle|^2 \, (2\mathbb{N})^{-k_0\alpha^j} < \infty. \tag{2.13}$$

Proof. Let $\Phi \in \mathcal{A}^*_k(H) = \mathcal{L}(\mathcal{A}_k, S(H)_{-1})$. There exists $k_0 \in \mathbb{N}_0$, such that $\Phi \in \mathcal{L}(\mathcal{A}_k, S(H)_{-1,-k_0})$. The mapping $f_{ij} : \mathcal{A}_k \to \mathbb{R}$ given by $\varphi \mapsto ([\Phi, \varphi]|H_{\alpha^j} e_i)_{-1,-k_0;H}$ is linear and continuous for each $H_{\alpha^j} e_i$, i.e. $f_{ij} \in \mathcal{A}'_k = \mathcal{A}_{-k}$ for each $i, j \in \mathbb{N}$. Thus,

$$\langle f_{ij}, \varphi\rangle = ([\Phi, \varphi]|H_{\alpha^j} e_i)_{-1,-k_0;H}\,, \quad \varphi \in \mathcal{A}_k, \quad j \in \mathbb{N}.$$

Also, $[\Phi, \varphi] \in S(H)_{-1,-k_0}$ has the expansion

$$[\Phi, \varphi] = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} ([\Phi, \varphi]|H_{\alpha^j} e_i)_{-1,-k_0;H}\, H_{\alpha^j} e_i. \tag{2.14}$$

The series on the right-hand side of (2.14) converges if and only if

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|([\Phi,\varphi]|H_{\alpha^j}e_i)_{-1,-k_0;H}\right|^2(2\mathbb{N})^{-k_0\alpha^j}=\sum_{j=1}^{\infty}|\langle f_j,\varphi\rangle|^2(2\mathbb{N})^{-k_0\alpha^j}<\infty,$$

which yields (2.13). Now, by Definition 2.2.1, (2.14) is equal to

$$[\Phi,\varphi]=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\langle f_{ij},\varphi\rangle H_{\alpha^j}e_i=\left[\sum_{j=1}^{\infty}f_{ij}\otimes H_{\alpha^j}e_i,\varphi\right],$$

and this implies (2.12).

Conversely, let $\Phi=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}f_{ij}\otimes H_{\alpha^j}e_i$, where $f_{ij}\in\mathcal{A}_{-k}$, $i,j\in\mathbb{N}$, and let (2.13) hold for any bounded set $B\subseteq\mathcal{A}_k$.

Since $[\Phi,\varphi]\in S(H)_{-1,-k_0}$, it has the expansion

$$\begin{aligned}[\Phi,\varphi]&=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}([\Phi,\varphi]|H_{\alpha^j}e_i)_{-1,-k_0;H}H_{\alpha^j}e_i\\&=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}([\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}f_{lk}\otimes H_{\alpha^k}e_l,\varphi]|H_{\alpha^j}e_i)_{-1,-k_0;H}H_{\alpha^j}e_i\\&=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\langle f_{lk},\varphi\rangle H_{\alpha^k}e_l|H_{\alpha^j}e_i)_{-1,-k_0;H}H_{\alpha^j}e_i\\&=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\langle f_{ij},\varphi\rangle(2\mathbb{N})^{-k_0\alpha^j}H_{\alpha^j}e_i,\ \varphi\in\mathcal{A}_k,\end{aligned}$$

where the orthogonality of the basis $H_{\alpha^j}e_i$ was used in the last step.

The sequence of partial sums $\Phi_m=\sum_{i=1}^{m}\sum_{j=1}^{m}f_{ij}\otimes H_{\alpha^j}e_i$, $m\in\mathbb{N}$, is a Cauchy sequence in $\mathcal{L}(\mathcal{A}_k,S(H)_{-1,-k_0})$ because, for given $\varepsilon>0$,

$$\|[\Phi_m,\varphi]-[\Phi_n,\varphi]\|^2_{-1,-k_0;H}=\sum_{i=n+1}^{m}\sum_{j=n+1}^{m}|\langle f_{ij},\varphi\rangle|^2(2\mathbb{N})^{-k_0\alpha^j}<\varepsilon,$$

if we choose n,m large enough. This yields that $\langle\Phi_m\rangle_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{A}(H)^*_k$. Also, (2.13) implies that

$$\sup_{m\in\mathbb{N}}\|\Phi_m\|^{*2}_{-k;H}\le\sup\{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|\langle f_{ij},\varphi\rangle|^2(2\mathbb{N})^{-k_0\alpha^j}\,:\,\varphi\in\mathcal{A}_k,\|\varphi\|_k\le1\}<\infty.$$

Thus, due to the Banach-Steinhaus theorem, $\Phi_0=\lim_{m\to\infty}\Phi_m\in\mathcal{A}(H)^*_k$. So it has to be of the form

$$\Phi_0=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\tilde{f}_{ij}\otimes H_{\alpha^j}e_i.$$

It remains to show that $\Phi_0 = \Phi$. Since

$$\begin{aligned}\langle \tilde{f}_{ij}, \varphi\rangle - \langle f_{ij}, \varphi\rangle &= ([\Phi_0, \varphi] | H_{\alpha^j} e_i)_{-1,-k_0;H} - ([\Phi, \varphi] | H_{\alpha^j} e_i)_{-1,-k_0;H} \\ &= ([\lim_{m\to\infty} \Phi_m, \varphi] | H_{\alpha^j} e_i)_{-1,-k_0;H} - ([\Phi, \varphi] | H_{\alpha^j} e_i)_{-1,-k_0;H} \\ &= \lim_{m\to\infty} ([\Phi_m - \Phi, \varphi] | H_{\alpha^j} e_i)_{-1,-k_0;H} \\ &= \lim_{m\to\infty} ([\sum_{i=m+1} \sum_{j=m+1}^{\infty} f_{ij} \otimes H_{\alpha^j} e_i, \varphi] | H_{\alpha^j} e_i)_{-1,-k_0;H} \\ &= \lim_{m\to\infty} \sum_{i=m+1} \sum_{j=m+1}^{\infty} |\langle f_{ij}, \varphi\rangle|^2 (2\mathbb{N})^{-k_0 \alpha^j} = 0\end{aligned}$$

for any $\varphi \in \mathcal{A}_k$, it implies that $\tilde{f}_{ij} = f_{ij}$, $i, j \in \mathbb{N}$. □

Corollary 2.2.1 *If Φ can be represented in the form (2.12) and there exists $k_1 \in \mathbb{N}_0$ such that*

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \|f_{ij}\|_{-k}^2 (2\mathbb{N})^{-k_1 \alpha^j} < \infty, \tag{2.15}$$

then $\Phi \in \mathcal{A}(H)_k^*$.

Proof. According to the Cauchy-Schwartz inequality, it is obvious that (2.15) implies (2.13). □

Remark: Note that in the finite dimensional valued case we had an equivalence between (2.5) and (2.6) in Theorem 2.1.1. But now (2.13) does not necessarily imply (2.15). This implication would be true only if $S(H)_{-1}$ were a nuclear space (see [GV, Theorem 1, p. 67]), which it is not.

Since $\mathcal{A}(H)^*$ is constructed as the inductive limit of the family $\mathcal{A}(H)_k^*$, $k \in \mathbb{N}_0$, we obtain the following expansion theorem for a H–valued GRP (I).

Theorem 2.2.2 $\Phi \in \mathcal{A}(H)^*$ *if and only if there exist k, $k_0 \in \mathbb{N}_0$ such that series expansion (2.12) and condition (2.13) hold.*

Let U be a H–valued GRP (O) given by the expansion

$$U(t, \omega) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}(t) H_{\alpha^j}(\omega) e_i, \qquad t \in \mathbb{R}, \quad \omega \in \mathcal{S}'(\mathbb{R}),$$

such that $a_{ij}(t) \in L^1_{loc}(\mathbb{R})$, $i, j \in \mathbb{N}$. Then there is a H–valued GRP (I), denoted by $\tilde{U}$ associated with U, such that

$$[\tilde{U}, \varphi](\omega) = \int_{\mathbb{R}} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}(t) H_{\alpha^j}(\omega) e_i \varphi(t) dt = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \langle \tilde{a}_{ij}, \varphi\rangle H_{\alpha^j}(\omega) e_i, \ \omega \in \mathcal{S}'(\mathbb{R}),$$

where $\tilde{a}_{ij} \in \mathcal{S}'(\mathbb{R})$ is the generalized function associated with the function $a_{ij}(t) \in L^1_{loc}(\mathbb{R})$, $i, j \in \mathbb{N}$. Thus, $\tilde{U}$ has expansion

$$\tilde{U} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \tilde{a}_{ij} \otimes H_{\alpha^j}\ e_i.$$

The expansion theorems for ${}^{exp}\mathcal{A}(H)^* = \mathcal{L}(\mathcal{A}, \exp S(H)_{-1})$ can also be stated as in the case of a one–dimensional state space:

Theorem 2.2.3 *Let $k \in \mathbb{N}_0$. The following conditions are equivalent:*

(i) $\Phi \in {}^{exp}\mathcal{A}(H)^*_k$.

(ii) Φ *can be represented in the form*

$$\Phi = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f_{ij} \otimes H_{\alpha^j}\ e_i, \qquad f_{ij} \in \mathcal{A}_{-k}, \quad i, j \in \mathbb{N}, \tag{2.16}$$

and there exists $k_0 \in \mathbb{N}_0$, such that for each bounded set $B \subseteq \mathcal{A}_k$

$$\sup_{\varphi \in B} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\langle f_{ij}, \varphi \rangle|^2 e^{-k_0 (2\mathbb{N})^{\alpha^j}} < \infty. \tag{2.17}$$

Corollary 2.2.2 *If Φ can be represented in the form (2.16) and there exists $k_1 \geq 0$, such that*

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \|f_{ij}\|^2_{-k} e^{-k_1 (2\mathbb{N})^{\alpha^j}} < \infty,$$

then $\Phi \in {}^{exp}\mathcal{A}(H)^*_k$.

Theorem 2.2.4 $\Phi \in {}^{exp}\mathcal{A}(H)^*$ *if and only if there exist k, $k_0 \in \mathbb{N}_0$ such that series expansion (2.16) and condition (2.17) hold.*

Example 2.2.1 *Let $\mathcal{R} = -\frac{d^2}{dx^2} + x^2 + 1$ and $I = \mathbb{R}$. Then $\mathcal{A} = \mathcal{S}(\mathbb{R})$, $\mathcal{A}' = \mathcal{S}'(\mathbb{R})$ and $\psi_k(t) = \xi_k(t)$, $k \in \mathbb{N}$, where ξ_k are the Hermite functions.*

(i) In Example 1.7.4, H–valued one-dimensional d-parameter singular white noise as a GRP (O) was defined by the formal sum

$$\mathbf{W}(t, \omega) = \sum_{k=1}^{\infty} \kappa_k(t) H_{\varepsilon_k}(\omega), \quad \kappa_k(t) = \delta_{n(i,j),k}\ \xi_j(t)\ e_i.$$

With the Hermite function ξ_j we associate a generalized function $\tilde{\xi}_j \in \mathcal{S}'(\mathbb{R})$ defined by $\langle \tilde{\xi}_j, \varphi \rangle = \int_{\mathbb{R}} \xi_j(t)\varphi(t)dt$, $\varphi \in \mathcal{S}(\mathbb{R})$. Define $\tilde{\kappa}_k = \delta_{n(i,j),k}\ \tilde{\xi}_j(t)\ e_i$. Then, white noise $\tilde{\mathbf{W}}$ as an H–valued GRP (I) has the expansion

$$\tilde{\mathbf{W}} = \sum_{k=1}^{\infty} \tilde{\kappa}_k \otimes H_{\varepsilon_k} \quad \in \mathcal{L}(\mathcal{S}(\mathbb{R}), S(H)_{-1}).$$

Condition (2.15) is also satisfied, because

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \delta_{n(i,j),k}\ \|\xi_j\|^2_{L^2(\mathbb{R})}(2\mathbb{N})^{-p\varepsilon_j} = \sum_{k=1}^{\infty}(2k)^{-p} < \infty,$$

if we choose $p > 1$.

Note also that $\tilde{\kappa}_k$ is an element of $\mathcal{S}'(\mathbb{R}; H)$.

(ii) Let $l > \frac{5}{12}$ and $t_1, t_2, t_3, \ldots \in \mathbb{R}$ such that $t_1 \le t_2 \le t_3 \le \cdots \to \infty$. It is known that the Dirac delta distributions δ_{t_j}, $j \in \mathbb{N}$, belong to $\mathcal{A}_{-l} = \mathcal{S}_{-l}(\mathbb{R})$. Let

$$\Delta_k = \delta_{n(i,j),k}\ \delta_{t_j}\ e_i, \quad k \in \mathbb{N}$$

(to avoid confusion: the first delta is the Kronecker symbol, the second one is the Dirac distribution). With

$$\sum_{k=1}^{\infty} \Delta_k \otimes H_{\alpha^k} \tag{2.18}$$

is given a GRP (I) which is not a GRP (O). We will check condition (2.15). Since $\delta_{t_j}(x) = \sum_{n=1}^{\infty} \xi_n(t_j)\xi_n(x)$, $j \in \mathbb{N}$, and $\xi_n = \mathcal{O}(n^{-\frac{1}{12}})$, $n \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \delta_{n(i,j),k}\|\delta_{t_j}\|^2_{-l}(2\mathbb{N})^{-p\alpha^j} &= \sum_{k=1}^{\infty} \|\delta_{t_k}\|^2_{-l}(2\mathbb{N})^{-p\alpha^k} \\
&= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} |\xi_n(t_k)|^2(2n)^{-2l}(2\mathbb{N})^{-p\alpha^k} \\
&\le C\sum_{k=1}^{\infty}(2\mathbb{N})^{-p\alpha^k} \sum_{n=1}^{\infty} n^{-\frac{1}{6}}(2n)^{-2l} < \infty,
\end{aligned}$$

for some constant $C > 0$ and $p > 1$. Hence, the process given by (2.18) meets the definition of a GRP (I).

2.2.1 GRPs (I) on nuclear spaces

Recall that since $(S)_{-1}$ is a nuclear space, we have $S(H)_{-1} \cong (S)_{-1} \otimes H$. Assume now, that $\mathcal{A}$ is also a nuclear space (this is not a strict restriction since in most cases it is one); then we have by Proposition 50.7. in [Tr] $\mathcal{L}(\mathcal{A}, S(H)_{-1}) \cong \mathcal{A}' \otimes S(H)_{-1}$. Combining this with the previous remark, we can now consider GRPs (I) as elements of $\mathcal{A}' \otimes (S)_{-1} \otimes H$, or if we regroup the spaces, also as elements of $\mathcal{A}' \otimes H \otimes (S)_{-1}$, which is again by nuclearity of $\mathcal{A}$ isomorphic to $\mathcal{A}'(I;H) \otimes (S)_{-1}$. In other words, it is equivalent whether we consider the state space H as the codomain of the generalized random variables, or as the codomain of the deterministic generalized functions representing the trajectories of the process.

Similarly as we did for GRPs (I) in Theorem 2.2.1, we have a representation for elements of $\mathcal{A}'(I;H)$. A function g belongs to $\mathcal{A}_{-k}(I;H)$ if and only if it is of the form $\sum_{i=1}^{\infty} g_i \otimes e_i$, $g_i \in \mathcal{A}_{-k}$ and $sup_{\varphi\in B} \sum_{i=1}^{\infty} |\langle g_i, \varphi\rangle|^2 < \infty$ holds for each bounded set $B \subseteq \mathcal{A}_k$. The sum $\sum_{i=1}^{\infty} g_i \otimes e_i$ is defined by the action $\langle \sum_{i=1}^{\infty} g_i \otimes e_i, \varphi\rangle = \sum_{i=1}^{\infty} \langle g_i, \varphi\rangle e_i$, $\varphi \in \mathcal{A}$, provided the latter sum converges in H.

Thus, if Φ is a GRP (I) given by the expansion $\Phi = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f_{ij} \otimes H_{\alpha^j} e_i$, $f_{ij} \in \mathcal{A}_{-k}$, we can rewrite its action in the following manner:

$$\left[\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f_{ij} \otimes H_{\alpha^j} e_i, \varphi\right] = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \langle f_{ij}, \varphi\rangle H_{\alpha^j} e_i = \sum_{j=1}^{\infty}\langle \sum_{i=1}^{\infty} f_{ij} \otimes e_i, \varphi\rangle H_{\alpha^j}.$$

Also, from $\sup_{\varphi\in B} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |\langle f_{ij}, \varphi\rangle|^2 (2\mathbb{N})^{-p\alpha^j} < \infty$, $B \subseteq \mathcal{A}_k$, we get $\sup_{\varphi\in B} \sum_{j=1}^{\infty} \|\langle g_j, \varphi\rangle\|_H^2 (2\mathbb{N})^{-p\alpha^j} < \infty$, where $g_j = \sum_{i=1}^{\infty} f_{ij} \otimes e_i \in \mathcal{A}_{-k}(I;H)$ and $\|g_j\|_{-k;H}^2 = \sum_{i=1}^{\infty} |f_{ij}|^2 \tilde{\lambda}_i^{-k}$.

In view of these facts we can now reformulate our representation theorem for GRPs (I):

Theorem 2.2.5 *Let $k \in \mathbb{N}_0$. The following conditions are equivalent:*

(i) $\Phi \in \mathcal{A}(H)_k^*$.

(ii) Φ *can be represented in the form*

$$\Phi = \sum_{j=1}^{\infty} f_j \otimes H_{\alpha^j}, \qquad f_j \in \mathcal{A}_{-k}(I;H), \quad j \in \mathbb{N} \tag{2.19}$$

and there exists $k_0 \in \mathbb{N}_0$ such that for each bounded set $B \subseteq \mathcal{A}_k$

$$\sup_{\varphi\in B} \sum_{j=1}^{\infty} \|\langle f_j, \varphi\rangle\|_H^2 \, (2\mathbb{N})^{-k_0\alpha^j} < \infty. \tag{2.20}$$

Corollary 2.2.3 *If Φ can be represented in the form (2.19) and there exists $k_1 \in \mathbb{N}_0$ such that*

$$\sum_{j=1}^{\infty} \|f_j\|_{-k;H}^2 (2\mathbb{N})^{-k_1\alpha^j} < \infty, \tag{2.21}$$

then $\Phi \in \mathcal{A}(H)_k^*$.

Example 2.2.2 *Denote by $\delta_y \in \mathcal{S}'(\mathbb{R};H)$ be the H–valued Dirac delta distribution at $y \in \mathbb{R}$. Let $t_j \in \mathbb{R}$, $j \in \mathbb{N}$, such that $t_1 \le t_2 \le t_3 \le \cdots \to \infty$. Then,*

$$\sum_{j=1}^{\infty} \delta_{t_j} \otimes H_{\alpha^j}$$

defines a H–valued GRP (I). Indeed, since $\{\xi_n e_i : n, i \in \mathbb{N}\}$ is an orthogonal basis of $\mathcal{S}'(\mathbb{R};H)$, we can write $\delta_y = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} d_{in}(y)\xi_n e_i$, $\|\delta_y\|_{-k;H}^2 = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} |d_{in}(y)|^2(2n)^{-k} < \infty$, and moreover $\|\delta_y\|_{-k;H}^2$ does not depend on y. Thus, $\sum_{j=1}^{\infty}\|\delta_{t_j}\|_{-k;H}^2(2\mathbb{N})^{-p\alpha^j} = Const\sum_{j=1}^{\infty}(2\mathbb{N})^{-p\alpha^j} < \infty$ for $p > 1$, proving that (2.21) holds.

2.2.2 Differentiation of GRPs (I)

Definition 2.2.2 *Let $F \in \mathcal{A}(H)^*$. The distributional derivative of F, denoted by $\frac{\partial}{\partial t}F$ is defined by $\left[\frac{\partial}{\partial t}F, \varphi\right] = -\left[F, \frac{\partial}{\partial t}\varphi\right]$, for all $\varphi \in \mathcal{A}$.*

Lemma 2.2.2 *Let $F = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f_{ij} \otimes H_{\alpha^j} e_i \in \mathcal{A}(H)^*$. Then,*

$$\frac{\partial}{\partial t}F = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{\partial}{\partial t} f_{ij} \otimes H_{\alpha^j} e_i, \tag{2.22}$$

where $\frac{\partial}{\partial t} f_{ij}$ is the distributional derivative of f_{ij} in $\mathcal{A}'$.

Proof. The assertion follows from the fact that

$$\begin{aligned}\left[\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{\partial}{\partial t} f_{ij} \otimes H_{\alpha^j} e_i, \varphi\right] &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \langle \frac{\partial}{\partial t} f_{ij}, \varphi\rangle H_{\alpha^j} e_i \\ &= -\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \langle f_{ij}, \frac{\partial}{\partial t}\varphi\rangle H_{\alpha^j} e_i = -\left[F, \frac{\partial}{\partial t}\varphi\right]\end{aligned}$$

for all $\varphi \in \mathcal{A}$. Obviously, condition (2.13) is satisfied. □

It is a well-known fact in the (deterministic) generalized functions theory that if a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R} \setminus \{x_0\}$ and has a jump in $x_0 \in \mathbb{R}$, then $Df = f'(x) + C\delta_{x_0}$, where Df is the distributional derivative in $\mathcal{S}'(\mathbb{R})$, $f'(x)$ is the classical derivative on $\mathbb{R} \setminus \{x_0\}$, $\delta_{x_0} \in \mathcal{S}'(\mathbb{R})$ is the Dirac delta distribution, and $C = f(x_0^+) - f(x_0^-)$. In this light, the GRP defined in Example 2.2.2 is a stochastic analogue of the Dirac delta distribution.

Example 2.2.3 *Let $F = \sum_{j=1}^{\infty} f_j H_{\alpha^j}$, $f_j : \mathbb{R} \to \mathbb{R}$ be a GRP (O), i.e. for each $t \in \mathbb{R}$ fixed $\sum_{j=1}^{\infty} |f_j(t)|^2 (2\mathbb{N})^{-p\alpha^j} < \infty$ for some $p > 0$. Denote by $\delta_y \in \mathcal{S}'(\mathbb{R})$ be the Dirac delta distribution at $y \in \mathbb{R}$. Assume that for each $j \in \mathbb{N}$, the function f_j is differentiable on $\mathbb{R} \setminus \{t_j\}$, has one jump in $t_j \in \mathbb{R}$, and $t_1 \leq t_2 \leq t_3 \leq \cdots \to \infty$. Assume that $\sum_{j=1}^{\infty} |f_j'(t)|^2 (2\mathbb{N})^{-p\alpha^j} < \infty$ for each fixed $t \in \mathbb{R} \setminus \{t_1, t_2, \ldots\}$. Let $c_j = f(t_j^+) - f(t_j^-)$, $j \in \mathbb{N}$, be the jump heights. Assume there exists $C > 0$ such that $|c_j| \leq C$, $j \in \mathbb{N}$ (i.e. the jump heights are bounded). Then,*

$$\frac{\partial}{\partial t} F = \sum_{j=1}^{\infty} \frac{\partial}{\partial t} f_j \otimes H_{\alpha^j} = \sum_{j=1}^{\infty} \left(f_j' + c_j \delta_{t_j} \right) \otimes H_{\alpha^j}.$$

Since $\sum_{j=1}^{\infty} c_j^2 \|\delta_{t_j}\|_{-k} (2j)^{-p} \leq C^2 \sum_{j=1}^{\infty} \|\delta_{t_j}\|_{-k} (2j)^{-p} < \infty$, the process above is well-defined. Note $\sum_{j=1}^{\infty} f_j'(t) H_{\alpha^j}$ is a GRP (O) and $\sum_{j=1}^{\infty} \delta_{t_j} \otimes H_{\alpha^j}$ is the GRP (I) defined in Example 2.2.2.

2.2.3 Application to a class of linear SDEs

Let $\mathcal{R}$ be of the form (1.1) and $P(t) = p_n t^n + p_{n-1} t^{n-1} + \cdots + p_1 t + p_0$, $t \in I$, be a polynomial with real coefficients. We give two examples of stochastic differential equations using GRPs (I), and a differential operator $P(\mathcal{R})$ defined as $P(\mathcal{R}) = p_n \mathcal{R}^n + p_{n-1} \mathcal{R}^{n-1} + \cdots + p_1 \mathcal{R} + p_0 I$. Note that if ψ_k is an eigenfunction of $\mathcal{R}$, then

$$\begin{aligned} P(\mathcal{R})\psi_k &= (p_k \mathcal{R}^n + p_{n-1} \mathcal{R}^{n-1} + \cdots + p_1 \mathcal{R} + p_0 I)\psi_k \\ &= p_n (\widetilde{\lambda}_k)^n \psi_k + p_{n-1} (\widetilde{\lambda}_k)^{n-1} \psi_k + \cdots + p_1 \widetilde{\lambda}_k \psi_k + p_0 \psi_k = P(\widetilde{\lambda}_k)\psi_k. \end{aligned}$$

Consider an SDE of the form

$$P(\mathcal{R})u = g \tag{2.23}$$

where $g \in \mathcal{A}(H)_r^*$ is a GRP (I).

Let u and g be given by series expansions

$$u(t,\omega) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} u_{ij}(t) \otimes H_{\alpha^j}(\omega) e_i, \quad g(t,\omega) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} g_{ij}(t) \otimes H_{\alpha^j}(\omega) e_i,$$

$t \in I, \omega \in \mathcal{S}'(\mathbb{R})$, respectively, where u_{ij}, $g_{ij} \in \mathcal{A}'$, $i, j \in \mathbb{N}$.

Let $u_{ij} = \sum_{k=1}^{\infty} a_{ij}^k \psi_k$, $g_{ij} = \sum_{k=1}^{\infty} b_{ij}^k \psi_k$, $i, j \in \mathbb{N}$. Then we have

$$
\begin{aligned}
P(\mathcal{R})u &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P(\mathcal{R})u_{ij} \otimes H_{\alpha^j} e_i \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(P(\mathcal{R}) \sum_{k=1}^{\infty} a_{ij}^k \psi_k \right) \otimes H_{\alpha^j} e_i = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{ij}^k P(\widetilde{\lambda}_k) \psi_k \otimes H_{\alpha^j} e_i.
\end{aligned} \tag{2.24}
$$

In order to solve (2.23) we may use the method of undetermined coefficients. From (2.23) and (2.24) we have

$$
\sum_{k=1}^{\infty} a_{ij}^k P(\widetilde{\lambda}_k) \psi_k = g_{ij}, \quad i, j \in \mathbb{N}.
$$

Finally, we obtain the system

$$
a_{ij}^k P(\widetilde{\lambda}_k) = b_{ij}^k, \quad i, j, k \in \mathbb{N}.
$$

First case: If $P(\widetilde{\lambda}_k) \neq 0$ for all $k \in \mathbb{N}$, then $a_{ij}^k = \frac{b_{ij}^k}{P(\widetilde{\lambda}_k)}$, $i, j, k \in \mathbb{N}$, and the solution of the equation is given by

$$
u = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{b_{ij}^k}{P(\widetilde{\lambda}_k)} \psi_k \right) \otimes H_{\alpha^j} e_i.
$$

In this case the solution exists and it is unique. Note, there exists a constant $C > 0$ such that $P(\widetilde{\lambda}_k) \geq C$, for all $k \in \mathbb{N}$. Thus,

$$
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\| \sum_{k=1}^{\infty} \frac{b_{ij}^k}{P(\widetilde{\lambda}_k)} \psi_k \right\|_{-r}^2 (2\mathbb{N})^{-p\alpha^j} \leq \frac{1}{C^2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\| \sum_{k=1}^{\infty} b_{ij}^k \psi_k \right\|_{-r}^2 (2\mathbb{N})^{-p\alpha^j} < \infty,
$$

for some $p > 1$, because $g \in \mathcal{A}(H)_r^*$. Thus, $u \in \mathcal{A}(H)_r^*$.

Second case: Let $P(\widetilde{\lambda}_k) = 0$ for $k = k_1, k_2, \ldots k_m$. Then a solution exists if and only if $b_{ij}^{k_1} = b_{ij}^{k_2} = \cdots = b_{ij}^{k_m} = 0$, $i, j \in \mathbb{N}$. The solution in this case is not unique, and the coefficients of the solution u are given by

$$
u_{ij} = \sum_{\substack{k=1 \\ P(\widetilde{\lambda}_k) \neq 0}}^{\infty} \frac{b_{ij}^k}{P(\widetilde{\lambda}_k)} \psi_k + \sum_{s=1}^{m} c_{ij}^s \psi_{k_s}, \qquad i, j \in \mathbb{N},
$$

where c_{ij}^s, $s = 1, 2, \ldots m$, $i, j \in \mathbb{N}$, are arbitrary real numbers.

Remark. In case $P(\widetilde{\lambda}_k) \neq 0$, $k \in \mathbb{N}$, the solution is unique in sense that in Section 1.3.1 we fixed $\widetilde{\lambda}_k = 1$ for all $k \in \mathbb{N}$ such that $\lambda_k = 0$. But we could have chosen also some other convention, thus in this sense the solution actually depends on the eigenvalues of the operator $\mathcal{R}$.

2.3 The Wick Product of Generalized Random Processes

It is a well-known problem that in general one can not define a pointwise multiplication of generalized functions; thus, it is not clear how to deal with nonlinearities. In framework of white noise analysis this difficulty is solved by introducing the Wick product. We will consider the Wick product for GRPs of type (I) and (O). Also, we carry over the Wick type product also to the spaces of deterministic generalized functions. In Example 2.3.1 we will show that in a special case (when the orthonoramal basis are trigonometric functions) the Wick product coincides with the ordinary product. For GRPs (II) the Wick product does not make sense, since GRPs (II) are generalized in the t parameter and not in the ω parameter.

Wick products in spaces of generalized random variables

First we recall the definition and some basic properties of the Wick product in the Kondratiev spaces (see [HØUZ]).

Definition 2.3.1 *Let $F, G \in (S)_{-1}$ be given by their chaos expansion $F(\omega) = \sum_{\alpha\in\mathcal{I}} f_\alpha H_\alpha(\omega)$, $G(\omega) = \sum_{\beta\in\mathcal{I}} g_\beta H_\beta(\omega)$, $f_\alpha, g_\beta \in \mathbb{R}$. The Wick product of F and G is the unique element in $(S)_{-1}$ defined by:*

$$F\lozenge G(\omega) = \sum_{\gamma\in\mathcal{I}} \left(\sum_{\alpha+\beta=\gamma} f_\alpha g_\beta \right) H_\gamma(\omega).$$

The Wick product is a commutative, associative operation, distributive with respect to addition. By the same formula we defined in [PS1] the Wick product for $F, G \in \exp(S)_{-1}$. It is known that the spaces $(S)_1$, $(S)_{-1}$, $\exp(S)_1$ and $\exp(S)_{-1}$ are closed under Wick multiplication.

In the Hilbert space valued case, the Wick product is defined analogously (see [MFA]); with a slight abuse of notation, we denote with the same symbol the Wick products in $(S)_{-1}$ and $S(H)_{-1}$.

Definition 2.3.2 *Let $F, G \in S(H)_{-1}$ be given by their chaos expansion $F(\omega) = \sum_{i=1}^{\infty} \sum_{\alpha\in\mathcal{I}} f_{i,\alpha} H_\alpha(\omega) e_i$, $G(\omega) = \sum_{i=1}^{\infty} \sum_{\beta\in\mathcal{I}} g_{i,\beta} H_\beta(\omega) e_i$, $f_{i,\alpha}, g_{i,\beta} \in \mathbb{R}$. The Wick product of F and G is the unique element in $S(H)_{-1}$ defined* *by:*

$$
\begin{aligned}
F\lozenge G(\omega) &= \sum_{i=1}^{\infty}\sum_{\gamma\in\mathcal{J}}\left(\sum_{\alpha+\beta=\gamma} f_{i,\alpha}g_{i,\beta}\right)H_\gamma(\omega)e_i\\
&= \sum_{i=1}^{\infty}(F_i\lozenge G_i(\omega))e_i,
\end{aligned}
$$

where $F_i(\omega)=\sum_{\alpha\in\mathcal{J}} f_{i,\alpha}H_\alpha(\omega)$, $G_i(\omega)=\sum_{\beta\in\mathcal{J}} g_{i,\beta}H_\beta(\omega)\in(S)_{-1}$.

This definition is legal, since $S(H)_1$ and $S(H)_{-1}$ are closed under Wick multiplication. In the same manner we can define $F\lozenge G$ for $F,G\in\exp S(H)_{-1}$ and it is an easy exercise to show (combining the methods in [MFA] and [PS1]) that $F\lozenge G\in\exp S(H)_{-1}$. Also, if $F,G\in\exp S(H)_1$ then $F\lozenge G\in\exp S(H)_1$.

Note that one can also Wick–multiply a $\mathbb{R}$–valued and a H–valued generalized random variable: If $F\in(S)_{-1}$ and $G\in S(H)_{-1}$ have the forms $F(\omega)=\sum_{\alpha\in\mathcal{J}} f_\alpha H_\alpha(\omega)$, $G(\omega)=\sum_{i=1}^{\infty}\sum_{\beta\in\mathcal{J}} g_{i,\beta}H_\beta(\omega)$, then $F\lozenge G\in S(H)_{-1}$ has expansion $\sum_{i=1}^{\infty}\sum_{\gamma\in\mathcal{J}}\left(\sum_{\alpha+\beta=\gamma} f_\alpha g_{i,\beta}\right)H_\gamma(\omega)e_i$.

The Wick product of GRPs (O) is defined pointwisely (as in [HØUZ]) by the formula

$$
F\lozenge G(\omega)=\sum_{\gamma\in\mathcal{J}}\left(\sum_{\alpha+\beta=\gamma} f_\alpha(t)g_\beta(t)\right)H_\gamma(\omega),
$$

for $F(t,\omega)=\sum_{\alpha\in\mathcal{J}} f_\alpha(t)H_\alpha(\omega)$, $G(t,\omega)=\sum_{\beta\in\mathcal{J}} g_\beta(t)H_\beta(\omega)$.

The Wick product in Zemanian spaces

Now we recall the definition of a deterministic multiplication of Wick type in $\mathcal{A}'$, which was introduced in [Se] and [PS1]. From now on, when we use Wick products, we will always assume that $\mathcal{A}$ is nuclear, i.e. there exists some $p\geq 0$, such that

$$
M:=\sum_{n=1}^{\infty}\widetilde{\lambda_n}^{-2p}<\infty.
$$

Definition 2.3.3 *Let $f,g\in\mathcal{A}'$ be generalized functions given by expansions $f=\sum_{k=1}^{\infty} a_k\psi_k$, $g=\sum_{k=1}^{\infty} b_k\psi_k$. Define $f\diamond g$ to be the generalized function from $\mathcal{A}'$ given by*

$$
f\diamond g=\sum_{n=1}^{\infty}\left(\sum_{\substack{i,j\in\mathbb{N}\\ i+j=n+1}} a_ib_j\right)\psi_n. \tag{2.25}
$$

Proposition 2.3.1 *If $f = \sum_{i=1}^{\infty} a_i\psi_i \in \mathcal{A}_{-k}$ and $g = \sum_{i=1}^{\infty} b_i\psi_i \in \mathcal{A}_{-l}$, then $f \diamond g \in \mathcal{A}_{-(k+l+p)}$. Moreover,*

$$|\langle f \diamond g, \varphi\rangle|^2 \leq M\|f\|_{-k}^2\|g\|_{-l}^2\|\varphi\|_{k+l+p}, \tag{2.26}$$

for each test function $\varphi \in \mathcal{A}$.

Similarly, one can also define the multiplication of test-functions, under an additional assumption:

Lemma 2.3.1 *Let there exist a constant $C > 0$, such that*

$$\widetilde{\lambda_{i+j}} \leq C\widetilde{\lambda_i}\widetilde{\lambda_j}, \qquad i, j \in \mathbb{N}.$$

If $f = \sum_{i=1}^{\infty} a_i\psi_i \in \mathcal{A}_k$ and $g = \sum_{i=1}^{\infty} b_i\psi_i \in \mathcal{A}_k$, then $f \diamond g \in \mathcal{A}_{(k-p)}$.

For example, we have $\psi_i \diamond \psi_j = \psi_{i+j-1}$ for arbitrary $i, j \in \mathbb{N}$.

Example 2.3.1 *Let $I = (-\pi, \pi)$, $\mathcal{R} = -iD$. Then, the orthonormal basis of $\mathcal{A}$ is given by the family of trigonometric functions $\psi_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}$, and $\lambda_n = n$, $n = 0, \pm1, \pm2, \pm3, \ldots$ etc. In this case, we leave the enumeration running through the set of integer numbers $\mathbb{Z}$. Instead doing a reordering in the space $\mathcal{A}$, we make a slight modification in the definition of the Wick product and put instead of (2.25) $f \diamond g = \sum_{n=1}^{\infty} \left(\sum_{\substack{i,j\in\mathbb{Z}\\ i+j=n}} a_i b_j\right)\psi_n$. Then,*

$$\psi_i \diamond \psi_j = \psi_{i+j}, \quad \psi_i \cdot \psi_j = \frac{1}{\sqrt{2\pi}}\psi_{i+j} \quad i, j \in \mathbb{Z},$$

i.e. the Wick product and the ordinary product coincide up to a constant.

The Wick product of GRPs (I)

The notion of the Wick product to GRPs (I) was also extended in [PS1]. We summarize the basic results here.

Definition 2.3.4 *Let $\Phi \in \mathcal{A}_k^*$, $\Psi \in \mathcal{A}_l^*$ be two GRPs (I) given by expansions $\Phi = \sum_{i=1}^{\infty} f_i \otimes H_{\alpha^i}$, $f_i \in \mathcal{A}_{-k}$, $i \in \mathbb{N}$, and $\Psi = \sum_{j=1}^{\infty} g_j \otimes H_{\alpha^j}$, $g_j \in \mathcal{A}_{-l}$, $j \in \mathbb{N}$. The Wick product of Φ and Ψ is defined to be*

$$\Phi\blacklozenge\Psi = \sum_{n=1}^{\infty}\left(\sum_{\substack{i,j\in\mathbb{N}\\ \alpha^i+\alpha^j=\alpha^{n+1}}} f_i \diamond g_j\right) \otimes H_{\alpha^n}. \tag{2.27}$$

Theorem 2.3.1 *The Wick product $\Phi \blacklozenge \Psi$ from the previous definition is a GRP (I), precisely $\Phi \blacklozenge \Psi \in \mathcal{A}^*_{k+l+p}$.*

The Wick product for GRPs (I) taking values in $\exp(S)_{-1}$ can be defined in an analogue way.

Theorem 2.3.2 *The Wick product $\Phi \blacklozenge \Psi$ defined by formula (2.27) of $\Phi \in {}^{exp}\mathcal{A}^*_k$ and $\Psi \in {}^{exp}\mathcal{A}^*_l$ is a GRP (I), precisely $\Phi \blacklozenge \Psi \in {}^{exp}\mathcal{A}^*_{k+l+p}$.*

The Wick product $\diamond$ in $\mathcal{A}'$ can be embedded into the Wick product $\blacklozenge$ in $\mathcal{A}^*$, since each deterministic function can be (trivially) regarded as a stochastic process. Also, the Wick product $\blacklozenge$ acting on $\mathcal{A}^*$ is an extension of the Wick product $\Diamond$ acting on $(S)_{-1}$.

2.3.1 The Wick product of H–valued GRPs (I)

Now, the main difference is that we are not able to define the Wick product $\blacklozenge$ for the whole class of H–valued GRPs (I). This is due to the fact that $S(H)_{-1}$ is not a nuclear space. Therefore, the Wick product will be defined for the class of H–valued GRPs (I) satisfying condition (2.15).

Definition 2.3.5 *Let $\Phi \in \mathcal{A}(H)^*_k$, $\Psi \in \mathcal{A}(H)^*_l$ be two H–valued GRPs (I) given by expansions $\Phi = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f_{i,j} \otimes H_{\alpha^j} e_i$, $f_{i,j} \in \mathcal{A}_{-k}$, $i,j \in \mathbb{N}$, and $\Psi = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} g_{i,j} \otimes H_{\alpha^j}$, $g_{i,j} \in \mathcal{A}_{-l}$, $i,j \in \mathbb{N}$, and let $r_1 \geq 0$, $r_2 \geq 0$ be such that $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \|f_{i,j}\|^2_{-k}(2\mathbb{N})^{-r_1\alpha^j} < \infty$ and $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \|g_{i,j}\|^2_{-l}(2\mathbb{N})^{-r_2\alpha^j} < \infty$. The Wick product of Φ and Ψ is defined to be*

$$\Phi \blacklozenge \Psi = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\left(\sum_{\substack{s,r\in\mathbb{N}\\ \alpha^s+\alpha^r=\alpha^{n+1}}} f_{i,s} \diamond g_{i,r}\right) \otimes H_{\alpha^n}\ e_i. \tag{2.28}$$

We may write (2.28) also as

$$\Phi \blacklozenge \Psi = \sum_{i=1}^{\infty}(F_i \blacklozenge G_i) e_i,$$

where $F_i = \sum_{j=1}^{\infty} f_{i,j} \otimes H_{\alpha^j} \in \mathcal{A}^*_k$, $G_i = \sum_{j=1}^{\infty} g_{i,j} \otimes H_{\alpha^j} \in \mathcal{A}^*_l$ and $F_i \blacklozenge G_i$ is defined as in Definition 2.3.4.

Theorem 2.3.3 *The Wick product $\Phi \blacklozenge \Psi$ from the previous definition is an H–valued GRP (I), precisely $\Phi \blacklozenge \Psi \in \mathcal{A}(H)^*_{k+l+p}$.*

Proof. Due to Lemma 2.3.1 it follows that $f_{i,s} \diamond g_{i,r} \in \mathcal{A}_{-k-l-p}$ for all $i, s, r \in \mathbb{N}$, and thus, $\sum_{\alpha^s+\alpha^r=\alpha^{n+1}} f_{i,s} \diamond g_{i,r} \in \mathcal{A}_{-k-l-p}$, $n \in \mathbb{N}$. Since $[\Phi\blacklozenge\Psi, \varphi] = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\alpha^s+\alpha^r=\alpha^{n+1}} \langle f_{i,s} \diamond g_{i,r}, \varphi\rangle H_{\alpha^n} e_i$, it remains to prove

$$\sup_{\varphi\in B}\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\left|\sum_{\alpha^s+\alpha^r=\alpha^{n+1}} \langle f_{i,s} \diamond g_{i,r}, \varphi\rangle\right|^2 (2\mathbb{N})^{-r_3\alpha^n} < \infty,$$

for any bounded set $B \subseteq \mathcal{A}$ and some $r_3 \geq 0$. Put $r_3 = r_1 + r_2 + q$, where $q > 1$. Then, according to (2.26)

$$\sup_{\varphi\in B}\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\left|\sum_{\alpha^s+\alpha^r=\alpha^{n+1}} \langle f_{i,s} \diamond g_{i,r}, \varphi\rangle\right|^2 (2\mathbb{N})^{-r_3\alpha^n}$$

$$\leq \sup_{\varphi\in B}\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\left|\sum_{\alpha^s+\alpha^r=\alpha^{n+1}} \|f_{i,s}\|_{-k}\|g_{i,r}\|_{-l}\|\varphi\|_{k+l+p}\sqrt{M}\right|^2 (2\mathbb{N})^{-r_3\alpha^n} \leq$$

$$M\sum_{n=1}^{\infty}(2\mathbb{N})^{-q\alpha^n}\sum_{i=1}^{\infty}\left|\sum_{\alpha^s+\alpha^r=\alpha^{n+1}} \|f_{i,s}\|_{-k}(2\mathbb{N})^{-\frac{r_1\alpha^s}{2}}\|g_{i,r}\|_{-l}(2\mathbb{N})^{-\frac{r_2\alpha^r}{2}}\sup_{\varphi\in B}\|\varphi\|_{k+l+p}\right|^2$$

$$\leq M\sum_{n=1}^{\infty}(2\mathbb{N})^{-q\alpha^n}\sum_{i=1}^{\infty}\left(\sum_{s=1}^{\infty}\|f_{i,s}\|^2_{-k}(2\mathbb{N})^{-r_1\alpha^s}\sum_{r=1}^{\infty}\|g_{i,r}\|^2_{-l}(2\mathbb{N})^{-r_2\alpha^r}\sup_{\varphi\in B}\|\varphi\|^2_{k+l+p}\right)$$

$$= M\sum_{n=1}^{\infty}(2\mathbb{N})^{-q\alpha^n}\sum_{i=1}^{\infty}\sum_{s=1}^{\infty}\|f_{i,s}\|^2_{-k}(2\mathbb{N})^{-r_1\alpha^s}\sum_{i=1}^{\infty}\sum_{r=1}^{\infty}\|g_{i,r}\|^2_{-l}(2\mathbb{N})^{-r_2\alpha^r}\sup_{\varphi\in B}\|\varphi\|^2_{k+l+p}$$

$< \infty$, where we used the property $(2\mathbb{N})^{\alpha^s+\alpha^r} = (2\mathbb{N})^{\alpha^s}(2\mathbb{N})^{\alpha^r}$. □

Note that the Wick product $\Diamond$ acting on $S(H)_{-1}$ can be embedded into $\blacklozenge$ acting on $\mathcal{A}(H)^*$.

Also, following theorem holds, similarly as in finite dimensional case.

Theorem 2.3.4 *The Wick product $\Phi\blacklozenge\Psi$ defined by formula (2.28) of $\Phi \in {}^{exp}\mathcal{A}(H)^*_k$ and $\Psi \in {}^{exp}\mathcal{A}(H)^*_l$ is a H–valued GRP (I), precisely $\Phi\blacklozenge\Psi \in {}^{exp}\mathcal{A}(H)^*_{k+l+p}$, provided there exist $r_1 \geq 0$, $r_2 \geq 0$ such that*
$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\|f_{i,j}\|^2_{-k}e^{-r_1(2\mathbb{N})^{\alpha^j}} < \infty$ *and* $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\|g_{i,j}\|^2_{-l}e^{-r_2(2\mathbb{N})^{\alpha^j}} < \infty$.

2.3.2 A Class of nonlinear SDEs

Now we will consider a class of a nonlinear SDEs with Wick products involving H–valued GRPs (I). Here we illustrate a simple method of solving by means of series expansions.

Let $P(\mathcal{R})$ be the differential operator defined as in (2.23). Assume that $P(\widetilde{\lambda}_k) - 1 \neq 0$ and $P(\widetilde{\lambda}_k) \neq 0$ for all $k \in \mathbb{N}$. Consider a nonlinear SDE of the form

$$P(\mathcal{R})X = X \blacklozenge \tilde{W} + g, \tag{2.29}$$

where $\tilde{W}$ is singular white noise in $\mathcal{A}(H)_0^*$ given by the expansion

$$\tilde{W}(t,\omega) = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} w_{i,n}(t) \otimes H_{\varepsilon_n}(\omega) e_i, \qquad t \in I,\, \omega \in \Omega,$$

where

$$w_{i,n} = \begin{cases} \psi_j(t), & n = m(i,j) \\ 0, & \text{else} \end{cases}$$

and $m(i,j)$ is defined as in Example 1.7.3.

Let $g \in \mathcal{A}(H)_r^*$ be of the form

$$g(t,\omega) = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} g_{i,n}(t) \otimes H_{\varepsilon_n}(\omega) e_i, \qquad t \in I,\, \omega \in \Omega,$$

such that $g_{i,n} \in \mathcal{A}_{-r}$, $i, n \in \mathbb{N}$; and assume there exist $q \geq 0$, such that

$$\sum_{i=1}^{\infty}\sum_{n=1}^{\infty} \|g_{i,n}\|_{-r}^2 (2\mathbb{N})^{-q\varepsilon_n} = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} \|g_{i,n}\|_{-r}^2 (2n)^{-q} < \infty. \tag{2.30}$$

Expanding each $g_{i,n}$ in $\mathcal{A}_{-r}$ we get

$$g_{i,n}(t) = \sum_{k=1}^{\infty} g_{i,n,k}\psi_k(t), \qquad g_{i,n,k} \in \mathbb{R},\, t \in I,\, i,n \in \mathbb{N},$$

which yields

$$\sum_{k=1}^{\infty} |g_{i,n,k}|^2 \widetilde{\lambda}_k^{-2r} < \infty. \tag{2.31}$$

We will look for the solution X of the equation in the form

$$X(t,\omega) = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} a_{i,n}(t) \otimes H_{\varepsilon_n}(\omega) e_i, \qquad t \in I,\, \omega \in \Omega,$$

where $a_{i,n} \in \mathcal{A}'$, $i, n \in \mathbb{N}$, are the coefficients to be determined. Let $a_{i,n}(t)$, $t \in I$, be given by expansion

$$a_{i,n}(t) = \sum_{k=1}^{\infty} a_{i,n,k}\psi_k(t), \qquad a_{i,n,k} \in \mathbb{R},\ t \in I,\ n \in \mathbb{N}.$$

Then,

$$P(\mathcal{R})X(t,\omega) = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{i,n,k}P(\widetilde{\lambda_k})\psi_k(t)\right) \otimes H_{\varepsilon_n}(\omega)e_i, \quad t \in I,\ \omega \in \Omega, \tag{2.32}$$

and due to the definition of Wick product:

$$X(t,\omega)\blacklozenge\tilde{W}(t,\omega) + g(t,\omega) = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\left(\sum_{r+s=n+1} a_{i,s}(t) \diamond w_{i,r}(t) + g_n(t)\right) \otimes H_{\varepsilon_n}(\omega)e_i, \qquad t \in I,\ \omega \in \Omega. \tag{2.33}$$

Since

$$\begin{aligned} a_{i,s}(t) \diamond w_{i,r}(t) &= \sum_{k=1}^{\infty} a_{i,s,k}\psi_k(t) \diamond w_{i,r}(t) = \sum_{k=1}^{\infty} a_{i,s,k}\psi_k(t) \diamond \psi_j(t) \\ &= \sum_{k=1}^{\infty} a_{i,s,k}\psi_{k+j-1}(t), \end{aligned}$$

for $j \in \mathbb{N}$ such that $r = m(i,j)$, relation (2.33) becomes

$$X(t,\omega)\blacklozenge\tilde{W}(t,\omega) + g(t,\omega) = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\left(\sum_{r+s=n+1}\sum_{k=1}^{\infty} a_{i,s,k}\psi_{k+j-1}(t) + g_n(t)\right) \otimes H_{\varepsilon_n}(\omega)e_i, \quad t \in I,\ \omega \in \Omega. \tag{2.34}$$

From (2.32) and (2.34) we obtain

$$\sum_{\substack{s,r\in\mathbb{N} \\ s+r=n+1,r=m(i,j)}} \sum_{k=1}^{\infty} a_{i,s,k}\psi_{k+j-1}(t) + \sum_{k=1}^{\infty} g_{i,n,k}\psi_k(t) = \sum_{k=1}^{\infty} a_{i,n,k}P(\widetilde{\lambda_k})\psi_k(t),\ i, n \in \mathbb{N}. \tag{2.35}$$

From the system of equations (2.35) one can recursively determine the coefficients $a_{i,n,k}$, $i,n,k \in \mathbb{N}$.

For $i = 1$, $n = 1$ we have only one possibility how to get $r + s = 2$ $(s = 1, r = 1)$ and for $r = 1$ the corresponding j is $j = 1$. Thus, (2.35) gives

$$\sum_{k=1}^{\infty} a_{1,1,k}\psi_k(t) + \sum_{k=1}^{\infty} g_{1,1,k}\psi_k(t) = \sum_{k=1}^{\infty} a_{1,1,k}P(\widetilde{\lambda}_k)\psi_k(t), \quad t \in I,$$

which implies

$$a_{1,1,k} = \frac{g_{1,1,k}}{P(\widetilde{\lambda}_k) - 1}, \qquad k = 1, 2, \ldots \tag{2.36}$$

For $i = 1, n = 2$ we have two possibilities how to get $r+s = 3$ $(s = 1, r = 2$ and $s = 2, r = 1)$. For $r = 1$ we have $j = 1$, and for $r = 2$ we also get $j = 1$. Thus, from (2.35) we obtain

$$\sum_{k=1}^{\infty} a_{1,1,k}\psi_k(t) + \sum_{k=1}^{\infty} g_{1,2,k}\psi_k(t) = \sum_{k=1}^{\infty} a_{1,2,k}P(\widetilde{\lambda}_k)\psi_k(t), \quad t \in I,$$

and since $a_{1,1,k}$ are known from the previous step, now we get

$$a_{1,2,k} = \frac{a_{1,1,k} + g_{1,2,k}}{P(\widetilde{\lambda}_k)}, \qquad k = 1, 2, \ldots$$

For $i = 1$, $n = 3$ and consequently $r + s = 4$ we get following triples: $s = 1, r = 3, j = 2$; $s = 3, r = 1, j = 1$ and $s = 2, r = 2, j = 1$. Thus,

$$\sum_{k=1}^{\infty} a_{1,1,k}\psi_{k+1} + \sum_{k=1}^{\infty} a_{1,3,k}\psi_k + \sum_{k=1}^{\infty} a_{1,2,k}\psi_k + \sum_{k=1}^{\infty} g_{1,3,k}\psi_k = \sum_{k=1}^{\infty} a_{1,3,k}P(\widetilde{\lambda}_k)\psi_k.$$

After reordering the indeces in the first sum we get

$$\begin{aligned} a_{1,3,1} &= \frac{a_{1,2,1} + g_{1,3,1}}{P(\widetilde{\lambda}_1) - 1}, \\ a_{1,3,k} &= \frac{a_{1,1,k-1} + a_{1,2,k} + g_{1,3,k}}{P(\widetilde{\lambda}_k) - 1}, \quad k = 2, 3, \ldots \end{aligned}$$

We follow this schedule for $n = 4, 5, \ldots$. Then we fix $i = 2$ and obtain the coefficients for $n = 1, 2, 3, \ldots$ given by following recursion formulae:

$$
\begin{aligned}
a_{2,1,k} &= \frac{g_{2,1,k}}{P(\widetilde{\lambda}_k) - 1}, \qquad k = 1, 2, \ldots \\
a_{2,2,k} &= \frac{a_{2,1,k} + g_{2,2,k}}{P(\widetilde{\lambda}_k)}, \qquad k = 1, 2, \ldots \\
a_{2,3,1} &= \frac{a_{2,2,1} + g_{2,3,1}}{P(\widetilde{\lambda}_1) - 1}, \\
a_{2,3,k} &= \frac{a_{2,1,k-1} + a_{2,2,k} + g_{2,3,k}}{P(\widetilde{\lambda}_k) - 1}, \quad k = 2, 3, \ldots
\end{aligned}
$$

Then we fix $i = 3$, and so on....

Since for each $r \in \mathbb{N}$ its corresponding $j \in \mathbb{N}$ from $r = m(i, j)$ always satisfies $j < r$, we get a general formula

$$
a_{i,n,k} = \frac{g_{i,n,k} + L}{P(\widetilde{\lambda}_k) - 1}, \qquad i \in \mathbb{N},\ n \in \mathbb{N};\ k = n, n+1, \ldots \tag{2.37}
$$

where L is a linear combination of a_{i,n_1,k_1} for $n_1 < n$, $k_1 < k$.

Since $P(\widetilde{\lambda}_k) - 1 \neq 0$, $k \in \mathbb{N}$, there exists a constant $K > 0$ such that $|P(\widetilde{\lambda}_k) - 1| \geq K$, $k \in \mathbb{N}$. Relation (2.31) yields that there exists $C(i, n) > 0$, such that

$$
|g_{i,n,k}| \leq C(i, n)\widetilde{\lambda}_k^{\,r}, \qquad k \in \mathbb{N}.
$$

Similarly, according to (2.30) there exists $D > 0$, such that

$$
\|g_{i,n}\|_{-r}^2 = \sum_{k=1}^{\infty} |g_{i,n,k}|^2 \widetilde{\lambda}_k^{\,-2r} \leq D(2n)^q, \qquad i, n \in \mathbb{N}.
$$

Hence, for each $i, n \in \mathbb{N}$ we have $C(i, n) \leq D(2n)^q$ and consequently

$$
|g_{i,n,k}| \leq D(2n)^q \widetilde{\lambda}_k^{\,r}, \qquad i, n, k \in \mathbb{N}. \tag{2.38}
$$

We prove now the estimate

$$
|a_{i,n,k}| \leq D(2n)^q \widetilde{\lambda}_k^{\,r} Q_n(\frac{1}{K}), \quad i, n \in \mathbb{N},\ k \geq n, \tag{2.39}
$$

where Q_n is a polynomial of order n. The proof can be done by induction. For $i = n = 1$ we get from (2.36) and (2.38) that $|a_{1,1,k}| \leq \frac{1}{K} D 2^q \widetilde{\lambda}_k^{\,r}$, $k \in \mathbb{N}$. Assume now (2.39) holds. Then, from (2.37) and (2.38) we get

$$
|a_{i+1,n+1,k}| \leq \frac{1}{K}\left(|L| + |g_{i+1,n+1,k}|\right) \leq \frac{1}{K}\left(|L| + D(2(n+1))^q \widetilde{\lambda_{k+1}}^{\,r}\right).
$$

Since L is a linear combination of n coefficients a_{i,n_1,k_1}, using the induction hypothesis we get $|L| \leq D(2n)^q \widetilde{\lambda_k}^r Q_n(\frac{1}{K})$. Thus, since $\widetilde{\lambda_k}^r \leq \widetilde{\lambda}_{k+1}^r$, and $(2n)^q \leq (2(n+1))^q$, we obtain

$$|a_{i+1,n+1,k}| \leq D(2(n+1))^q \widetilde{\lambda_{k+1}}^r \frac{1}{K}\left(1 + Q_n(\frac{1}{K})\right) = D(2(n+1))^q \widetilde{\lambda_{k+1}}^r R_{n+1}(\frac{1}{K}),$$

where R_{n+1} is some polynomial of order $n+1$. This proves (2.39).

Let p be given as in (2.3). Then, due to (2.39)

$$\sum_{k=1}^{\infty} |a_{i,n,k}|^2 \widetilde{\lambda_k}^{-2r-2p} \leq D^2(2n)^{2q} Q_n^2(\frac{1}{K}) \sum_{k=1}^{\infty} \widetilde{\lambda_k}^{-2p} < \infty,$$

and thus,

$$\|a_{i,n}\|^2_{-(r+p)} < \infty, \qquad i, n \in \mathbb{N},$$

which yields that all coefficients $a_{i,n}$, $i, n \in \mathbb{N}$, belong to $\mathcal{A}_{-(r+p)}$.

Also, it holds that,

$$\sum_{i=1}^{\infty}\sum_{n=1}^{\infty} \|a_{i,n}\|^2_{-(r+p)} e^{-s(2\mathbb{N})^{\varepsilon n}} \leq D^2 M \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} (2n)^{2q} Q_n^2(\frac{1}{K}) e^{-2sn}.. \qquad (2.40)$$

The series on the right-hand side of (2.40) can be made convergent if we choose s large enough. Thus, there exists a solution X of equation (2.29) in the space ${}^{exp}\mathcal{A}(H)^*_{r+p}$. Since every GRP (I) is uniquely determined by the coefficients in its expansion, it follows that the solution is unique. (Again, we assume the fixed convention $\widetilde{\lambda}_k = 1$ for $\lambda_k = 0$).

2.4 Colombeau Algebras for Generalized Random Processes of Type (I) and (O)

In this section we construct Colombeau type extensions for the spaces of generalized random processes; we consider the case of GRPs (I) and (O).

2.4.1 Colombeau extension of $\mathcal{L}(\mathcal{A}, (S)_{-1})$

Throughout this section we assume that $\mathcal{A}$ is nuclear ($M = \sum_{i=1}^{\infty} \widetilde{\lambda_i}^{-2s} < \infty$ for some $s > 0$) and that there exists a constant $C > 0$, such that $\widetilde{\lambda_{i+j}} \leq C\widetilde{\lambda_i}\widetilde{\lambda_j}$, $i, j \in \mathbb{N}$. These assumptions are necessary for the Wick–multiplication of Zemanian test functions (see Lemma 2.3.1).

- Let $\mathcal{E}_M(\mathcal{A};(S)_{-1})$ be the vector space of functions $R:(0,1)\to\mathcal{A}\otimes(S)_{-1}$, $\epsilon\mapsto R_\epsilon(t,\omega)=\sum_{\alpha\in\mathcal{I}} r_{\alpha,\epsilon}(t)\otimes H_\alpha(\omega)$, $r_{\alpha,\epsilon}\in\mathcal{A}, t\in I,\omega\in\Omega$, such that for every $k\in\mathbb{N}_0$ there exist a sequence $\{C_\alpha\}_{\alpha\in\mathcal{I}}$ of positive numbers, $\epsilon_0\in(0,1)$, $p\in\mathbb{N}_0$, and there exists a sequence $\{a_\alpha\}_{\alpha\in\mathcal{I}}$ bounded from above (i.e. there exists $a\in\mathbb{R}$ such that $a_\alpha\le a$, $\alpha\in\mathcal{I}$), with following properties:

$$\|r_{\alpha,\epsilon}\|_k\le C_\alpha\epsilon^{-a_\alpha},\quad \text{for all } \alpha\in\mathcal{I},\epsilon<\epsilon_0, \tag{2.41}$$

$$\sum_{\alpha\in\mathcal{I}} C_\alpha^2(2\mathbb{N})^{-p\alpha}<\infty. \tag{2.42}$$

Clearly, from (2.41) and (2.42) we have

$$\sum_{\alpha\in\mathcal{I}}\|r_{\alpha,\epsilon}\|_k^2(2\mathbb{N})^{-p\alpha}\le\sum_{\alpha\in\mathcal{I}} C_\alpha^2\epsilon^{-2a_\alpha}(2\mathbb{N})^{-p\alpha}\le\epsilon^{-2a}\sum_{\alpha\in\mathcal{I}} C_\alpha^2(2\mathbb{N})^{-p\alpha}=K\epsilon^{-2a},$$

for $\epsilon<\epsilon_0$,, where $K=\sum_{\alpha\in\mathcal{I}} C_\alpha^2(2\mathbb{N})^{-p\alpha}$.

- Let $\mathcal{N}(\mathcal{A};(S)_{-1})$ denote the vector space of functions $R_\epsilon\in\mathcal{E}_M(\mathcal{A};(S)_{-1})$ with the property that for every $k\in\mathbb{N}_0$ there exist a sequence $\{C_\alpha\}_{\alpha\in\mathcal{I}}$ of positive numbers, $\epsilon_0\in(0,1)$, $p\in\mathbb{N}_0$, and for all sequences $\{a_\alpha\}_{\alpha\in\mathcal{I}}$ bounded from below (i.e. there exists $a\in\mathbb{R}$ such that $a_\alpha\ge a$, $\alpha\in\mathcal{I}$), following hold:

$$\|r_{\alpha,\epsilon}\|_k\le C_\alpha\epsilon^{a_\alpha},\quad \text{for all } \alpha\in\mathcal{I},\epsilon<\epsilon_0, \tag{2.43}$$
$$\sum_{\alpha\in\mathcal{I}} C_\alpha^2(2\mathbb{N})^{-p\alpha}<\infty. \tag{2.44}$$

Clearly, from (2.43) and (2.44) we have

$$\sum_{\alpha\in\mathcal{I}}\|r_{\alpha,\epsilon}\|_k^2(2\mathbb{N})^{-p\alpha}\le\sum_{\alpha\in\mathcal{I}} C_\alpha^2\epsilon^{2a_\alpha}(2\mathbb{N})^{-p\alpha}\le\epsilon^{2a}\sum_{\alpha\in\mathcal{I}} C_\alpha^2(2\mathbb{N})^{-p\alpha}=K\epsilon^{2a},$$

for $\epsilon<\epsilon_0$, where $K=\sum_{\alpha\in\mathcal{I}} C_\alpha^2(2\mathbb{N})^{-p\alpha}$.

- Define the multiplication in $\mathcal{E}_M(\mathcal{A};(S)_{-1})$ and $\mathcal{N}(\mathcal{A};(S)_{-1})$ in the following way: For $F_\epsilon(x,\omega)=\sum_{\alpha\in\mathcal{I}} f_{\alpha,\epsilon}(x)\otimes H_\alpha(\omega)$, $G_\epsilon(x,\omega)=\sum_{\alpha\in\mathcal{I}} g_{\alpha,\epsilon}(x)\otimes H_\alpha(\omega)$ let

$$F_\epsilon\Diamond G_\epsilon(x,\omega)=\sum_{\gamma\in\mathcal{I}}\left(\sum_{\alpha+\beta=\gamma} f_{\alpha,\epsilon}(x)\diamond g_{\alpha,\epsilon}(x)\right)\otimes H_\gamma(\omega), \tag{2.45}$$

i.e. we use the Wick product for multiplication in $(S)_{-1}$ and in $\mathcal{A}$ as well. We recall following estimate from [Se] (see also Lemma 2.3.1): If $f, g \in \mathcal{A}_k$, then $\|f \diamond g\|_{k-s} \leq C^k \sqrt{M} \|f\|_k \|g\|_k$.

Lemma 2.4.1 *(i) $\mathcal{E}_M(\mathcal{A}; (S)_{-1})$ is an algebra under the multiplication rule given by* (2.45).

(ii) $\mathcal{N}(\mathcal{A}; (S)_{-1})$ is an ideal of $\mathcal{E}_M(\mathcal{A}; (S)_{-1})$.

Proof. (i) Let $F_\epsilon, G_\epsilon \in \mathcal{E}_M(\mathcal{A}; (S)_{-1})$ and prove that $F_\epsilon \Diamond G_\epsilon \in \mathcal{E}_M(\mathcal{A}; (S)_{-1})$. Let $k \in \mathbb{N}_0$ be arbitrary. Since $F_\epsilon = \sum_{\alpha \in \mathcal{I}} f_{\alpha,\epsilon}(x) \otimes H_\alpha(\omega) \in \mathcal{E}_M(\mathcal{A}; (S)_{-1})$ and $G_\epsilon = \sum_{\beta \in \mathcal{I}} g_{\beta,\epsilon}(x) \otimes H_\beta(\omega) \in \mathcal{E}_M(\mathcal{A}; (S)_{-1})$, for $\tilde{k} = k + s$ there exist $a, b \in \mathbb{R}$, $p, q \in \mathbb{N}_0$, $\epsilon_1, \epsilon_2 \in (0,1)$, and there exist sequences $\{C_\alpha\}_{\alpha \in \mathcal{I}}$, $\{D_\alpha\}_{\alpha \in \mathcal{I}}$, $\{a_\alpha\}_{\alpha \in \mathcal{I}}$, $\{b_\alpha\}_{\alpha \in \mathcal{I}}$, such that $a_\alpha \leq a$, $b_\alpha \leq b$ for all $\alpha \in \mathcal{I}$, $\|f_{\alpha,\epsilon}\|_{k+s} \leq C_\alpha \epsilon^{-a_\alpha}$ for $\epsilon < \epsilon_1$, $\|g_{\alpha,\epsilon}\|_{k+s} \leq D_\alpha \epsilon^{-b_\alpha}$ for $\epsilon < \epsilon_2$, and $\sum_{\alpha \in \mathcal{I}} C_\alpha^2 (2\mathbb{N})^{-p\alpha} < \infty$, $\sum_{\alpha \in \mathcal{I}} D_\alpha^2 (2\mathbb{N})^{-q\alpha} < \infty$. We will prove that for $\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$ there exist $r > 0$, a sequence $\{M_\gamma\}_{\gamma \in \mathcal{I}}$ and a sequence $\{m_\gamma\}_{\gamma \in \mathcal{I}}$ bounded from above such that $\| \sum_{\alpha+\beta=\gamma} f_{\alpha,\epsilon}(x) \diamond g_{\alpha,\epsilon}(x)\|_k \leq M_\gamma \epsilon^{-m_\gamma}$ for $\epsilon < \epsilon_0$ and $\sum_{\gamma \in \mathcal{I}} M_\gamma^2 (2\mathbb{N})^{-r\alpha} < \infty$.

For a fixed multiindex $\gamma \in \mathcal{I}$ put $M_\gamma = C^k \sqrt{M} \sum_{\alpha+\beta=\gamma} C_\alpha D_\beta$ and $m_\gamma = \max\{a_\alpha + b_\beta : \ \alpha, \beta \in \mathcal{I}, \ \alpha + \beta = \gamma\}$ (note, for γ fixed, there are only finite many α and β which give the sum γ). Now using the fact that $(\mathcal{A}, \diamond)$ is an algebra, we get

$$\| \sum_{\alpha+\beta=\gamma} f_{\alpha,\epsilon}(x) \diamond g_{\alpha,\epsilon}(x)\|_k \leq C^k \sqrt{M} \sum_{\alpha+\beta=\gamma} \|f_{\alpha,\epsilon}(x)\|_{k+s} \|g_{\alpha,\epsilon}(x)\|_{k+s} \leq$$

$$C^k \sqrt{M} \sum_{\alpha+\beta=\gamma} C_\alpha D_\beta \epsilon^{-(a_\alpha + b_\beta)} \leq \epsilon^{-m_\gamma} C^k \sqrt{M} \sum_{\alpha+\beta=\gamma} C_\alpha D_\beta = \epsilon^{-m_\gamma} M_\gamma.$$

Clearly, $m_\gamma \leq a + b$, $\gamma \in \mathcal{I}$, thus the sequence $\{m_\gamma\}_{\gamma \in \mathcal{I}}$ is bounded from above. Let $r = p + q + 2$. Then, using the nuclearity of $(S)_{-1}$, we obtain

$$\sum_{\gamma \in \mathcal{I}} M_\gamma^2 (2\mathbb{N})^{-r\alpha} \leq C^{2k} M \sum_{\gamma \in \mathcal{I}} \left(\sum_{\alpha+\beta=\gamma} C_\alpha D_\beta \right)^2 (2\mathbb{N})^{-(p+q+2)\gamma}$$

$$\leq C^{2k} M \sum_{\gamma \in \mathcal{I}} (2\mathbb{N})^{-2\gamma} \sum_{\alpha \in \mathcal{I}} C_\alpha^2 (2\mathbb{N})^{-p\alpha} \sum_{\beta \in \mathcal{I}} D_\beta^2 (2\mathbb{N})^{-q\beta} < \infty.$$

(ii) Let us check first that $\mathcal{N}(\mathcal{A}; (S)_{-1})$ is a subalgebra of $\mathcal{E}_M(\mathcal{A}; (S)_{-1})$. Let $F_\epsilon, G_\epsilon \in \mathcal{N}(\mathcal{A}; (S)_{-1})$ be of the form as in (i). Let $k \in \mathbb{N}_0$

and $\{m_\gamma\}_{\gamma\in\mathcal{I}}$ be an arbitrary sequence bounded from below. Put $a_\gamma = b_\gamma = \frac{m_\gamma}{2}$, $\gamma \in \mathcal{I}$. Now for the sequences $\{a_\gamma\}_{\gamma\in\mathcal{I}}, \{b_\gamma\}_{\gamma\in\mathcal{I}}$ also bounded from below, and for $\tilde{k} = k + s$, since $F_\epsilon, G_\epsilon \in \mathcal{N}(\mathcal{A}; (S)_{-1})$, there must exist $p, q \in \mathbb{N}_0$, $\epsilon_1, \epsilon_2 \in (0,1)$, and $\{C_\alpha\}_{\alpha\in\mathcal{I}}$, $\{D_\alpha\}_{\alpha\in\mathcal{I}}$ such that $\|f_{\alpha,\epsilon}\|_{k+s} \leq C_\alpha \epsilon^{m_\alpha/2}$ for $\epsilon < \epsilon_1$, $\|g_{\alpha,\epsilon}\|_{k+s} \leq D_\alpha \epsilon^{m_\alpha/2}$ for $\epsilon < \epsilon_2$, and $\sum_{\alpha\in\mathcal{I}} C_\alpha^2 (2\mathbb{N})^{-p\alpha} < \infty$, $\sum_{\alpha\in\mathcal{I}} D_\alpha^2 (2\mathbb{N})^{-q\alpha} < \infty$. Let $\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$, $r = p + q + 2$ and $M_\gamma = C^k\sqrt{M}\sum_{\alpha+\beta=\gamma} C_\alpha D_\beta$, $\gamma \in \mathcal{I}$. Now we may proceed as in (i) to get $\sum_{\alpha+\beta=\gamma} \|f_{\alpha,\epsilon} \diamond g_{\alpha,\epsilon}\|_k \leq C^k\sqrt{M}\sum_{\alpha+\beta=\gamma} \|f_{\alpha,\epsilon}\|_{k+s}\|g_{\alpha,\epsilon}\|_{k+s} \leq M_\gamma \epsilon^{m_\gamma}$ and $\sum_{\gamma\in\mathcal{I}} M_\gamma^2 (2\mathbb{N})^{-r\alpha} < \infty$.

In order to prove that $\mathcal{N}(\mathcal{A}; (S)_{-1})$ is an ideal of $\mathcal{E}_M(\mathcal{A}; (S)_{-1})$, we must check that for all $G_\epsilon \in \mathcal{N}(\mathcal{A}; (S)_{-1})$ and all $F_\epsilon \in \mathcal{E}_M(\mathcal{A}; (S)_{-1})$, we have $G_\epsilon \Diamond F_\epsilon = F_\epsilon \Diamond G_\epsilon \in \mathcal{N}(\mathcal{A}; (S)_{-1})$. Let $k \in \mathbb{N}_0$ and $\{n_\gamma\}_{\gamma\in\mathcal{I}}$ be an arbitrary sequence bounded from below (i.e. $n_\gamma \geq n$, $\gamma \in \mathcal{I}$). Since $F_\epsilon \in \mathcal{E}_M(\mathcal{A}; (S)_{-1})$, there exist $p \in \mathbb{N}_0$, $\epsilon_1 \in (0,1)$, $\{D_\beta\}_{\beta\in\mathcal{I}}$, $b \in \mathbb{R}$ and a sequence $\{b_\beta\}_{\beta\in\mathcal{I}}$ such that $b_\beta \leq b$, $\beta \in \mathcal{I}$, $\|f_{\beta,\epsilon}\|_{k+s} \leq D_\beta \epsilon^{-b_\beta}$, $\epsilon < \epsilon_1$, and $\sum_\beta D_\beta^2 (2\mathbb{N})^{-p\beta} < \infty$.

For a fixed multiindex $\alpha \in \mathcal{I}$, let $a_\alpha = b + n_\alpha$. Clearly, the sequence $\{a_\alpha\}_{\alpha\in\mathcal{I}}$ is bounded from below by $a = b + n$, thus since $G_\epsilon \in \mathcal{N}(\mathcal{A}; (S)_{-1})$, there exist $q \in \mathbb{N}_0$, $\epsilon_2 \in (0,1)$, $\{C_\alpha\}_{\alpha\in\mathcal{I}}$, such that $\|g_{\alpha,\epsilon}\|_{k+s} \leq C_\alpha \epsilon^{-a_\alpha}$, $\epsilon < \epsilon_2$, and $\sum_\alpha C_\alpha^2 (2\mathbb{N})^{-q\alpha} < \infty$. Let $\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$ and $N_\gamma = C^k\sqrt{M}\sum_{\alpha+\beta=\gamma} C_\alpha D_\beta$, $\gamma \in \mathcal{I}$.

Now for $G_\epsilon \Diamond F_\epsilon = \sum_{\gamma\in\mathcal{I}}\sum_{\alpha+\beta=\gamma} g_{\alpha,\epsilon} \diamond f_{\alpha,\epsilon} \otimes H_\gamma$ we have that

$$\sum_{\alpha+\beta=\gamma} \|g_{\alpha,\epsilon}\|_{k+s}\|f_{\beta,\epsilon}\|_{k+s} \leq \sum_{\alpha+\beta=\gamma} C_\alpha D_\beta \epsilon^{a_\alpha - b_\beta}$$

$$\leq \sum_{\alpha+\beta=\gamma} C_\alpha D_\beta \epsilon^{b+n_\alpha-b_\beta} \leq \sum_{\alpha+\beta=\gamma} C_\alpha D_\beta \epsilon^{b_\beta+n_\alpha-b_\beta} \leq C^{-k} M^{-\frac{1}{2}} N_\gamma \epsilon^{n_\gamma},\ \epsilon < \epsilon_0.$$

It is clear that $\sum_{\gamma\in\mathcal{I}} N_\gamma^2 (2\mathbb{N})^{-r\gamma} < \infty$ for $r = p+q+2$. Thus, $G_\epsilon \Diamond F_\epsilon \in \mathcal{N}(\mathcal{A}; (S)_{-1})$. □

- The quotient space

$$\mathcal{G}(\mathcal{A}; (S)_{-1}) = \mathcal{E}_M(\mathcal{A}; (S)_{-1}) \ / \ \mathcal{N}(\mathcal{A}; (S)_{-1})$$

is the $(S)_{-1}$–valued Colombeau extension of $\mathcal{A}$, and its elements are Colombeau generalized random processes. We denote the elements of $\mathcal{G}(\mathcal{A}; (S)_{-1})$ (equivalence classes) by $[R_\epsilon]$.

Note that $(S)_{-1}$ can be embedded into $\mathcal{G}(\mathcal{A};(S)_{-1})$ by

$$(S)_{-1} \ni F(\omega) = \sum_{\alpha\in\mathcal{J}} f_\alpha H_\alpha(\omega) \rightsquigarrow \sum_{\alpha\in\mathcal{J}} f_\alpha(t)\otimes H_\alpha(\omega) \in \mathcal{G}(\mathcal{A};(S)_{-1}),$$

where $f_\alpha(t) = f_\alpha$ is the constant mapping.

Remark. In some special cases, one can take the ordinary product instead of the Wick product in the multiplication rule (2.45). If $\mathcal{A}$ is the Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing functions (see Example 1.3.1), then $(\mathcal{S}(\mathbb{R}),\cdot)$ is also an algebra and one can construct a Colombeau algebra $\mathcal{G}(\mathcal{S}(\mathbb{R});(S)_{-1})$ in the same way but with following multiplication rule:

$$F_\epsilon \Diamond G_\epsilon(x,\omega) = \sum_{\gamma\in\mathcal{J}} \left(\sum_{\alpha+\beta=\gamma} f_{\alpha,\epsilon}(x)\cdot g_{\alpha,\epsilon}(x)\right) \otimes H_\gamma(\omega), \tag{2.46}$$

The same holds true if $\mathcal{A}$ is generated by the operator $\mathcal{R} = -iD$ (see Example 2.3.1). In this case the Wick product and the ordinary product coincide up to a constant.

2.4.2 Colombeau extension of $C^\infty(I;(S)_{-1})$

In the previous case we constructed a Colombeau extension for the space of GRPs (I). Now, we consider the space of infinitely differentiable GRPs (O) i.e. $C^\infty(I;(S)_{-1})$. Recall that $F \in C^\infty(I;(S)_{-1})$ if it is of the form $F(t,\omega) = \sum_{\alpha\in\mathcal{J}} f_\alpha(t)H_\alpha(\omega)$, $f_\alpha \in C^\infty(I)$, and for each $t \in I$ fixed, $\sum_{\alpha\in\mathcal{J}} |f_\alpha(t)|^2(2\mathbb{N})^{-p\alpha} < \infty$ for some $p \geq 0$.

The (usual) topology on $C^\infty(I)$ is the topology of uniform convergence of the function and its partial derivatives of each order on compact subsets of I. Endowed with this topology $C^\infty(I)$ is a nuclear space.

Note that since I is an open subset of $\mathbb{R}^d$ and $(S)_{-1}$ is complete, we have (see [Tr, Theorem 44.1]) $C^\infty(I;(S)_{-1}) \cong C^\infty(I)\otimes(S)_{-1}$ and $C_0^\infty(I)\otimes(S)_{-1}$ is dense in $C^\infty(I;(S)_{-1})$.

For example, singular white noise and Brownian motion defined in Example 1.7.2 and Example 1.7.1 are infinitely differentiable GRPs (O).

The algebra now to be constructed is the $(S)_{-1}$–valued version of the simplified Colombeau algebra for deterministic generalized functions. Recall that the simplified Colombeau algebra was also constructed on $C^\infty(I)$.

- Let $\mathcal{E}_M(I;(S)_{-1})$ be the vector space of functions $R : (0,1) \to C^\infty(I)\otimes(S)_{-1}$, $\epsilon \mapsto R_\epsilon(t,\omega) = \sum_{\alpha\in\mathcal{J}} r_{\alpha,\epsilon}(t)H_\alpha(\omega)$, $r_{\alpha,\epsilon} \in C^\infty(I), t \in I, \omega \in \Omega$, such that for each compact subset $U_k \subset\subset I$, $k \in \mathbb{N}$, there exist a sequence $\{C_\alpha\}_{\alpha\in\mathcal{I}}$ of positive numbers, $\epsilon_0 \in (0,1)$, $p \in \mathbb{N}_0$, and there exists a sequence $\{a_\alpha\}_{\alpha\in\mathcal{I}}$ bounded from above with the property that for all $|\beta| \le k$,

$$\sup_{t\in U_k} |D^\beta r_{\alpha,\epsilon}(t)| \le C_\alpha \epsilon^{-a_\alpha}, \quad \text{for all } \alpha \in \mathcal{I}, \epsilon < \epsilon_0, \tag{2.47}$$

$$\sum_{\alpha\in\mathcal{I}} C_\alpha^2 (2\mathbb{N})^{-p\alpha} < \infty. \tag{2.48}$$

- Let $\mathcal{N}(I;(S)_{-1})$ denote the vector space of functions $R_\epsilon \in \mathcal{E}_M(I;(S)_{-1})$ with the property that for each compact subset $U_k \subset\subset I$, $k \in \mathbb{N}$, there exist a sequence $\{C_\alpha\}_{\alpha\in\mathcal{I}}$ of positive numbers, $\epsilon_0 \in (0,1)$, $p \in \mathbb{N}_0$, and for all sequences $\{a_\alpha\}_{\alpha\in\mathcal{I}}$ bounded from below, and for all $|\beta| \le k$ following hold:

$$\sup_{t\in U_k} |D^\beta r_{\alpha,\epsilon}(t)| \le C_\alpha \epsilon^{a_\alpha}, \quad \text{for all } \alpha \in \mathcal{I}, \epsilon < \epsilon_0, \tag{2.49}$$

$$\sum_{\alpha\in\mathcal{I}} C_\alpha^2 (2\mathbb{N})^{-p\alpha} < \infty. \tag{2.50}$$

- Define the multiplication in $\mathcal{E}_M(I;(S)_{-1})$ and $\mathcal{N}(I;(S)_{-1})$ in the following way: For $F_\epsilon(x,\omega) = \sum_{\alpha\in\mathcal{I}} f_{\alpha,\epsilon}(x) \otimes H_\alpha(\omega)$, $G_\epsilon(x,\omega) = \sum_{\alpha\in\mathcal{I}} g_{\alpha,\epsilon}(x) \otimes H_\alpha(\omega)$ let

$$F_\epsilon \lozenge G_\epsilon(x,\omega) = \sum_{\gamma\in\mathcal{I}} \left(\sum_{\alpha+\beta=\gamma} f_{\alpha,\epsilon}(x) \cdot g_{\alpha,\epsilon}(x) \right) \otimes H_\gamma(\omega), \tag{2.51}$$

  i.e. we use the Wick product for multiplication in $(S)_{-1}$ and the ordinary product in $C^\infty(I)$. Clearly, $(C^\infty(I), \cdot)$ is an algebra.

- Then $\mathcal{N}(I;(S)_{-1})$ is an ideal of $\mathcal{E}_M(I;(S)_{-1})$ and the quotient space

$$\mathcal{G}(I;(S)_{-1}) = \mathcal{E}_M(I;(S)_{-1}) \,/\, \mathcal{N}(I;(S)_{-1}),$$

  the $(S)_{-1}$–valued Colombeau extension of $C^\infty(I)$ is also an algebra. Its elements are Colombeau generalized random processes.

Note that $(S)_{-1}$ can be embedded into $\mathcal{G}(I;(S)_{-1})$ by

$$(S)_{-1} \ni F(\omega) = \sum_{\alpha\in\mathcal{I}} f_\alpha H_\alpha(\omega) \rightsquigarrow \sum_{\alpha\in\mathcal{I}} f_\alpha(t) \otimes H_\alpha(\omega) \in \mathcal{G}(I;(S)_{-1}),$$

where $f_\alpha(t) = f_\alpha$ is the constant mapping.

The space $\mathcal{G}(I;(S)_{-1})$ is constructed so that it involves singular GRPs (O) i.e. GRPs (I). Like in deterministic case, where $\mathcal{S}'(I)$ can be embedded into $\mathcal{G}(I)$, here $\mathcal{L}(\mathcal{S}(I),(S)_{-1}) = \mathcal{S}'(I) \otimes (S)_{-1}$ can also be embedded into $\mathcal{G}(I;(S)_{-1})$ via convolution. For a fixed mollifier function ρ_ϵ, $\epsilon \in (0,1)$ we have that if $f \in \mathcal{S}(I) \otimes (S)_{-1}$, then $f_\epsilon(t,\omega) = f(\cdot,\omega) * \rho_\epsilon(t) \in \mathcal{S}(I) \otimes (S)_{-1}$ and clearly, $f - f_\epsilon \in \mathcal{N}(I;(S)_{-1})$. Thus, $[f_\epsilon] \in \mathcal{G}(I;(S)_{-1})$. The embedding $\iota_\rho : \mathcal{S}(I) \otimes (S)_{-1} \to \mathcal{G}(I;(S)_{-1})$, $f \mapsto [f_\epsilon]$, can be extended to an embedding $\iota_\rho : \mathcal{S}'(I) \otimes (S)_{-1} \to \mathcal{G}(I;(S)_{-1})$, defined by

$$\mathcal{S}'(I) \otimes (S)_{-1} \ni F \mapsto [F * \rho_\epsilon] \in \mathcal{G}(I;(S)_{-1}).$$

Also, $\mathcal{G}(I)$ can be embedded into $\mathcal{G}(I;(S)_{-1})$ by

$$\mathcal{G}(I) \ni f = [f_\epsilon(t)] \rightsquigarrow \sum_{\alpha=(0,0,0,\ldots)} [f_\epsilon(t)] H_\alpha(\omega) \in \mathcal{G}(I;(S)_{-1}).$$

In Chapter 3 we will construct also some other algebras as polynomial Colombeau extensions of GRP (I) spaces; especially we will make use of Sobolev spaces instead of the Zemanian test function spaces. We also note that this Colombeau–type construction cannot be carried out for H–valued GRPs (I): Since $S(H)_{-1}$ is not nuclear, Lemma 2.4.1 need not hold true.

2.5 Generalized Random Processes of Type (II)

The results presented in this section were mainly subject of [PS2]. We begin with an example showing that the classes of GRPs of type (I) and (II) are not contained in each other. As main topic, we examine the structure representation of GRPs (II) and their correlation operators on the space of test functions $\mathcal{K}\{M_p\}$. In order to analyze assumptions for general representation theorems, we give in Theorem 2.5.1 a representation theorem for a GRP (II) on $L^r(G)$, $r > 1$, where G is an open interval in $\mathbb{R}^n$. Analyzing the continuity condition $|\xi(\omega,\varphi)| \leq C(\omega)\|\varphi\|_{p(\omega),2}, \phi \in \mathcal{K}\{M_p\}, \omega \in \Omega$, for a GRP (II), we obtain a structural characterization for it and for its correlation operator on $\mathcal{K}\{M_p\}$. More precisely, we show that for each $\varepsilon > 0$ there exists a set M with measure $P(M) \geq 1 - \varepsilon$, and there exist $d \in \mathbb{N}_0$ such that $|\xi(\omega,\varphi)| \leq C(\omega)\|\varphi\|_{d,2}, \phi \in \mathcal{K}\{M_p\}, \omega \in M$. Thus, if $C \in L^p(\Omega)$, this assures the continuity of ξ as a mapping $\mathcal{K}\{M_p\} \to L^p(\Omega)$, where $L^p(\Omega)$ is equipped with convergence in mean of order p for $p \geq 1$, and convergence in probability for $p = 0$.

Especially, in Theorem 2.5.3 we consider Gaussian GRPs, and using another technique obtain generalized Gaussian GRPs (II) on $\mathcal{K}\{M_p\}$ as a sum of derivatives of classical Gaussian processes, which have appropriate growth rate at $|t| \to \infty$.

Several examples show that the assumptions which we made are substantial for the characterization of a GRP (II).

In order to motivate our considerations and demonstrate the preferences of our structural theorems, we will present the simplest equation $y' = f$ in the frame of GRPs (II). The notation we will use further on is given below.

Let $f : \Omega \times \mathcal{K}\{M_p\} \to \mathbb{R}$ be such that for all $\varphi \in \mathcal{K}\{M_p\}$, $f(\cdot, \varphi)$ is a Gaussian random variable, and for every $\omega \in \Omega$, $f(\omega, \cdot)$ is an element in $\mathcal{K}'\{M_p\}$. Let $\omega \mapsto X(\omega), \omega \in \Omega$, be a Gaussian random variable. Consider the stochastic differential equation

$$\frac{d}{dt}\, y(\omega, t) = f(\omega, t),\ y(\omega, 0) = X(\omega), \tag{2.52}$$

where the value at zero, $y(\omega, t)$ at $t = 0$, is understood in the sense of Lojasiewicz.

To solve (2.52) we use the decomposition of $\mathcal{K}\{M_p\}$: Every $\phi \in \mathcal{K}\{M_p\}$ is of the form

$$\phi = C_\phi \phi_0 + \psi, \quad \text{where } \phi_0, \psi \in \mathcal{K}\{M_p\},\ C_\phi = \int_{\mathbb{R}} \phi(t)dt,$$

$$\int_{\mathbb{R}} \phi_0 = 1, \quad \text{and } \psi = \frac{d}{dt}\theta, \quad \text{for some } \ \theta \in \mathcal{K}\{M_p\}.$$

Now, fixing $\omega \in \Omega$, and using the continuity of $f(\omega, \cdot) : \mathcal{K}\{M_p\} \to \mathbb{R}$, we solve this equation in the usual way:

$$y(\omega, \phi) = C_\phi X(\omega) - \langle f(\omega, t), \theta(t)\rangle,\ \phi \in \mathcal{K}\{M_p\}. \tag{2.53}$$

But now we obtain that

$$\lim_{\varepsilon \to 0} \langle f(\omega, \varepsilon t), \theta(t)\rangle = 0, \qquad \text{for } \theta \in \mathcal{K}\{M_p\} \tag{2.54}$$

is a necessary condition for the existence of y.

Another procedure is to use the representation stated in Theorem 2.5.3, which can be used in more general cases: Under certain conditions on f, which are weaker than (2.54), we can still find a solution y on a somewhat smaller set $M \subset \Omega$. We will show this explicitly in Corollary 2.5.1.

GRPs (II) vs. GRPs (I)

As already defined at the beginning of Chapter 1, we denote by Z^2 the space of random variables with finite second moment. Recall now the definition of GRPs of type (II), which we study in this section:

Definition 2.5.1 *A GRP (II) is a mapping $\xi : \Omega \times V \to \mathbb{C}$ such that for every $\varphi \in V$, $\xi(\cdot, \varphi)$ is a complex random variable and for every $\omega \in \Omega$, $\xi(\omega, \cdot)$ is an element in V'.*

Example 2.5.1 *Let V be a separable Hilbert space and $(\phi_n)_{n\in\mathbb{N}}$ be a complete orthogonal system in V converging to zero in V – e.g. if $(e_n)_{n\in\mathbb{N}}$ is a CONS in V, then we can take $\phi_n = \frac{e_n}{n}$, $n \in \mathbb{N}$. We will choose a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ in Z^2 satisfying certain conditions - depending on whether we want to construct a GRP (I) or (II) – and use the following construction: Define a GRP on $\Omega \times \{\phi_i : i \in \mathbb{N}\}$ by putting $\xi(\cdot, \phi_n) = X_n$, $n \in \mathbf{N}$, and extending this by linearity and continuity (type of continuity depending on the conditions we implied on the sequence X_n) onto V.*

(a) We construct a GRP of type (I) but not of type (II).

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables in Z^2 such that:

- $E(X_n X_m) = 0$ *for* $n \neq m$,
- $\|X_n\|_{Z^2} \leq \|\phi_n\|_V$, $n \in \mathbb{N}$,
- $(X_n)_{n\in\mathbb{N}}$ *does not converge to zero almost surely.*

Note that from the second condition we have that $(X_n)_{n\in\mathbb{N}}$ converges to zero in Z^2. Such a sequence exists, e.g. one can take a sequence of independent random variables with distribution law

$$X_n : \begin{pmatrix} -1 & 0 & 1 \\ \frac{1}{2n} & 1 - \frac{1}{n} & \frac{1}{2n} \end{pmatrix}, \; n \in \mathbb{N}.$$

The extension by continuity onto V is well defined (we take convergence in Z^2), since

$$\|\xi(\omega, \sum_{i=1}^{\infty} \alpha_i \phi_i)\|_{Z^2}^2 = \|\sum_{i=1}^{\infty} \alpha_i X_i(\omega)\|_{Z^2}^2 = \sum_{i=1}^{\infty} \alpha_i^2 \|X_i\|_{Z^2}^2$$

$$\leq \sum_{i=1}^{\infty} \alpha_i^2 \|\phi_i\|_V^2 = \|\sum_{i=1}^{\infty} \alpha_i \phi_i\|_V^2.$$

For any sequence $(\varphi_n)_{n\in\mathbb{N}}$ converging to zero in V we have $\|\xi(\omega, \varphi_n)\|_{Z^2} \leq \|\varphi_n\|_V$, thus ξ is a GRP (I). But, $|\xi(\omega, \phi_n)| = |X_n(\omega)|$ does not converge to zero for a.e. $\omega \in \Omega$, thus $\xi(\omega, \cdot) \notin V'$.

(b) We construct a GRP of type (II) but not of type (I).

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables in Z^2 such that:

- *$(X_n)_{n\in\mathbb{N}}$ does not converge to zero in Z^2.*
- *$P(\Omega_0) = 1$, where*
  $\Omega_0 = \{\omega \in \Omega : (\exists k_0 \in \mathbb{N})(\forall k \in \mathbb{N})(k \geq k_0 \Rightarrow X_k(\omega) = 0)\}$.

For example, the sequence of random variables with distribution law

$$X_n : \left(\begin{array}{cc} 0 & n \\ 1 - \frac{1}{n^2} & \frac{1}{n^2} \end{array} \right), \; n \in \mathbb{N}.$$

meets both conditions given above.

The extension by continuity onto V is well defined (we take convergence in a.e. sense), since for $v = \sum_{i=1}^{\infty} \alpha_i \phi_i \in V$ and $\omega \in \Omega_0$ we have

$$|\xi(\omega, v)|^2 = |\sum_{i=1}^{\infty} \alpha_i X_i(\omega)|^2$$

$$\leq \sum_{i=1}^{\infty} \alpha_i^2 \|\phi_i\|_V^2 \cdot \sum_{i=1}^{\infty} \frac{|X_i(\omega)|^2}{\|\phi_i\|_V^2} = \|v\|_V^2 \sum_{i=1}^{k_0} \frac{|X_i(\omega)|^2}{\|\phi_i\|_V^2}.$$

For any sequence $(\varphi_n)_{n\in\mathbb{N}}$ converging to zero in V we have $|\xi(\omega, \varphi_n)| \leq \|\varphi_n\|_V \sum_{i=1}^{\infty} \frac{|X_i(\omega)|^2}{\|\phi_i\|_V^2}$, which converges to zero for each $\omega \in \Omega_0$. Thus, $\xi(\omega, \cdot) \in V'$. On the other hand, $\|\xi(\omega, \phi_n)\|_{Z^2} = \|X_n(\omega)\|_{Z^2}$, which does not converge to zero. Thus, ξ is not a GRP (I).

Further examples will be given in Section 2.5.3.

As in [GV], we will assume that the expectation of ξ, denoted by $E(\xi(\omega, \varphi)) = m(\varphi)$, $\varphi \in V$, exists and belongs to V'. Due to this fact, throughout the paper, we will assume that $E(\xi(\cdot, \varphi)) = 0$ for every $\varphi \in V$. If $E(\xi(\omega, \varphi)\overline{\xi(\omega, \psi)})$ exists for all φ and ψ, and if it is continuous in each argument φ and ψ, then the correlation operator of the GRP ξ is defined by $C_\xi(\varphi, \psi) = E(\xi(\cdot, \varphi)\overline{\xi(\cdot, \psi)})$, $\varphi, \psi \in V$.

2.5.1 Generalized random processes on $L^r(G)$, $r > 1$

Denote by $\kappa(E)$ the characteristic function of $E \subset \mathbb{R}^n$. Let $x_0 \in \mathbb{R}^n$ and let $(E_n)_{n\in\mathbb{N}_0}$ be a sequence of Borel sets, which shrinks nicely to x_0. Recall from [Ru, Chapter 8.] that a sequence $(E_n)_{n\in\mathbb{N}_0}$ of Borel sets in $\mathbb{R}^n$ shrinks nicely to $x_0 \in \mathbb{R}^n$, if there exists $\alpha > 0$ such that each E_n lies in an open ball $B(x_0, r_n)$ and $m(E_n) \geq \alpha m(B(x_0, r_n))$, $n \in \mathbb{N}$, $r_n \to 0$.

Put

$$\delta_n(x - x_0) = \frac{\kappa(E_n)(x)}{m(E_n)}, \quad n \in \mathbb{N},\ x \in \mathbb{R}^n. \tag{2.55}$$

The proof of the following results can be found in [LP2] and [LPP].

Theorem 2.5.1 *Let $G = \prod_{i=1}^n (\alpha_i, \beta_i) \subset \mathbb{R}^n$, $-\infty \leq \alpha_i < \beta_i \leq \infty$, $i = 1, 2, ..., n$, and let ξ be a GRP on $\Omega \times L^r(G)$, $r > 1$.*

a) There exists a function $f : \Omega \times G \to \mathbb{C}$ such that

(i) for every $x \in G$, $f(\cdot, x)$ is measurable and for every $\omega \in \Omega$, $f(\omega, \cdot) \in L^p(G)$, $p = r/(r-1)$.

(ii)

$$\xi(\omega, \varphi) = \int_G f(\omega, t)\varphi(t)dt, \quad \omega \in \Omega, \quad \varphi \in L^r(G).$$

b) If ξ is a Gaussian GRP on $\Omega \times L^r(G)$ such that for every $\omega \in \Omega$, $x_0 \in G$ and every delta sequence $\delta_n(x - x_0)$, $n \in \mathbb{N}$, of the form (2.55)

$$\lim_{n\to\infty} \xi(\omega, \delta_n(x - x_0)) \ \text{ exists,}$$

then there exists a Gaussian random process $f : \Omega \times G \to \mathbb{C}$ such that (i) and (ii) in a) hold.

c) Let $G = \mathbb{R}^n$. If there exists $A \in \mathfrak{F}$ such that $P(A) = 0$ and

$$|\xi(\omega, \varphi)| \leq C(\omega)\|\varphi\|_r\ ,\ \omega \in \Omega\backslash A,$$

then the correlation operator $C_\xi(\cdot, \cdot)$ has a representation

$$C_\xi(\varphi, \psi) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \varphi(t)E(f(\omega, t)\overline{f(\omega, s)})\overline{\psi(s)}dtds, \quad \varphi, \psi \in L^r(G).$$

Proposition 2.5.1 *Let G be an open subset of $\mathbb{R}^n$ and let ξ be a GRP on $\Omega \times L^r(G)$. Then for every $\omega \in \Omega$ there exists a set $A(\omega) \in G$ such that $m(A(\omega)) = 0$ and for every $x_0 \in G \setminus A(\omega)$ there exists the limit*

$$\lim_{n\to\infty} \xi(\omega, \delta_n(\cdot - x_0)),$$

where $\delta_n(x - x_0)$ is defined as in (2.55).

2.5.2 Generalized random processes on $\mathcal{K}\{M_p\}$

Here we will use the techniques given in Section 1.5 for $\mathcal{K}\{M_p\}$ spaces.

Theorem 2.5.2 *a) Let ξ be a GRP on $\mathcal{K}\{M_p\}$. Then for every $\varepsilon > 0$ there exist $d \in \mathbb{N}_0$, $M \in \mathcal{F}$ satisfying $P(M) \geq 1 - \varepsilon$, and functions $f_\alpha : \Omega \times \mathbb{R} \to \mathbb{C}$, $\alpha = 0, 1, ..., d$, such that $f_\alpha(\cdot, t)$ is measurable for every $t \in \mathbb{R}$, $f_\alpha(\omega, \cdot)$ is in $L^2(\mathbb{R})$ for every $\omega \in M$, $\alpha = 0, 1, ..., d$ and*

$$\xi(\omega, \varphi) = \sum_{\alpha=0}^{d} \int_{\mathbb{R}} f_\alpha(\omega, t) M_d(t) \varphi^{(\alpha)}(t) dt, \quad \omega \in M, \quad \varphi \in \mathcal{K}\{M_p\}, \tag{2.56}$$

$$\sum_{\alpha=0}^{d} \|f_\alpha(\omega, \cdot)\|_{L^2} \leq d, \quad \omega \in M. \tag{2.57}$$

In particular, if there exist $C(\omega) > 0$, $\omega \in \Omega$, and $d \in \mathbb{N}$ such that

$$|\xi(\omega, \varphi)| \leq C(\omega) \|\varphi\|_{d,2}, \ \omega \in \Omega, \ \varphi \in \mathcal{K}\{M_p\}, \tag{2.58}$$

then representation (2.56) is valid on the whole Ω.

b) Moreover, if ξ is also a continuous mapping from $\mathcal{K}\{M_p\}$ to Z^2, then for almost every $t, s \in \mathbb{R}$ there exist $E(f_\alpha(\cdot, t)\overline{f_\beta(\cdot, s)})$, $\alpha \leq d$, $\beta \leq d$ and the correlation operator $C_\xi(\varphi, \psi)$, $\varphi, \psi \in \mathcal{K}\{M_p\}$ has the representation

$$C_\xi(\varphi, \psi)$$

$$= \sum_{\alpha=0}^{d} \sum_{\beta=0}^{d} \int_{\mathbb{R}} \int_{\mathbb{R}} E(f_\alpha(\cdot, t)\overline{f_\beta(\cdot, s)}) M_d(t) \overline{M_d(s)} \varphi^{(\alpha)}(t) \overline{\psi^{(\beta)}(s)} dt ds.$$

c) If ξ is a GRP on $\mathcal{K}\{M_p\}$ such that (2.58) holds and $\omega \mapsto C(\omega)$ is in Z^2, then $\xi : \mathcal{K}\{M_p\} \to Z^2$ is continuous and (2.56) holds for every $\omega \in \Omega$. Condition $C(\cdot) \in Z^2$ is sufficient but not necessary for the continuity of $\xi : \mathcal{K}\{M_p\} \to Z^2$.

Proof. a) Note that in [LP] and [LP1] more restricted problems were considered (see also [Ha] and [Ul]).

Since for every $\omega \in \Omega$, $\xi(\omega, \cdot)$ is in $\mathcal{K}'\{M_p\}$, it follows that for every $\omega \in \Omega$ there exist $C(\omega) > 0$ and $p(\omega) \in \mathbb{N}$ such that

$$|\xi(\omega, \varphi)| \leq C(\omega) \|\varphi\|_{p(\omega),2}, \ \varphi \in \mathcal{K}\{M_p\}.$$

We can assume that $p(\omega) \geq C(\omega)$. For every $\varphi \in \mathcal{K}\{M_p\}$ and $N \in \mathbb{N}$, put

$$A_N(\varphi) = \{\omega \in \Omega : \ |\xi(\omega, \varphi)| < N\|\varphi\|_{N,2}\}, \ \ A_N = \bigcap_{\varphi \in \mathcal{K}\{M_p\}} A_N(\varphi).$$

Since $\mathcal{K}\{M_p\}$ is separable, it contains a countable dense subset D and $A_N = \bigcap_{\varphi \in D} A_N(\varphi) \in \mathcal{F}$. Thus, from

$$\Omega = \bigcup_{N=1}^{\infty} A_N \quad \text{and} \quad A_N \subset A_{N+1}, \ \ N \in \mathbb{N},$$

it follows that for given $\varepsilon > 0$ there exists an integer d such that $P(A_d) \geq 1 - \varepsilon$. Denote $M = A_d$. It follows

$$|\xi(\omega, \varphi)| \leq d\|\varphi\|_{d,2}, \ \ \omega \in M, \ \varphi \in \mathcal{K}\{M_p\}.$$

We extend ξ on the whole Ω by

$$\xi_1(\omega, \varphi) = \begin{cases} \xi(\omega, \varphi), & \omega \in M \\ 0, & \omega \notin M \end{cases}, \ \varphi \in \mathcal{K}\{M_p\}. \tag{2.59}$$

Further, put $R = \{\varphi \in \mathcal{K}\{M_p\} : \ \ \|\varphi\|_{d,2} \leq 1\}$ and

$$S(\omega) = \sup_{\varphi \in R} |\xi_1(\omega, \varphi)| = \sup_{\varphi \in D \bigcap R} |\xi_1(\omega, \varphi)|, \ \omega \in \Omega.$$

It follows that S is measurable on Ω, $S(\omega) \leq d, \omega \in \Omega$. Thus,

$$|\xi_1(\omega, \varphi)| \leq S(\omega)\|\varphi\|_{d,2}, \ \ \varphi \in \mathcal{K}\{M_p\}, \ \omega \in \Omega. \tag{2.60}$$

Inequality (2.60) holds also for the space $H^d_M(\mathbb{R}) \subset H^d(\mathbb{R})$, where $H^d(\mathbb{R})$ is the Sobolev space, and $H^d_M = \{\varphi \in H^d(\mathbb{R}) : M_d\varphi^{(\alpha)} \in L^2(\mathbb{R}), \ \alpha = 0, 1, \ldots, d\}$, equipped with the topology induced by the norm $\|\varphi\|_{d,L^2} = \sum_{\alpha=0}^{d} \|M_d\varphi^{(\alpha)}\|_{L^2}$.

We need the following consequence of (2.60):

$$\text{if } (\varphi_\nu)_{\nu \in \mathbb{N}} \text{ is a sequence in } \mathcal{K}\{M_p\} \ \text{ and } \ \varphi_\nu \to 0 \text{ in } \ H^d_M, \tag{2.61}$$

$$\text{then } \ \xi_1(\omega, \varphi_\nu) \to 0, \nu \to \infty.$$

Let $\Gamma_d = \prod_{i=0}^{d} L^2(\mathbb{R})$ and endow it with the scalar product $((\varphi_\alpha), (\psi_\alpha)) = \sum_{\alpha=0}^{d} \int_{\mathbb{R}} \varphi_\alpha \overline{\psi_\alpha} dt$, $(\varphi_\alpha), (\psi_\alpha) \in \Gamma_d$. Clearly, Γ_d is a Hilbert space. Define a mapping $\theta : \mathcal{K}\{M_p\} \to \Gamma_d$ by $\theta(\varphi) = (M_d\varphi, M_d\varphi', \ldots, M_d\varphi^{(d)})$, $\varphi \in \mathcal{K}\{M_p\}$, which is injective, and denote $\Delta = \theta(\mathcal{K}\{M_p\})$. Note that

$$\overline{\Delta} = \theta(H^d_M). \tag{2.62}$$

Define a mapping $\Omega \times \Gamma_d \to \mathbb{C}$, for every $\omega \in \Omega$, by

$$F(\omega, \psi) = \begin{cases} \xi_1(\omega, \theta^{-1}(\psi)), & \psi \in \Delta \\ \lim_{\nu\to\infty} \xi_1(\omega, \theta^{-1}(\psi_\nu)), & \psi \in \bar{\Delta}, \quad \psi_\nu \in \Delta, \ \ \psi_\nu \xrightarrow{L^2} \psi, \\ 0, & \psi \in \bar{\Delta}^\perp. \end{cases}$$

The existence of the limit follows from (2.61) and (2.62). Thus,

$$F(\omega, \tilde{\psi}) = F(\omega, \psi), \ \tilde{\psi} \in \Gamma_d, \ \ \tilde{\psi} = \psi + \psi^\perp, \ \psi \in \bar{\Delta}, \psi^\perp \in \bar{\Delta}^\perp.$$

Clearly $F(\cdot, \tilde{\psi})$ is measurable, for any $\tilde{\psi} \in \Gamma_d$. Let $\varphi \in \mathcal{K}\{M_p\}$, $\omega \in \Omega$. We have

$$|F(\omega, \theta(\varphi))| \leq S(\omega)\|\varphi\|_{d,2} = S(\omega)\|\theta(\varphi)\|_{\Gamma_d}.$$

So, for every $\omega \in \Omega$, $F(\omega, \cdot)$ is a continuous linear functional on Γ_d and it is of the form

$$F(\omega, \cdot) = \sum_{\alpha=0}^{d} F_\alpha(\omega, \cdot), \ \omega \in \Omega.$$

Here $F_\alpha(\omega, \cdot)$, $\in \Omega$, $\alpha \leq d$, are continuous linear functionals on subspaces $\Gamma_{d,\alpha} \subset \Gamma_d, \alpha \leq d$, where

$$\Gamma_{d,\alpha} = \{\psi = (\psi_\beta) \in \Gamma_d : \ \psi_\beta \in L^2(\mathbb{R}), \ \psi_\beta \equiv 0, \ \beta \neq \alpha\}.$$

It is endowed with the natural norm such that it is isometric to $L^2(\mathbb{R})$, for every $\alpha \in \{0, 1, ..., d\}$. Let $\psi \in \Gamma_d$. Denote by $[\psi]_\alpha$ the corresponding element in $\Gamma_{d,\alpha}$. The βth coordinates of $[\psi]_\alpha$ are equal to zero for $\beta \neq \alpha$ and the αth coordinate is equal to ψ_α. Since F is a GRP, it follows that $F_\alpha = F|_{\Gamma_{d,\alpha}}$ is a GRP on $\Omega \times \Gamma_{d,\alpha}$ i.e. on $\Omega \times L^2(\mathbb{R})$, for every α, $\alpha \leq d$.

By Theorem 2.5.1, for every $\alpha = 0, 1, \ldots, d$, there exists a function $f_\alpha : \Omega \times \mathbb{R} \to \mathbb{C}$ such that $f_\alpha(\cdot, t)$ is measurable for every $t \in \mathbb{R}$, $f_\alpha(\omega, \cdot) \in L^2(\mathbb{R})$, $\omega \in \Omega$, and

$$F_\alpha(\omega, \varphi) = \int_{\mathbb{R}} f_\alpha(\omega, t)\varphi(t)dt, \quad \varphi \in L^2(\mathbb{R}), \ \omega \in \Omega.$$

Thus, if $\omega \in \Omega$ and $\psi = \theta(\varphi)$ for $\varphi \in \mathcal{K}\{M_p\}$, then

$$F(\omega, \psi) = \sum_{\alpha=0}^{d} F_\alpha(\omega, [\psi]_\alpha) = \sum_{\alpha=0}^{d} \int_{\mathbb{R}^n} f_\alpha(\omega, t) M_d(t)\psi^{(\alpha)} dt \tag{2.63}$$

and

$$\|F(\omega, \cdot)\|'_{\Gamma_d} = \sum_{\alpha=0}^{d} \|f_\alpha(\omega, \cdot)\|_{L^2(\mathbb{R})} \leq S(\omega) \leq d, \quad \omega \in \Omega,$$

where $\|\cdot\|'_{\Gamma_d}$ is the dual norm. Now, the assertion follows by (2.59).

The proof of the last assertion in a) follows by repeating the previous proof starting from relation (2.60). It follows that ξ is of the form (2.56) for every $\omega \in \Omega$.

b) Obviously, $C_\xi(\varphi, \psi) = E(\xi(\cdot, \varphi)\overline{\xi(\cdot, \psi)})$, $\varphi, \psi \in \mathcal{K}\{M_p\}$ is bilinear. The continuity follows from

$$C_\xi(\varphi, \psi) = |E(\xi(\cdot, \varphi)\overline{\xi(\cdot, \psi)})| \le \|\xi(\cdot, \varphi)\|_{Z^2}\|\xi(\cdot, \psi)\|_{Z^2} \le$$

$$\le \|\varphi\|_{d,2}\|\psi\|_{d,2} \sup\{\|\xi(\cdot, \varphi)\|_{Z^2}, \varphi \in \mathcal{K}\{M_p\}, \|\varphi\|_{d,2} \le 1\}$$

$$\cdot \sup\{\|\xi(\cdot, \psi)\|_{Z^2}, \psi \in \mathcal{K}\{M_p\}, \|\psi\|_{d,2} \le 1\}.$$

Fubini's theorem implies

$$C_\xi(\varphi, \psi) = E(\xi(\cdot, \varphi)\overline{\xi(\cdot, \psi)})$$

$$= E((\sum_{\alpha=0}^{d} \int_{\mathbb{R}} f_\alpha(\cdot, t) M_d(t) \varphi^{(\alpha)}(t) dt)(\sum_{\alpha=0}^{d} \int_{\mathbb{R}} \overline{f_\alpha(\cdot, s) M_d(s)}\overline{\psi^{(\alpha)}(s)} ds))$$

$$= \sum_{\alpha=0}^{d} \sum_{\beta=0}^{d} E(\int_{\mathbb{R}} \int_{\mathbb{R}} f_\alpha(\cdot, t) M_d(t) \varphi^{(\alpha)}(t) \overline{f_\beta(\cdot, s) M_d(s)}\overline{\psi^{(\beta)}(s)} dt ds)$$

$$= \sum_{\alpha=0}^{d} \sum_{\beta=0}^{d} (\int_{\mathbb{R}} \int_{\mathbb{R}} E(f_\alpha(\cdot, t)\overline{f_\beta(\cdot, s)}) M_d(t)\overline{M_d(s)} \varphi^{(\alpha)}(t)\overline{\psi^{(\beta)}(s)} dt ds.$$

This proves the last assertion in b).
c) If ξ is a GRP on $\mathcal{K}\{M_p\}$ which satisfies (2.58) and $C(\cdot) \in Z^2$, then ξ is a continuous mapping $\mathcal{K}\{M_p\} \to Z^2$. Namely, for any sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathcal{K}\{M_p\}$ such that $\varphi_n \to 0, n \to \infty$, it follows

$$\|\xi(\cdot, \varphi_n)\|_{Z^2} = E(|\xi(\cdot, \varphi_n)|^2) \le E(C^2(\cdot))\|\varphi_n\|^2_{d,2} \to 0.$$

Example 2.5.3 in the next subsection shows that $\xi : \mathcal{K}\{M_p\} \to Z^2$ may be a continuous mapping from $\mathcal{K}\{M_p\}$ to Z^2 although $C(\cdot) \notin Z^2$. □

Restriction (2.58) is not a draconian one; it holds for a large class of processes but not for the whole class of GRPs (II) as it will be seen in Example 2.5.5. The following theorem for Gaussian GRPs (II) represents the main result of [PS2]. The point value of a distribution is defined in sense of Lojasiewicz (see Section 1.5).

Theorem 2.5.3 *a) Let ξ be a Gaussian GRP on $\mathcal{K}\{M_p\}$ and $\varepsilon \in (0,1)$. Then there exist $M \in \mathfrak{F}$ such that $P(M) \geq 1-\varepsilon$, $s_i, \tilde{d}_i \in \mathbb{N}$, $i = 1,2,3$, and there exist Gaussian processes $\Xi_1(\omega,t)$, $\Xi_2(\omega,t)$ and $\Xi_3(\omega,t)$, $(\omega,t) \in \Omega \times \mathbb{R}$, with the following properties:*

$$\textit{for every } \omega \in \Omega, \quad t \mapsto \Xi_i(\omega,t) \textit{ is continuous, } i = 1,2,3;$$

$$\operatorname{supp} \Xi_1(\omega,\cdot) \subset [-\infty,-a], \ \operatorname{supp} \Xi_2(\omega,\cdot) \subset [-2a,2a], \ \operatorname{supp} \Xi_3(\omega,\cdot) \subset [a,\infty];$$

$$\sup_{t\in\mathbb{R}} \frac{|\Xi_i(\omega,t)|}{M_{\tilde{d}_i}(t)} < \infty, \ i = 1,2,3;$$

$$\xi(\omega,\cdot) = \Xi_1(\omega,\cdot)^{(s_1)} + \Xi_2(\omega,\cdot)^{(s_2)} + \Xi_3(\omega,\cdot)^{(s_3)}\ , \ \omega \in M. \tag{2.64}$$

b) In particular, if there exist $D(\omega) > 0$, $\omega \in \Omega$ and $d \in \mathbb{N}_0$ such that

$$|\xi(\omega,\varphi)| \leq D(\omega)\|\varphi\|_{d,2}\ , \quad \omega \in \Omega, \ \varphi \in \mathcal{K}\{M_p\}, \tag{2.65}$$

then representation (2.64) is valid on the whole Ω.

c) Let ξ be as in part a) and $\varepsilon \in (0,1)$. Let $C(\omega)$, $\omega \in \Omega$, be a Gaussian random variable. The GRP ξ has the point value $C(\omega)$ at $t = t_0$, denoted by $\xi(\omega,t_0) = C(\omega)$, $\omega \in \Omega$, if and only if there exist $M \in \mathfrak{F}$ such that $P(M) \geq 1-\varepsilon$, $s_3, d_3 \in \mathbb{N}$, $a \in \mathbb{R}$ and a Gaussian process $\Xi_3(\omega,t)$, $(\omega,t) \in \Omega \times \mathbb{R}$ with following properties:

$$\operatorname{supp} \Xi_3(\omega,\cdot) \subset [t_0-2a,\infty),$$

$$\xi(\omega,\cdot) = \Xi_3(\omega,\cdot)^{(s_3)}, \ \omega \in M, \ \textit{in a neighborhood of } t_0,$$

$$\sup_{t\in\mathbb{R}} \frac{|\Xi_3(\omega,t)|}{M_{d_3}(t)} < \infty,$$

and

$$\lim_{t\to t_0} \frac{\Xi_3(\omega,t)}{(t-t_0)^{s_3}} = \frac{C(\omega)}{s_3!}, \quad \omega \in M. \tag{2.66}$$

In particular, if there exist $D \in Z^2$, $D(\omega) > 0$, $\omega \in \Omega$ and $d \in \mathbb{N}_0$ such that

$$|\xi(\omega,\varphi)| \leq D(\omega)\|\varphi * f_{d_3+\frac{1}{2}}\|_{d,2}\ , \quad \omega \in \Omega, \ \varphi \in \mathcal{K}\{M_p\}, \tag{2.67}$$

then (2.66) is equivalent to

$$\lim_{t\to t_0} \left\| \frac{\Xi_3(\omega,t)}{(t-t_0)^{s_3}} - \frac{C(\omega)}{s_3!} \right\|_{Z^2} = 0. \tag{2.68}$$

Proof. a) Let ψ_1, ψ_2 and ψ_3 be smooth functions on $\mathbb{R}^d$ such that supp $\psi_1 \subset (-\infty, -a]$, supp $\psi_2 \subset [-2a, 2a]$ and supp $\psi_3 \subset [a, \infty)$, $a > 0$, $\psi_1 = 1$ on $(-\infty, -2a)$, $\psi_3 = 1$ on $(2a, \infty)$ and $\psi_1 + \psi_2 + \psi_3 = 1$. With this partition of unity we obtain Gaussian stochastic processes

$$\xi_1(\omega, \phi) = \xi(\omega, \phi\psi_1), \quad \phi \in \mathcal{K}\{M_p\},$$

$$\xi_2(\omega, \phi) = \xi(\omega, \phi\psi_2), \quad \phi \in \mathcal{K}\{M_p\},$$

$$\xi_3(\omega, \phi) = \xi(\omega, \phi\psi_3), \quad \phi \in \mathcal{K}\{M_p\}.$$

With the same proof as of Theorem 2.5.2 a),we have the existence of $d_i \in \mathbb{N}_0$, a set $A_{i,d}$ such that $P(A_{i,d}) \geq 1 - \varepsilon/3$, $M^i = A_{i,d}$, $i = 1, 2, 3$, such that

$$|\xi_i(\omega, \phi)| \leq d_i \|\phi\|_{d_i,2}\ ,\ \omega \in M^i,\ \phi \in \mathcal{K}\{M_p\},\ i = 1, 2, 3.$$

We extend ξ_i out of M_i to be equal to zero, denote this extension with $\tilde{\xi}_i$, $i = 1, 2, 3$, and (as in the proof of part a)) we have the existence of measurable functions $S_i(\omega), \omega \in \Omega$ such that $S_i(\omega) \leq d_i, \omega \in \Omega$, $i = 1, 2, 3$ and

$$|\tilde{\xi}_i(\omega, \phi)| \leq S_i(\omega)\|\phi\|_{d_i,2},\ \omega \in \Omega,\ \phi \in \mathcal{K}\{M_p\},\ i = 1, 2, 3. \tag{2.69}$$

Denote by $\mathcal{K}_d$ the completion of $\mathcal{K}\{M_p\}$ with respect to the norm $\|\cdot\|_{d,2}$. Since the limit of a sequence of Gaussian processes is Gaussian, we extend (2.69) to elements of $\mathcal{K}_{d_i}$ and obtain

$$|\tilde{\xi}_i(\omega, \phi)| \leq S_i(\omega)\|\phi\|_{d_i,2},\ \omega \in \Omega,\ \phi \in \mathcal{K}_{d_i},\ i = 1, 2, 3.$$

In the sequel we consider $\tilde{\xi}_3$. Let f_s be given as in (1.4) and let $\kappa \in C^\infty(\mathbb{R})$, supp $\kappa \subset [-a, \infty)$, $\kappa \geq 0, \kappa(t) = 1, t \geq 0$.

We have that for $s_3 = d_3 + 2$, and every $x \in \mathbb{R}$,

$$t \mapsto \kappa(t) f_{s_3}^{x-}(t) = \kappa(t) f_{s_3}(x - t) \in \mathcal{K}_{d_3}^-$$

and moreover, $x \mapsto \kappa f_{s_3}^{x-}$, $x \in \mathbb{R}$, is continuous, because

$$\kappa f_{s_3}^{x-} \to \kappa f_{s_3}^{x_0-} \text{ in } \mathcal{K}_{d_3}^-,\ \text{as } x \to x_0.$$

So, for every $\omega \in \Omega,\ x \in \mathbb{R}$, we have

$$\tilde{\xi}_3(\omega, \cdot) * \delta(x) = \tilde{\xi}_3(\omega, \cdot) * f_{s_3}^{(s_3)}(x) = \frac{d^{s_3}}{dx^{s_3}}\, \tilde{\xi}_3(\omega, \kappa f_{s_3}^{x-}).$$

For every $x \in \mathbb{R}$,

$$\omega \mapsto \tilde{\xi}_3(\omega, \kappa f_{s_3}^{x-})$$

is a Gaussian random variable and for every $\omega \in \Omega$ we have that

$$x \mapsto \tilde{\xi}_3(\omega, \kappa f_{s_3}^{x-}), \quad x \in \mathbb{R},$$

is continuous. Denote it by Ξ_3, i.e. $\Xi_3(\omega, x) = \tilde{\xi}_3(\omega, \kappa f_{s_3}^{x-})$. By (1.5) we have

$$\Xi_3(\omega, \cdot)^{(s_3)} = \tilde{\xi}_3(\omega, \cdot), \quad \omega \in \Omega.$$

From

$$|\tilde{\xi}_3(\omega, \kappa f_{s_3}^{x-})| \le S_3(\omega) \|\kappa f_{s_3}^{x-}\|_{d_3,2}\,, \quad \omega \in \Omega$$

and

$$\begin{aligned}
\|\kappa f_{s_3}^{x-}\|_{d_3,2}^2 &= \sup_{i \le d_3} \int_{-a}^{x} |M_{d_3}(t)[\kappa(t) f_{s_3}(x-t)]^{(i)}|^2 dt \\
&= \sup_{i \le d_3} \int_0^{x+a} \left| M_{d_3}(x-u) \left[\frac{\kappa(x-u) u^{d_3+1}}{\Gamma(d_3+2)} \right]^{(i)} \right|^2 du \\
&\le C_1 \cdot M_{\tilde{d}_3}^2(x) \sup_{i \le d_3} \int_0^{x+a} \left| \frac{1}{M_{d_3}(u)} \sum_{n=0}^{i} \binom{i}{n} \left[\frac{u^{d_3+1}}{\Gamma(d_3+2)} \right]^{(n)} \right|^2 du
\end{aligned}$$

we have

$$\sup_{x \in \mathbb{R}} \frac{|\Xi_3(\omega, x)|}{M_{\tilde{d}_3}(x)} \le C_2 \sqrt{ \sup_{i \le d_3} \int_{\mathbb{R}} \left| \frac{1}{M_{d_3}(u)} \left[\frac{u^{d_3+1}}{\Gamma(d_3+2)} \right]^{(i)} \right|^2 du } < \infty,$$

for some constants $C_1, C_2 > 0$.

In a similar way we construct continuous Gaussian processes Ξ_1 and Ξ_2 such that $\Xi_1(\omega, x) = \tilde{\xi}_1(\omega, f_{s_1}^{x+})$, $\Xi_2(\omega, x) = \tilde{\xi}_2(\omega, f_{s_2}^{x+})$, $\omega \in \Omega$, $x \in \mathbb{R}$, and for every $\omega \in \Omega$ the functions $x \mapsto \Xi_i(\omega, x)$, $x \in \mathbb{R}$, $i = 1, 2, 3$, are of $M_{\tilde{d}_1}, M_{\tilde{d}_2}$ and $M_{\tilde{d}_3}$ growth rate respectively.

This completes the proof of the theorem, since

$$\xi(\omega, \cdot) = \Xi_1(\omega, \cdot)^{(s_1)} + \Xi_2(\omega, \cdot)^{(s_2)} + \Xi_3(\omega, \cdot)^{(s_3)}, \quad \omega \in M = \bigcap_{i=1}^{3} M^i.$$

b) The proof is similar as in Theorem 2.5.2.

c) Without loss of generality, we prove the assertion only for $t_0 = 0$. Using a similar partition of unity as in part a) but choosing supp $\psi_1 \subset (-\infty, -3a]$, supp $\psi_2 \subset [-4a, -a]$ and supp $\psi_3 \subset [-2a, \infty)$, we obtain a representation

$$\xi(\omega, \cdot) = \Xi_1(\omega, \cdot)^{(s_1)} + \Xi_2(\omega, \cdot)^{(s_2)} + \Xi_3(\omega, \cdot)^{(s_3)},\ \omega \in M.$$

Since we are interested in values in the neighborhood of $t_0 = 0$, we restrict our attention to $\Xi_3(\omega, \cdot)$, where supp $\Xi_3(\omega, \cdot) \subset [-2a, \infty)$, and $\Xi_3(\omega, x) = \tilde{\xi}_3(\omega, f_{s_3}^{x-}\kappa)$ as in part a). Thus,

$$\lim_{\varepsilon\to 0}\langle \xi(\omega, \varepsilon x), \phi(x)\rangle = \lim_{\varepsilon\to 0}\sum_{i=1}^{3}\langle \tilde{\xi}_i(\omega, \varepsilon x), \phi(x)\rangle = \lim_{\varepsilon\to 0}\langle \tilde{\xi}_3(\omega, \varepsilon x), \phi(x)\rangle.$$

We prove now that

$$\lim_{\varepsilon\to 0}\langle \tilde{\xi}_3(\omega, \varepsilon x), \phi(x)\rangle = C(\omega)\int_{\mathbb{R}}\phi(x)dx, \quad \omega \in M, \phi \in \mathcal{K}\{M_p\} \tag{2.70}$$

is equivalent to

$$\lim_{x\to 0}\frac{\Xi_3(\omega, x)}{x^{s_3}} = \frac{C(\omega)}{s_3!},\ \omega \in M. \tag{2.71}$$

Assume that (2.70) holds. Then, for $\phi(x) = f_{s_3}^{x-}$, $x \in \mathbb{R}$, we obtain

$$\lim_{x\to 0}\frac{\Xi_3(\omega, x)}{x^{s_3}} = \lim_{x\to 0}\langle \tilde{\xi}_3(\omega, t), \frac{1}{x\Gamma(s_3)}(1-\frac{t}{x})^{s_3-1}\rangle$$

$$= \lim_{x\to 0}\langle \tilde{\xi}_3(\omega, tx), \frac{1}{\Gamma(s_3)}(1-t)^{s_3-1}\rangle = C(\omega)\int_0^1\frac{1}{\Gamma(s_3)}(1-t)^{s_3-1}dt = \frac{C(\omega)}{s_3!}.$$

Conversely, assume (2.71) holds. Then, for $\phi \in \mathcal{K}\{M_p\}$ we have

$$\lim_{\varepsilon\to 0}\langle \tilde{\xi}_3(\omega, \varepsilon x), \phi(x)\rangle = \lim_{\varepsilon\to 0}\langle \tilde{\xi}_3(\omega, x), \frac{1}{\varepsilon}\phi(\frac{x}{\varepsilon})\rangle$$

$$= \lim_{\varepsilon\to 0}\langle \Xi_3(\omega, x)^{(s_3)}, \frac{1}{\varepsilon}\phi(\frac{x}{\varepsilon})\rangle = \lim_{\varepsilon\to 0}(-1)^{s_3}\langle \Xi_3(\omega, x), \frac{1}{\varepsilon^{s_3+1}}\phi^{(s_3)}(\frac{x}{\varepsilon})\rangle$$

$$= \lim_{\varepsilon\to 0}(-1)^{s_3}\langle \frac{\Xi_3(\omega, \varepsilon x)x^{s_3}}{\varepsilon^{s_3}x^{s_3}}, \phi^{(s_3)}(x)\rangle = (-1)^{s_3}\langle \frac{C(\omega)x^{s_3}}{s_3!}, \phi^{(s_3)}(x)\rangle$$

$$= C(\omega)\langle \left(\frac{x^{s_3}}{s_3!}\right)^{(s_3)}, \phi(x)\rangle = C(\omega)\int_{\mathbb{R}}\phi(x)dx.$$

Thus, ξ has a point value in sense of Lojasiewicz at $t_0 = 0$ if and only if (2.66) holds. If condition (2.67), and thus also (2.65), is satisfied and $D \in Z^2$, then ξ is a continuous mapping $\mathcal{K}\{M_p\} \to Z^2$. Also, from (2.67) we have

$$|\frac{\Xi_3(\omega, x)}{x^{s_3}}| = \frac{|\tilde{\xi}_3(\omega, \kappa f_{s_3}^{x-})|}{|x|^{s_3}} \leq \frac{D(\omega)}{|x|^{s_3}}\|\kappa f_{s_3+d_3+\frac{1}{2}}^{x-}\|_{d,2}$$

$$= \frac{D(\omega)}{|x|^{d_3+2}} \sqrt{\sup_{i \le d_3} \int_{-2a}^{x} \left| M_d(t) \left(\kappa(t) \frac{(x-t)^{2d_3+\frac{3}{2}}}{\Gamma(2d_3+\frac{5}{2})} \right)^{(i)} \right|^2 dt}$$

$$\le K_1 \frac{D(\omega)}{|x|^{d_3+2}} \sqrt{\int_{-2a}^{x} (x-t)^{2d_3+3} dt} \le K_2 \cdot D(\omega),$$

for some constants $K_1, K_2 > 0$ and $x \to 0$. Thus, the vector-valued version (2.68) follows from the Lebesgue dominated convergence. □

Now, concerning equation (2.52), we will assume that the GRP on the right hand side of the equation has a primitive with a point value at $t = 0$ in sense of Lojasiewicz.

Corollary 2.5.1 *Let f be a Gaussian GRP $f : \mathcal{K}\{M_p\} \to Z^2$ such that its primitive has a point value at $t = 0$ in sense of Lojasiewicz. Let $X(\omega)$, $\omega \in \Omega$ be a Gaussian random variable. Then for every $\varepsilon > 0$ there exists $M \in \mathcal{F}$ satisfying $P(M) \ge 1 - \varepsilon$, and there exists a Gaussian GRP $y : \mathcal{K}\{M_p\} \to Z^2$ such that for every $\omega \in M$*

$$\frac{d}{dt} y(\omega, \cdot) = f(\omega, \cdot) \text{ and } y(\omega, 0) = X(\omega).$$

Proof. Theorem 2.5.3 c) ensures the existence of $F(\omega, t)$, $\omega \in \Omega$, $t \in [-2, \infty)$ and $k \in \mathbb{N}$ such that

$$f(\omega, \cdot) = F(\omega, \cdot)^{(k)}, \ \omega \in M,$$

where $F(\omega, \cdot)$ is continuous for every $\omega \in \Omega$, and $F(\cdot, t)$ is Gaussian for every $t \in [-2, \infty)$. From the assumption on the initial point $t = 0$ we get that $\frac{F(\omega, t)}{t^k}$ must be bounded in a neighborhood of zero. Define

$$y(\omega, t) = \left(\frac{X(\omega) t^k}{k!} + \int_0^t F(\omega, s) ds \right)^{(k)}, \quad \omega \in M, \ t \in [-2, \infty). \tag{2.72}$$

Let us check now if y is a solution of (2.52) on M. Indeed, from (2.72) we have $\frac{d}{dt} y(\omega, t) = \left(\int_0^t F(\omega, s) ds \right)^{(k+1)} = F(\omega, t)^{(k)} = f(\omega, t)$. Thus, y is a primitive for f, and therefore it has a point value at $t = 0$. Also,

$$\lim_{t \to 0} \frac{1}{t^k} \left(\frac{X(\omega) t^k}{k!} + \int_0^t F(\omega, s) ds \right) = \frac{X(\omega)}{k!} + \lim_{t \to 0} \frac{1}{t^k} \int_0^t F(\omega, s) ds$$

$$= \frac{X(\omega)}{k!} + \lim_{t\to 0} \frac{1}{kt^{k-1}} F(\omega,t) = \frac{X(\omega)}{k!} + \lim_{t\to 0} \frac{t}{k} \frac{F(\omega,t)}{t^k} = \frac{X(\omega)}{k!}.$$

This means $y(\omega, 0) = X(\omega)$. □

Note that the solution given in (2.72) coincides with the solution given in (2.53) up to the set M, provided (2.54) holds and thus (2.53) exists.

2.5.3 Examples of GRPs

In the first four examples which are to follow we let $\Omega = \mathbb{R}$, $\mathcal{F}$ be the Borel field, $P(A) = \int_A \frac{dx}{\pi(1+x^2)}$ for $A \in \mathcal{F}$, and X be the identity mapping from $\mathbb{R}$ to $\mathbb{R}$. As usual, δ denotes the Dirac delta distribution.

Example 2.5.2 *Let, for $x \in \mathbb{R}$, $\varphi \in \mathcal{K}\{M_p\}$,*

$$\xi(x, \varphi) = X(x)\langle\delta, \varphi\rangle.$$

It is a GRP of type (II) but not of type (I).

Example 2.5.3 *Let, for $x \in \mathbb{R}$, $\varphi \in \mathcal{K}\{M_p\}$,*

$$\xi(x, \varphi) = X(x)\varphi(X(x)) = X(x)\langle\delta(y - X(x)), \varphi(y)\rangle. \tag{2.73}$$

It is a GRP (II) and moreover, it is a continuous mapping from $\mathcal{K}\{M_p\}$ to Z^2, i.e. it is also a GRP (I). For every $x \in \mathbb{R}$ we have

$$|\xi(x, \varphi)| = |X(x)\varphi(x)| \leq |X(x)|\|\varphi\|_{p,2}\ , \tag{2.74}$$

although $|X| \notin Z^2$.

Example 2.5.4 *Let*

$$\xi(x, \varphi) = \begin{cases} X(x)\varphi(\frac{1}{X(x)}), & x \neq 0 \\ 0, & x = 0 \end{cases}, x \in \mathbb{R},\ \varphi \in \mathcal{K}\{M_p\}.$$

This is a GRP (II) which does not map $\mathcal{K}\{M_p\}$ to Z^2, since if the support of a test function ϕ equals $[0, a]$ for some $a > 0$, then for $\frac{1}{x}$ being in this interval, one has $1/X(x) = 1/x \in [0, a)$ and $X(x) = x \in [1/a, \infty)$, thus $\xi(\cdot, \phi)$ is not in Z^2.

Example 2.5.5 *We construct a GRP (II) which does not satisfy (2.58), and thus can not be represented as a sum of continuous processes on a set of probability measure one, only on a set of arbitrary large probability measure.*

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers strictly increasing to infinity. For $x \in \mathbb{R}$, $\varphi \in \mathcal{K}\{M_p\}$, put

$$\xi(x,\varphi) = \begin{cases} \dfrac{1}{1+X^2(x)}\varphi(0), & x \neq x_n,\ n \in \mathbb{N}, \\ \varphi^{(n)}(0), & x = x_n,\ n \in \mathbb{N}. \end{cases}$$

For every $\varepsilon > 0$, this is a continuous mapping from $\mathcal{K}\{M_p\}$ to Z^2 on $x \in \mathbb{R}\backslash(-\varepsilon,\varepsilon)$. There do not exist $C(x) > 0$ and $p \in \mathbb{N}$ such that (2.58) holds, because it would imply that for every $x \in \mathbb{R}$ the corresponding distribution is of finite order, while $f = \xi(x_n,\cdot) = (-1)^n\delta^{(n)}$, $n \in \mathbb{N}$.

In the following examples we consider the space of tempered distributions $\mathcal{S}'(\mathbb{R})$ and $Z^2 = L^2(\mathcal{S}'(\mathbb{R}),\mathcal{B},\mu)$ as defined in Section 1.7.1.

Example 2.5.6 *Two examples of GRPs (I) which are also GRPs (II) are singular pointwise one-dimensional one-parameter white noise, and Brownian motion. Recall from Example 1.7.1 and Example 1.7.2 , they are given respectively, by*

$$W(\omega,x) = \sum_{k=1}^{\infty}\xi_k(x)H_{\varepsilon_k}(\omega),\ \ \omega \in \mathcal{S}'(\mathbb{R}),\ x \in \mathbb{R}$$

and

$$B(\omega,x) = \sum_{k=1}^{\infty}\int_0^x \xi_k(t)dtH_{\varepsilon_k}(\omega),\ \ \omega \in \mathcal{S}'(\mathbb{R}),\ x \in \mathbb{R}$$

where $\{\xi_n, n \in \mathbb{N}_0\}$ is the Hermite orthonormal basis of $L^2(\mathbb{R})$.

Actually, they act on $\mathcal{S}(\mathbb{R})$ as follows:

$$\mathcal{S}(\mathbb{R}) \ni \varphi \mapsto \langle W(\omega,x),\varphi(x)\rangle = \sum_{k=1}^{\infty}\int_{\mathbb{R}}\xi_k(x)\varphi(x)dxH_{\varepsilon_k}(\omega).$$

$$\mathcal{S}(\mathbb{R}) \ni \varphi \mapsto \langle B(\omega,x),\varphi(x)\rangle = \sum_{k=1}^{\infty}\int_{\mathbb{R}}\int_0^x \xi_k(t)dt\varphi(x)dxH_{\varepsilon_k}(\omega).$$

Since for fixed $\omega \in \Omega$ we have $\frac{d}{dx}B(\omega,x) = W(\omega,x)$ in $\mathcal{S}'(\mathbb{R})$ and $W(\omega,x)$ is continuous with respect to $x \in \mathbb{R}$ (see [HØUZ]), this is indeed compatible with the assertion in Theorem 3.5.3, i.e. continuity ensures representation on the whole Ω.

Another process described in [HØUZ] and [HKPS] is the coordinate process of white noise defined as

$$W_\phi(\omega, x) = \langle \omega(y), \phi(y-x) \rangle = \sum_{k=1}^{\infty} \int_{\mathbb{R}} \xi_k(y)\varphi(y-x) dy H_{\varepsilon_k}(\omega),$$

where $x \in \mathbb{R},\ \omega \in \mathcal{S}'(\mathbb{R}),\ \phi \in \mathcal{S}(\mathbb{R})$. *Note that the same process is obtained if we apply our partition of unity from Theorem 2.5.3.*

In a similar manner we can define a process

$$B_\phi(\omega, x) = \sum_{k=1}^{\infty} \int_{\mathbb{R}} (\int_0^y \xi_k(t)dt)\varphi(y-x) dy H_{\varepsilon_k}(\omega),\ x \in \mathbb{R},\ \omega \in \mathcal{S}'(\mathbb{R}),\ \phi \in \mathcal{S}(\mathbb{R}).$$

Since both $W_\phi(\omega, x)$ *and* $B_\phi(\omega, x)$ *belong to* Z^2, *we have in classical sense* $\frac{d}{dx} B_\phi(\omega, x) = W_\phi(\omega, x)$. *In notation of Theorem 2.5.3 (see the proof after applying the partition of unity) this means*

$$W(\omega, \tilde{\phi}_x) = \frac{d}{dx} B(\omega, \tilde{\phi}_x), \quad \textit{where} \quad \tilde{\phi}_x(y) = f_1^{x-}(y) = H(x-y).$$

This means that in distributional sense i.e. in $\mathcal{S}'(\mathbb{R})$ *we obtain again the well-known result* $\frac{d}{dx} B(\omega, x) = W(\omega, x)$.

2.6 Hilbert Space Valued Generalized Random Processes of Type (II)

Recall that H is a separable Hilbert space over $\mathbb{C}$ with orthonormal basis $\{e_n : n \in \mathbb{N}\}$. While for GRPs (I) on nuclear spaces we had $\mathcal{L}(\mathcal{A}(H); (S)_{-1}) \cong \mathcal{L}(\mathcal{A}; S(H)_{-1})$, i.e. it was equivalent whether H was the codomain of the x–variable function space or the ω–variable function space, for GRPs (II) we have a different situation. Now we state the definition of a Hilbert space valued GRP (II) and the corresponding structure theorem for it. We will restrict our attention to GRPs on $\mathcal{K}\{M_p\}$ spaces.

Definition 2.6.1 *A* H*–valued GRP (II) is a mapping* $\xi : \Omega \times \mathcal{K}\{M_p\}(H) \to \mathbb{C}$ *such that:*

(i) for every $\varphi \in \mathcal{K}\{M_p\}(H)$, $\xi(\cdot, \varphi)$ *is a complex random variable,*

(ii) for every $\omega \in \Omega$, $\xi(\omega, \cdot)$ *is an element in* $\mathcal{K}'\{M_p\}(H)$.

For $r > 1$ denote $L^r(\mathbb{R}^n; H) = L^r(\mathbb{R}^n) \otimes H$ and recall that its dual is $L^p(\mathbb{R}^n; H)$, $p = r/(r-1)$. The dual pairing of $f \in L^p(\mathbb{R}^n; H)$, $\varphi \in L^r(\mathbb{R}^n; H)$ can be written as $\int_{\mathbb{R}^n} \langle f(t), \varphi(t)\rangle_H dt$. It can easily be checked that following H–valued version of Theorem 2.5.1 holds.

Theorem 2.6.1 *Let $G = \prod_{i=1}^n (\alpha_i, \beta_i) \subset \mathbb{R}^n$, $-\infty \le \alpha_i < \beta_i \le \infty$, $i = 1, 2, ..., n$, and let ξ be a GRP on $\Omega \times L^r(G; H)$, $r > 1$. There exists a function $f : \Omega \times G \to H$ such that*

(i) for every $x \in G$, $f(\cdot, x)$ is measurable and for every $\omega \in \Omega$, $f(\omega, \cdot) \in L^p(G; H)$, $p = r/(r-1)$.

(ii)

$$\xi(\omega, \varphi) = \int_G \langle f(\omega, t), \varphi(t)\rangle_H dt, \quad \omega \in \Omega, \quad \varphi \in L^r(G; H).$$

Following H–valued analogue of Theorem 2.5.2 holds:

Theorem 2.6.2 *a) Let ξ be a H–valued GRP (II). Then for every $\varepsilon > 0$ there exist $d \in \mathbb{N}_0$, $M \in \mathcal{F}$ satisfying $P(M) \ge 1 - \varepsilon$, and functions $f_\alpha : \Omega \times \mathbb{R} \to H$, $\alpha = 0, 1, ..., d$, such that $f_\alpha(\cdot, t)$ is measurable for every $t \in \mathbb{R}$, $f_\alpha(\omega, \cdot)$ is in $L^2(\mathbb{R}; H)$ for every $\omega \in M$, $\alpha = 0, 1, ..., d$ and*

$$\xi(\omega, \varphi) = \sum_{\alpha=0}^{d} \int_{\mathbb{R}} \langle f_\alpha(\omega, t), M_d(t)\varphi^{(\alpha)}(t)\rangle_H dt, \quad \omega \in M, \ \ \varphi \in \mathcal{K}\{M_p\}(H), \tag{2.75}$$

$$\sum_{\alpha=0}^{d} \|f_\alpha(\omega, \cdot)\|_{L^2(\mathbb{R}; H)} \le d, \quad \omega \in M. \tag{2.76}$$

In particular, if there exist $C(\omega) > 0$, $\omega \in \Omega$, and $d \in \mathbb{N}$ such that

$$|\xi(\omega, \varphi)| \le C(\omega)\|\varphi\|_{d,2;H}, \ \omega \in \Omega, \ \varphi \in \mathcal{K}\{M_p\}(H), \tag{2.77}$$

then representation (2.75) is valid on the whole Ω.

b) Moreover, if ξ is also a continuous mapping from $\mathcal{K}\{M_p\}(H)$ to Z^2, then for almost every $t, s \in \mathbb{R}$ there exist $E(\langle f_\alpha(\cdot, t), \overline{f_\beta(\cdot, s)}\rangle_H)$, $\alpha \le d$, $\beta \le d$ and the correlation operator $C_\xi(\varphi, \psi)$, $\varphi, \psi \in \mathcal{K}\{M_p\}$ has the representation

$$C_\xi(\varphi, \psi)$$

$$= \sum_{\alpha=0}^{d} \sum_{\beta=0}^{d} \int_{\mathbb{R}} \int_{\mathbb{R}} E(\langle f_\alpha(\cdot, t), \overline{f_\beta(\cdot, s)}\rangle_H) M_d(t)\overline{M_d(s)} \langle \varphi^{(\alpha)}(t), \overline{\psi^{(\beta)}(s)}\rangle_H dt ds.$$

c) If ξ is a GRP on $\mathcal{K}\{M_p\}(H)$ such that (2.77) holds and $\omega \mapsto C(\omega)$ is in Z^2, then $\xi : \mathcal{K}\{M_p\}(H) \to Z^2$ is continuous and (2.75) holds for every $\omega \in \Omega$. Condition $C(\cdot) \in Z^2$ is sufficient but not necessary for the continuity of $\xi : \mathcal{K}\{M_p\}(H) \to Z^2$.

Proof. a) Since for every $\omega \in \Omega$, $\xi(\omega, \cdot)$ is in $\mathcal{K}'\{M_p\}(H)$, it follows that for every $\omega \in \Omega$ there exist $C(\omega) > 0$ and $p(\omega) \in \mathbb{N}$ such that

$$|\xi(\omega, \varphi)| \leq C(\omega)\|\varphi\|_{p(\omega),2;H}, \ \varphi \in \mathcal{K}\{M_p\}(H).$$

We can assume that $p(\omega) \geq C(\omega)$. For every $\varphi \in \mathcal{K}\{M_p\}(H)$ and $N \in \mathbb{N}$, put

$$A_N(\varphi) = \{\omega \in \Omega : \ |\xi(\omega, \varphi)| < N\|\varphi\|_{N,2;H}\}, \ \ A_N = \bigcap_{\varphi \in \mathcal{K}\{M_p\}(H)} A_N(\varphi).$$

Since $\mathcal{K}\{M_p\}(H)$ is separable, it contains a countable dense subset D and $A_N = \bigcap_{\varphi \in D} A_N(\varphi) \in \mathcal{F}$. Thus, from

$$\Omega = \bigcup_{N=1}^{\infty} A_N \quad \text{and} \quad A_N \subset A_{N+1}, \ \ N \in \mathbb{N},$$

it follows that for given $\varepsilon > 0$ there exists an integer d such that $P(A_d) \geq 1 - \varepsilon$. Denote $M = A_d$. It follows

$$|\xi(\omega, \varphi)| \leq d\|\varphi\|_{d,2;H}, \ \ \omega \in M, \ \varphi \in \mathcal{K}\{M_p\}(H).$$

We extend ξ on the whole Ω by

$$\xi_1(\omega, \varphi) = \begin{cases} \xi(\omega, \varphi), & \omega \in M \\ 0, & \omega \notin M \end{cases}, \ \varphi \in \mathcal{K}\{M_p\}(H). \tag{2.78}$$

Further, put $R = \{\varphi \in \mathcal{K}\{M_p\}(H) : \ \ \|\varphi\|_{d,2;H} \leq 1\}$ and

$$S(\omega) = \sup_{\varphi \in R} |\xi_1(\omega, \varphi)| = \sup_{\varphi \in D \bigcap R} |\xi_1(\omega, \varphi)|, \ \omega \in \Omega.$$

It follows that S is measurable on Ω, $S(\omega) \leq d, \omega \in \Omega$. Thus,

$$|\xi_1(\omega, \varphi)| \leq S(\omega)\|\varphi\|_{d,2;H}, \ \ \varphi \in \mathcal{K}\{M_p\}(H), \ \omega \in \Omega. \tag{2.79}$$

Inequality (2.79) holds also for the space $H^d_M(\mathbb{R}; H) \subset H^d(\mathbb{R}; H)$, where $H^d(\mathbb{R}; H) \cong H^d(\mathbb{R}) \otimes H$ is the H–valued Sobolev space, and $H^d_M = \{\varphi \in$

$H^d(\mathbb{R}; H) : M_d\varphi^{(\alpha)} \in L^2(\mathbb{R}; H),\ \alpha = 0, 1, \ldots, d\}$, equipped with the topology induced by the norm $\|\varphi\|_{d,L^2;H} = \sum_{\alpha=0}^{d} \|M_d\varphi^{(\alpha)}\|_{L^2(\mathbb{R};H)}$.

We need the following consequence of (2.79):

$$\text{if } (\varphi_\nu)_{\nu\in\mathbb{N}} \text{ is a sequence in } \mathcal{K}\{M_p\}(H) \text{ and } \varphi_\nu \to 0 \text{ in } H_M^d, \tag{2.80}$$

$$\text{then } \xi_1(\omega, \varphi_\nu) \to 0, \nu \to \infty.$$

Let $\Gamma_d = \prod_{i=0}^{d} L^2(\mathbb{R}; H)$ and endow it with the scalar product $((\varphi_\alpha), (\psi_\alpha)) = \sum_{\alpha=0}^{d} \int_{\mathbb{R}} \langle \varphi_\alpha, \overline{\psi_\alpha}\rangle_H dt$, $(\varphi_\alpha), (\psi_\alpha) \in \Gamma_d$. Clearly, Γ_d is a Hilbert space. Define a mapping $\theta : \mathcal{K}\{M_p\}(H) \to \Gamma_d$ by $\theta(\varphi) = (M_d\varphi, M_d\varphi', \ldots, M_d\varphi^{(d)})$, $\varphi \in \mathcal{K}\{M_p\}(H)$, which is injective, and denote $\Delta = \theta(\mathcal{K}\{M_p\}(H))$. Note that

$$\overline{\Delta} = \theta(H_M^d). \tag{2.81}$$

Define a mapping $\Omega \times \Gamma_d \to \mathbb{C}$, for every $\omega \in \Omega$, by

$$F(\omega, \psi) = \begin{cases} \xi_1(\omega, \theta^{-1}(\psi)), & \psi \in \Delta \\ \lim_{\nu\to\infty} \xi_1(\omega, \theta^{-1}(\psi_\nu)), & \psi \in \bar{\Delta}, \quad \psi_\nu \in \Delta, \quad \psi_\nu \overset{L^2(\mathbb{R};H)}{\longrightarrow} \psi, \\ 0, & \psi \in \bar{\Delta}^\perp. \end{cases}$$

The existence of the limit follows from (2.80) and (2.81). Thus,

$$F(\omega, \tilde{\psi}) = F(\omega, \psi),\ \tilde{\psi} \in \Gamma_d,\ \ \tilde{\psi} = \psi + \psi^\perp,\ \psi \in \bar{\Delta}, \psi^\perp \in \bar{\Delta}^\perp.$$

Clearly $F(\cdot, \tilde{\psi})$ is measurable, for any $\tilde{\psi} \in \Gamma_d$. Let $\varphi \in \mathcal{K}\{M_p\}(H)$, $\omega \in \Omega$. We have

$$|F(\omega, \theta(\varphi))| \le S(\omega)\|\varphi\|_{d,2;H} = S(\omega)\|\theta(\varphi)\|_{\Gamma_d}.$$

So, for every $\omega \in \Omega$, $F(\omega, \cdot)$ is a continuous linear functional on Γ_d and it is of the form

$$F(\omega, \cdot) = \sum_{\alpha=0}^{d} F_\alpha(\omega, \cdot),\ \omega \in \Omega.$$

Here $F_\alpha(\omega, \cdot)$, $\in \Omega$, $\alpha \le d$, are continuous linear functionals on subspaces $\Gamma_{d,\alpha} \subset \Gamma_d, \alpha \le d$, where

$$\Gamma_{d,\alpha} = \{\psi = (\psi_\beta) \in \Gamma_d :\ \psi_\beta \in L^2(\mathbb{R}; H),\ \psi_\beta \equiv 0,\ \beta \ne \alpha\}.$$

It is endowed with the natural norm such that it is isometric to $L^2(\mathbb{R}; H)$, for every $\alpha \in \{0, 1, ..., d\}$. Let $\psi \in \Gamma_d$. Denote by $[\psi]_\alpha$ the corresponding element in $\Gamma_{d,\alpha}$. The βth coordinates of $[\psi]_\alpha$ are equal to zero for $\beta \ne \alpha$ and the αth coordinate is equal to ψ_α. Since F is a GRP (II), it follows that $F_\alpha = F|_{\Gamma_{d,\alpha}}$ is a GRP (II) on $\Omega \times \Gamma_{d,\alpha}$ i.e. on $\Omega \times L^2(\mathbb{R}; H)$, for every α, $\alpha \leq d$.

By Theorem 2.6.1, for every $\alpha = 0, 1, \dots, d$, there exists a function $f_\alpha : \Omega \times \mathbb{R} \to H$ such that $f_\alpha(\cdot, t)$ is measurable for every $t \in \mathbb{R}$, $f_\alpha(\omega, \cdot) \in L^2(\mathbb{R}; H)$, $\omega \in \Omega$, and

$$F_\alpha(\omega, \varphi) = \int_{\mathbb{R}} \langle f_\alpha(\omega, t), \varphi(t) \rangle_H dt, \quad \varphi \in L^2(\mathbb{R}; H),\ \omega \in \Omega.$$

Thus, if $\omega \in \Omega$ and $\psi = \theta(\varphi)$ for $\varphi \in \mathcal{K}\{M_p\}(H)$, then

$$F(\omega, \psi) = \sum_{\alpha=0}^{d} F_\alpha(\omega, [\psi]_\alpha) = \sum_{\alpha=0}^{d} \int_{\mathbb{R}^n} \langle f_\alpha(\omega, t), M_d(t) \psi^{(\alpha)} \rangle_H dt \tag{2.82}$$

and

$$\|F(\omega, \cdot)\|'_{\Gamma_d} = \sum_{\alpha=0}^{d} \|f_\alpha(\omega, \cdot)\|_{L^2(\mathbb{R}; H)} \leq S(\omega) \leq d, \quad \omega \in \Omega,$$

where $\| \cdot \|'_{\Gamma_d}$ is the dual norm. Now, the assertion follows by (2.78).

The proof of the last assertion in a) follows by repeating the previous proof starting from relation (2.79). It follows that ξ is of the form (2.56) for every $\omega \in \Omega$.

b) Obviously, $C_\xi(\varphi, \psi) = E(\langle \xi(\cdot, \varphi), \overline{\xi(\cdot, \psi)} \rangle_H), \varphi, \psi \in \mathcal{K}\{M_p\}(H)$ is bilinear. The continuity follows from

$$C_\xi(\varphi, \psi) = |E(\langle \xi(\cdot, \varphi), \overline{\xi(\cdot, \psi)} \rangle_H)| \leq \|\xi(\cdot, \varphi)\|_{Z^2} \|\xi(\cdot, \psi)\|_{Z^2}$$

$$\leq \|\varphi\|_{d,2;H} \|\psi\|_{d,2;H} \sup\{\|\xi(\cdot, \varphi)\|_{Z^2}, \varphi \in \mathcal{K}\{M_p\}(H), \|\varphi\|_{d,2;H} \leq 1\}$$
$$\cdot \sup\{\|\xi(\cdot, \psi)\|_{Z^2}, \psi \in \mathcal{K}\{M_p\}(H), \|\psi\|_{d,2;H} \leq 1\}.$$

Fubini's theorem implies

$$C_\xi(\varphi, \psi) = E(\langle \xi(\cdot, \varphi), \overline{\xi(\cdot, \psi)} \rangle_H)$$

$$= E\Big(\Big(\sum_{\alpha=0}^{d} \int_{\mathbb{R}} \langle f_\alpha(\cdot, t), M_d(t) \varphi^{(\alpha)}(t) \rangle_H dt\Big)\Big(\sum_{\alpha=0}^{d} \int_{\mathbb{R}} \langle \overline{f_\alpha(\cdot, s)}, \overline{M_d(s)} \overline{\psi^{(\alpha)}(s)} \rangle_H ds\Big)\Big)$$

$$= \sum_{\alpha=0}^{d} \sum_{\beta=0}^{d} E\Big(\int_{\mathbb{R}} \int_{\mathbb{R}} \langle f_\alpha(\cdot, t), M_d(t) \varphi^{(\alpha)}(t) \rangle_H \langle \overline{f_\beta(\cdot, s)}, \overline{M_d(s)} \overline{\psi^{(\beta)}(s)} \rangle_H dt ds\Big)$$

$$= \sum_{\alpha=0}^{d} \sum_{\beta=0}^{d} \Big(\int_{\mathbb{R}} \int_{\mathbb{R}} E(\langle f_\alpha(\cdot, t), \overline{f_\beta(\cdot, s)} \rangle_H) M_d(t) \overline{M_d(s)} \langle \varphi^{(\alpha)}(t), \overline{\psi^{(\beta)}(s)} \rangle_H dt ds.$$

This proves the last assertion in b).
c) If ξ is a GRP on $\mathcal{K}\{M_p\}(H)$ which satisfies (2.58) and $C(\cdot) \in Z^2$, then ξ is a continuous mapping $\mathcal{K}\{M_p\}(H) \to Z^2$. Namely, for any sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathcal{K}\{M_p\}(H)$ such that $\varphi_n \to 0, n \to \infty$, it follows

$$\|\xi(\cdot, \varphi_n)\|_{Z^2} = E(|\xi(\cdot, \varphi_n)|^2) \leq E(C^2(\cdot))\|\varphi_n\|^2_{d,2;H} \to 0. \qquad \square$$

You see things; and you say, 'Why?'
But I dream things that never were;
and I say, 'Why not?'

(George Bernard Shaw)

Chapter 3

Applications to Singular Stochastic Partial Differential Equations

In this chapter we present some applications of generalized random processes to obtain solutions of some classes of SPDEs where the data and the boundary conditions are modeled by generalized random processes. We begin with a class of linear elliptic equations and note that for technical reasons, the space dimension is now denoted by n instead of d. Throughout the whole chapter, I is assumed to be an open, bounded subset of $\mathbb{R}^n$.

3.1 A Linear Elliptic Dirichlet Problem with Deterministic Coefficients and Stochastic Data

Our aim is to solve the stochastic Dirichlet problem

$$\begin{aligned} L\, u(x,\omega) &= h(x,\omega) + \sum_{i=1}^{n} D_i f^i(x,\omega), \qquad x \in I, \omega \in \Omega, \\ u(x,\omega) \restriction_{\partial I} &= g(x,\omega) \end{aligned} \tag{3.1}$$

where $I \subset \mathbb{R}^n$ is an open set, $(\Omega, \mathcal{F}, P)$ is a probability space, h, g and f^i, $i = 1, 2 \dots n$ are GRPs (I) and L is a linear elliptic operator of the form

$$\begin{aligned} L\,u(x,\cdot) = \sum_{i=1}^{n} D_i\Big(\sum_{j=1}^{n} a^{ij}(x,\cdot)D_j u(x,\cdot) + b^i(x,\cdot)u(x,\cdot)\Big) \\ + \sum_{i=1}^{n} c^i(x,\cdot)D_i u(x,\cdot) + d(x,\cdot)u(x,\cdot). \end{aligned} \tag{3.2}$$

The simplest example is when the coefficients are constants, e.g. for $a^{ij} = \delta_{ij}$ (the Kronecker symbol), $b^i = c^i = d = 0$, $i, j = 1, 2 \dots, n$, we obtain the Laplace operator $L = \Delta$. Note that the operator L in (3.2) is given in divergence form, which will make it suitable to work with in Sobolev spaces in terms of weak derivatives, since its coefficients are singular. If its principal coefficients a^{ij}, b^i, $i, j = 1, 2, \dots, n$ are assumed to be differentiable, then L can also be written in form

$$L\,u = \sum_{i,j=1}^{n} \hat{a}^{ij} D_{ij} u + \sum_{i=1}^{n} \hat{b}^i D_i u + \hat{d} u, \tag{3.3}$$

where $\hat{d} = d + \sum_{i=1}^{n} D_i b^i$ and $\hat{b}^i = b^i + c^i + \sum_{k=1}^{n} D_k a^{ki}$, $i = 1, 2, \dots, n$.

A physical interpretation of equation (3.1) in nondivergent form (3.3) can be given as in [Ev]: u can be interpreted as the density of some quantity (e.g. a chemical concentration) at equilibrium within a region I. The principal term $\sum_{i,j=1}^{n} \hat{a}^{ij} D_i D_j u$ represents the diffusion of u within I, and the coefficients $\hat{a}^{ij}$ describe the anisotropic, heterogeneous nature of the medium (e.g. wood or liquid crystal). The vector $F^i = \sum_{j=1}^{n} a^{ij} D_j u$, $i = 1, 2, \dots, n$ may be interpreted as the flux density. The ellipticity condition will imply $\sum_{j=1}^{n} F^j D_j u \geq 0$, meaning the flow direction is from regions of lower to higher concentration (to get the opposite direction, one has to put a minus sign in front of the coefficients $\hat{a}^{ij}$). The first order term $\sum_{i=1}^{n} \hat{b}^i D_i u$ represents transport within I, while the zeroth order term du describes local creation or depletion (owing, for example, to reactions).

In the framework we consider, the coefficients of L will be stochastic processes; thus in physical interpretation equation (3.1) can be understood as a diffusion process in a stochastic anisotropic medium, with transport and creation also dependent on some random factors, and with a stochastic boundary value. Example of a stochastic anisotropic medium is a medium consisting of two randomly mixed immiscible fluids.

In neurology, elliptic PDEs showed as a good model for brain function measurements. The elliptic PDE defined in the brain at a few points (the location of the sensors) involves the divergence form operator $L = \sum_{i=1}^n D_i(\sum_{j=1}^n a^{ij} D_j u)$. The fact that the matrix $[a^{ij}]$ is discontinuous along the boundaries separating different layers of the brain, means that the PDE must deal with singular data. Now, if there are also some random perturbations happening in the brain, the PDE is no longer a PDE but a SPDE.

If the coefficients of L are deterministic functions, then our assumptions on L will follow the approach in [GT]: We assume that L is strictly elliptic in I and has bounded measurable coefficients; that is, there exist $\lambda > 0$, $\Lambda > 0$, and $\nu \geq 0$ such that

$$\sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad x \in I, \xi \in \mathbb{R}^n, \tag{3.4}$$

$$\sum_{i,j=1}^n |a^{ij}(x)|^2 \leq \Lambda^2, \quad x \in I, \tag{3.5}$$

$$\frac{1}{\lambda^2}\sum_{i=1}^n \left(|b^i(x)|^2 + |c^i(x)|^2\right) + \frac{1}{\lambda}|d(x)| \leq \nu^2, \quad x \in I. \tag{3.6}$$

In fact, λ and Λ are the minimal and maximal, respectively, eigenvalues of the matrix $[a_{ij}]$. Without loss of generality we may and will assume that $\lambda \leq 1$ (else we divide equation (3.1) by λ to obtain this situation). We seek for a weak solution of (3.1) in the Hilbert space of generalized random processes $\mathcal{L}(W_0^{1,2}(I), (S)_{-1})$, where $W_0^{1,2}(I)$ denotes the Sobolev space and $(S)_{-1}$ denotes the Kondratiev space. As we will see in Theorem 3.1.6, this generalized solution exists and is unique. In Proposition 3.1.1 we investigate some stability and regularity properties of this solution. In [LØUZ] regularization was used and an error-function tending to zero was introduced to give meaning to a distributional valued solution, as well as to give meaning to the product of a non-smooth deterministic function (the coefficients of L) and the distribution $u(x, \cdot)$. The solution was expressed in terms of the Green function and the Itô integral. We will have a similar stability result, but using the Hilbert space structure. The goal in this section is to extend the approach for even more singular coefficients and data, namely Colombeau generalized functions. We develop the necessary Sobolev–Colombeau spaces for generalized stochastic processes and prove existence and uniqueness of the solution for the Dirichlet problem also in this setting.

However, if the coefficients of the operator L are also generalized random processes, then a further problem arises: How to interpret the product between two generalized random processes? In [Va] the product was interpreted as the Wick product, and the solution was found as an element of the Sobolev-Kondratiev space. A similar approach can be found in [Be], where the tensor product of the Sobolev space and the space of stochastic trigonometric functions was used, and the product was taken pointwisely. In [Bu] the author uses the direct product of the classical Sobolev space and a generalized Sobolev space as a probability space, which allows the product to be interpreted as ordinary product of two functions.

In [HØUZ] stochastic partial differential equations are solved by a general receipt: The H-transform and Wick products are used to convert the SPDE into a deterministic PDE, which can be solved by known PDE methods, then one must check the conditions to apply the inverse H-transform, which then defines the solution of the starting SPDE. This method has the disadvantage that each SPDE must be solved separately, and it can treat only random processes which are generalized by the random parameter ω, but continuous (at least) in the time-space variable x. The Hilbert space methods used in [Va], [Be], [Bu], have the advantage to solve a wider class of equations, also dealing with random processes that are more singular at the x-variable. Existence and uniqueness of the solution are obtained, but one does not obtain an explicit form of the solution. Due to our definition of generalized random processes as linear continuous mappings from the Sobolev space into the Kondratiev space, we are also able to reduce the SPDE into a PDE and make use of the deterministic theory, even without using the H-transform, which makes the approach simpler than in [HØUZ].

Preliminary review of the deterministic Dirichlet problem

In [GT] the deterministic case, i.e. a Dirichlet problem of the form

$$\begin{aligned} L\,u(x) &= h(x) + \sum_{i=1}^{n} D_i f^i(x), \qquad x \in I, \\ u(x) \restriction_{\partial I} &= g(x) \end{aligned} \tag{3.7}$$

is considered. A function $u \in W^{1,2}(I)$ is called a weak solution of (3.7) if

$$\begin{aligned} \int_I \sum_{i=1}^{n} (\sum_{j=1}^{n} (a^{ij} D_j u + b^i u) D_i v - (\sum_{i=1}^{n} c^i D_i u + du) v) dx \\ = \int_I (\sum_{i=1}^{n} f^i D_i v - hv) dx, \end{aligned} \tag{3.8}$$

for all $v \in W_0^{1,2}(I)$, and $u - g \in W_0^{1,2}(I)$. The concept of the weak solution is based on the fact that one can identify the operator L with its unique extension (denoted by the same symbol) $L : W^{1,2}(I) \to W^{-1,2}(I)$ which generates a bilinear form $(u,v) \mapsto \mathcal{B}(u,v) = -\langle Lu, v\rangle$, $(u,v) \in W^{1,2}(I) \times W_0^{1,2}(I)$ where $\langle Lu, v\rangle$ is defined by the left hand side of (3.8). Thus, $u \in W^{1,2}(I)$ is a weak solution if $\mathcal{B}(u,v) = \langle h + \nabla \cdot f, v\rangle = \langle h, v\rangle - \langle f, \nabla v\rangle$, for every $v \in W_0^{1,2}(I)$. Here $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $L^2(I)$, which can be extended to the canonical dual pairing on $W^{-1,2}(I) \times W_0^{1,2}(I)$. Thus, existence of a weak solution is equivalent to surjectivity, while uniqueness is equivalent to injectivity of the mapping L. The standard norm on $L^2(I)$ is denoted by $\| \cdot \|_2$.

We state the following theorem from [GT], which will be needed in the sequel:

Theorem 3.1.1 *Let the operator L given by (3.2) satisfy conditions (3.4), (3.5) and (3.6). Moreover, assume that*

$$\int_I \Big(d(x)v(x) - \sum_{i=1}^{n} b^i(x) D_i v(x)\Big)\, dx \le 0, \quad v \ge 0, v \in W_0^{1,2}(I). \tag{3.9}$$

Then for $g \in W^{1,2}(I)$ and $h, f^i \in L^2(I)$, $i = 1, 2, \ldots, n$ the Dirichlet problem (3.7) has a unique weak solution u. Moreover, there exists a constant $C > 0$ (depending only on L and I) such that

$$\|u\|_{W^{1,2}} \le C \left(\|\mathbf{h}\|_2 + \|g\|_{W^{1,2}}\right), \tag{3.10}$$

where $\mathbf{h} = (h, f^1, f^2, \ldots f^n)$.

These results are based on Hilbert space methods for PDEs. The Lax-Milgram theorem and the Fredholm alternative are used to prove existence and uniqueness of the weak solution. For details, refer to [GT, Chapter 8]. Here we state the three main classical theorems, since they are the keystones that cap the arch of our proofs.

Theorem 3.1.2 (Lax–Milgram) *Let $B : H \times H \to \mathbb{R}^n$ be a bilinear form on a Hilbert space H, satisfying following properties: There exist $K, C > 0$ such that*

- $|B(x,y)| \le K\|x\|\,\|y\|$, *for all $x, y \in H$,* *(boundedness)*
- $|B(x,x)| \ge C\|x\|^2$, *for all $x \in H$.* *(coercivity)*

Then for every bounded linear functional $F \in H'$, there exists a unique element $f \in H$ such that

$$B(x,f) = F(x), \quad \text{for all} \quad x \in H.$$

Theorem 3.1.3 (Fredholm alternative) *Let V be a normed vector space and $T : V \to V$ be a compact linear mapping. Then,*

(i) either the homogeneous equation $x - Tx = 0$ has a nontrivial solution $x \in V$,

(ii) or for each $y \in V$ the equation $x - Tx = y$ has a unique solution $x \in V$.

In case (ii), the operator $(I - T)^{-1}$ whose existence is asserted there is also bounded.

Theorem 3.1.4 (Fredholm alternative – spectral behavior) *Let V be a Hilbert space and $T : V \to V$ be a compact linear mapping. There exists a countable set $\Lambda \subset \mathbb{R}$ having no limit points except possibly $\lambda = 0$, such that if $\lambda \neq 0, \lambda \notin \Lambda$ the equations*

$$\lambda\, x - Tx = y, \qquad \lambda\, x - T^* x = y \tag{3.11}$$

have uniquely determined solutions $x \in V$ for every $y \in V$, and the inverse mappings $(\lambda I - T)^{-1}$, $(\lambda I - T^*)^{-1}$ are bounded. If $\lambda \in \Lambda$, the null spaces of the mappings $\lambda I - T$, $\lambda I - T^*$ have positive finite dimension and the equations (3.11) are solvable if and only if y is orthogonal to the null space of $\lambda I - T^*$ in the first case and $\lambda I - T$ in the second case.

Preliminary review of generalized random processes

Let the basic probability space $(\Omega, \mathcal{F}, P)$ be $(\mathcal{S}'(\mathbb{R}^n), \mathcal{B}, \mu)$ and consider the Kondratiev space of generalized random variables $(S)_{-1}$. Recall that the generalized expectation of $F \in (S)_{-1}$ is defined by $E(F) = \langle F, 1 \rangle$.

We consider generalized random processes as linear continuous mappings from the Sobolev space $W_0^{1,2}(I)$ into the space of Kondratiev generalized random variables $(S)_{-1}$. In Chapter 2 we considered GRPs (I) as linear continuous mappings from the Zemanian space $\mathcal{A}$ into $(S)_{-1}$, but since the Sobolev spaces are constructed in a similar manner, only using the triple $W_0^{1,2}(I) \subseteq L^2(I) \subseteq W^{-1,2}(I)$ and the Laplace operator, we can state the results of Theorem 2.1.1 also in this context. Denote further on the space of GRPs by

$$\mathcal{W}\mathcal{S}^* = \mathcal{L}(W_0^{1,2}(I), (S)_{-1}).$$

Note that $\mathcal{W}\mathcal{S}^* \cong \mathcal{L}((S)_1, W^{-1,2}(I)) \cong \mathcal{B}((S)_1 \times W_0^{1,2}(I), \mathbb{R}) \cong \mathcal{L}((S)_1 \otimes W_0^{1,2}(I), \mathbb{R}) \cong W^{-1,2}(I) \otimes (S)_{-1}$. Thus, we may regard a GRP $u(x, \omega)$ as a bilinear continuous mapping $u(\varphi, \theta)$ by $\varphi(x) \in W_0^{1,2}(I)$, $x \in I$ and $\theta(\omega) \in (S)_1$, $\omega \in \Omega$, as well as an element of the tensor product space

$W^{-1,2}(I) \otimes (S)_{-1}$. The latter isomorphisms hold since $(S)_1$ is a nuclear space and $W^{-1,2}(I)$ is a Frèchet space. Also note that the notation $\otimes$ stands for the π–completition of the tensor product space, which is in this case equivalent to the ϵ–completition by the nuclearity of $(S)_1$.

By $\mathcal{W}_0\mathcal{S}$ we denote $W_0^{1,2}(I) \otimes (S)_1$, and by $\mathcal{W}\mathcal{S}$ we denote $W^{1,2}(I) \otimes (S)_1$.

We state now Theorem 2.1.1 in this new context:

Theorem 3.1.5 *Following conditions are equivalent:*

(i) $u \in \mathcal{W}\mathcal{S}^*$.

(ii) *u can be represented in the form*

$$u(x,\omega) = \sum_{\alpha\in\mathcal{J}} f_\alpha(x) \otimes H_\alpha(\omega), \quad x \in I, \omega \in \Omega,\ f_\alpha \in W^{-1,2}(I), \alpha \in \mathcal{J}, \tag{3.12}$$

and there exists $p \in \mathbb{N}_0$ *such that for each bounded set* $B \subset W_0^{1,2}(I)$

$$\sup_{\varphi\in B} \sum_{\alpha\in\mathcal{J}} |\langle f_\alpha, \varphi\rangle|^2 (2\mathbb{N})^{-p\alpha} < \infty. \tag{3.13}$$

(iii) *u can be represented in the form (3.12) and there exists* $p \in \mathbb{N}_0$ *such that*

$$\sum_{\alpha\in\mathcal{J}} \|f_\alpha\|^2_{W^{-1,2}} (2\mathbb{N})^{-p\alpha} < \infty. \tag{3.14}$$

Note that $\sum_{\alpha\in\mathcal{J}} f_\alpha(x) H_\alpha(\omega)$, $f_\alpha \in W^{\pm 1,2}$ is also a generalized random process as defined in [Va]. The tensor product space $W^{-1,2}(I) \otimes (S)_{-1}$ was used also in [Be]; we find it more convenient to deal with the structure given in Theorem 3.1.5 and also later in the next section to deal with Wick products, which are all special cases (Sobolev versions) of the structures introduced in [PS1] and [Se].

3.1.1 Solvability of the stochastic Dirichlet problem

Now we return to our boundary problem (3.1). The idea is to prove that there exists a mapping $\Xi \in \mathcal{L}((S)_1, W^{-1,2}(I))$ such that $\Xi(\theta)$ is a weak solution of (3.7) for each $\theta \in (S)_1$. This will induce a GRP $u(x,\omega)$, $x \in I$, $\omega \in \Omega$ given by $\langle u(x,\omega), \theta(\omega)\rangle$, which will be called a *generalized solution* of (3.1).

Theorem 3.1.6 *Let the operator L given by (3.2) satisfy conditions (3.4), (3.5), (3.6) and (3.9). Then for* $g \in \mathcal{W}\mathcal{S}$, $h, f^i \in \mathcal{L}((S)_1, L^2(I))$, $i = 1, 2, \ldots, n$ *the Dirichlet problem (3.1) has a unique generalized solution* $u \in \mathcal{W}\mathcal{S}^*$.

Proof. Let $\theta \in (S)_1$ and $p > 0$. Put $g_\theta(x) = \langle g(x,\omega), \theta(\omega)\rangle$, $h_\theta(x) = \langle h(x,\omega), \theta(\omega)\rangle$, $f^i_\theta(x) = \langle f^i(x,\omega), \theta(\omega)\rangle$, $i = 1, 2, \dots, n$. Clearly, $g_\theta \in W^{1,2}(I)$, while $h_\theta, f^i_\theta \in L^2(I)$, $i = 1, 2, \dots, n$. According to Theorem 3.1.1 there exists a unique weak solution $u_\theta(x), x \in I$ in $W^{1,2}(I)$ of the boundary problem

$$\begin{aligned} L\, u_\theta(x) &= h_\theta(x) + \sum_{i=1}^{n} D_i f^i_\theta(x), \qquad x \in I, \\ u_\theta(x) \restriction_{\partial I} &= g_\theta(x). \end{aligned} \tag{3.15}$$

Due to (3.10) and continuity of h and g, there exist constants $C > 0$, $M > 0$ (note: C may depend on the coefficients of L and on I, but not on θ) such that

$$\begin{aligned} \|u_\theta\|_{W^{1,2}} &\le C\,(\|\mathbf{h}_\theta\|_2 + \|g_\theta\|_{W^{1,2}}) \\ &\le C\,(\|\mathbf{h}\| \cdot \|\theta\|_{1,p} + \|g\| \cdot \|\theta\|_{1,p}) \\ &\le M\|\theta\|_{1,p}, \end{aligned}$$

where $\mathbf{h} = (h, f^1, f^2, \dots f^n) \in \mathcal{L}((S)_1, \bigoplus_{i=1}^{n+1} L^2(I))$ and $\mathbf{h}_\theta = (h_\theta, f^1_\theta, f^2_\theta, \dots f^n_\theta)$, while $\|\cdot\|$ stands for the operator norm in $\mathcal{L}(W_0^{1,2}(I), (S)_{-1})$. Since $u_\theta \in W^{1,2}(I)$, we may identify it as an element of $W^{-1,2}(I)$ and by the Riesz isometry we get

$$\|u_\theta\|_{W^{-1,2}} \le M\|\theta\|_{1,p}. \tag{3.16}$$

Define a mapping $\Xi : (S)_1 \to W^{-1,2}(I)$ by $\theta \mapsto u_\theta$. Clearly, Ξ is linear, and by (3.16) it is bounded. Thus, $\Xi \in \mathcal{W}\mathcal{S}^*$. □

Note, since $u_\theta(x) = \langle u(x,\omega), \theta(\omega)\rangle$ is constructed as a weak solution of the deterministic Dirichlet problem, i.e. as a linear continuous mapping, we also have continuity in the second variable: There exists $N > 0$ such that $\|\langle u(x,\cdot), \varphi(x)\rangle\|_{-1,-p} \le N\|\varphi\|_{W^{1,2}}$, for $\varphi \in W_0^{1,2}$. Thus, there also exists a constant $K > 0$ such that

$$|\langle u(x,\omega), \varphi(x)\theta(\omega)\rangle| \le K\|\varphi\|_{W^{1,2}}\|\theta\|_{1,p}, \quad \varphi \in W_0^{1,2},\ \theta \in (S)_1,$$

i.e. our solution is a bilinear mapping on $\mathcal{W}_0\mathcal{S}$.

Remark. We can obtain the same result using the chaos expansion from Theorem 3.1.5. Thus, we provide an alternate proof of Theorem 3.1.6.

Assume g, h, f^i are given by following expansions, respectively:

$$g(x,\omega) = \sum_{\alpha\in\mathcal{J}} g_\alpha(x) \otimes H_\alpha(\omega), \quad x \in I, \omega \in \Omega,\ g_\alpha \in W^{1,2}(I), \alpha \in \mathcal{J},$$
$$h(x,\omega) = \sum_{\alpha\in\mathcal{J}} h_\alpha(x) \otimes H_\alpha(\omega), \quad x \in I, \omega \in \Omega,\ h_\alpha \in L^2(I), \alpha \in \mathcal{J},$$
$$f^i(x,\omega) = \sum_{\alpha\in\mathcal{J}} f^i_\alpha(x) \otimes H_\alpha(\omega), \quad x \in I, \omega \in \Omega,\ f^i_\alpha \in L^2(I), \alpha \in \mathcal{J},\ i = 1, 2, \dots, n.$$

We seek for the solution u in form of

$$u(x,\omega) = \sum_{\alpha\in\mathcal{J}} u_\alpha(x) \otimes H_\alpha(\omega), \quad x \in I, \omega \in \Omega,\ u_\alpha \in W^{1,2}(I), \alpha \in \mathcal{J},$$

where the coefficients u_α, $\alpha \in \mathcal{J}$ are to be determined. By linearity of L and consequently

$$L\ u(x,\omega) = \sum_{\alpha\in\mathcal{J}} L\ u_\alpha(x) \otimes H_\alpha(\omega) = \sum_{\alpha\in\mathcal{J}} \left(h_\alpha(x) + \sum_{i=1}^{n} D_i f^i_\alpha(x) \right) \otimes H_\alpha(\omega),$$

we obtain following system of equations:

$$\begin{aligned} L\ u_\alpha(x) &= h_\alpha(x) + \sum_{i=1}^{n} D_i f^i_\alpha(x), \qquad x \in I, \qquad \alpha \in \mathcal{J}, \\ u_\alpha(x) \restriction_{\partial I} &= g_\alpha(x). \end{aligned} \tag{3.17}$$

Due to Theorem 3.1.1, for every $\alpha \in \mathcal{J}$ there exists a unique $u_\alpha \in W^{1,2}(I)$, which solves (3.17), and there exists $C > 0$ (uniformly in α) such that

$$\|u_\alpha\|^2_{W^{1,2}} \le C \left(\|\mathbf{h}_\alpha\|_2^2 + \|g_\alpha\|^2_{W^{1,2}} \right)$$

where $\mathbf{h}_\alpha = (h_\alpha, f^1_\alpha, f^2_\alpha, \dots f^n_\alpha)$. Now, according to the assumptions made on h, f^i, $i = 1, 2 \dots, n$, there exists $p > 0$ such that

$$\sum_{\alpha\in\mathcal{J}} \|u_\alpha\|^2_{W^{1,2}} (2\mathbb{N})^{-p\alpha} \le C \left(\sum_{\alpha\in\mathcal{J}} \|\mathbf{h}_\alpha\|_2^2 (2\mathbb{N})^{-p\alpha} + \sum_{\alpha\in\mathcal{J}} \|g_\alpha\|^2_{W^{1,2}} (2\mathbb{N})^{-p\alpha} \right) < \infty.$$

Thus, $u \in \mathcal{WS}$. Again, one can consider it also as an element of the dual space i.e. $u \in \mathcal{WS}^*$.

Corollary 3.1.1 *Let $u \in \mathcal{WS}^*$ be the generalized solution of (3.1). Then its generalized expectation $E(u)$ coincides with the weak solution of the deterministic Dirichlet problem*

$$\begin{aligned} L\, v(x) &= \tilde{h}(x) + \sum_{i=1}^{n} D_i \tilde{f}^i(x), \qquad x \in I, \\ v(x) \restriction_{\partial I} &= \tilde{g}(x), \end{aligned} \tag{3.18}$$

where $\tilde{h} = E(h)$, $\tilde{f}^i = E(f^i)$, $i = 1, 2, \ldots, n$ and $\tilde{g} = E(g)$.

Proof. The assertion follows from the proof of Theorem 3.1.6 if we choose $\theta = 1$. $\square$

3.1.2 Stability properties of the Dirichlet problem

First we prove that our generalized solution of the stochastic Dirichlet problem is continuously dependent on the data. Let L be a strictly elliptic operator of the form (3.2) with coefficients a^{ij}, b^i, c^i, d, satisfying conditions (3.4), (3.5) and (3.6). Let $\tilde{L}$ be another strictly elliptic operator of the form (3.2) with coefficients $\tilde{a}^{ij}$, $\tilde{b}^i$, $\tilde{c}^i$, $\tilde{d}$, satisfying all given conditions. Let h, $\tilde{h}$, f^i and $\tilde{f}^i$, $i = 1, 2, \ldots, n$ be generalized random processes from $\mathcal{L}((S)_1, L^2(I))$. Let g and $\tilde{g}$ be generalized random processes from $\mathcal{WS}$. For $\theta \in (S)_1$ fixed, denote by u_θ the solution of the Dirichlet problem

$$\begin{aligned} L\, u_\theta(x) &= h_\theta(x) + \sum_{i=1}^{n} D_i f^i_\theta(x), \qquad x \in I, \\ u_\theta(x) \restriction_{\partial I} &= g_\theta(x) \end{aligned} \tag{3.19}$$

as it was obtained in Theorem 3.1.6. Respectively, let $\tilde{u}_\theta$ be the generalized solution of the Dirichlet problem

$$\begin{aligned} \tilde{L}\, \tilde{u}_\theta(x) &= \tilde{h}_\theta(x) + \sum_{i=1}^{n} D_i \tilde{f}^i_\theta(x), \qquad x \in I, \\ \tilde{u}_\theta(x) \restriction_{\partial I} &= \tilde{g}_\theta(x). \end{aligned} \tag{3.20}$$

Due to the construction of the generalized solutions as elements of Sobolev spaces, we can prove stability (in the x variable) in weak sense of these solutions. Every Dirichlet problem can be transformed into a problem of the same form, but with zero boundary conditions (see [GT]). Thus, for technical simplicity first we assume that $g = \tilde{g} = 0$. We will prove that for every $\varphi \in W_0^{1,2}(I)$ the expression $|\langle u_\theta - \tilde{u}_\theta, \varphi\rangle|$ is bounded by the operator norms of $\|L - \tilde{L}\|$ and $\|\mathbf{h}_\theta - \tilde{\mathbf{h}}_\theta\|$, where $\mathbf{h} = (h, f^1, f^2, \dots f^n)$. Here $\langle\cdot,\cdot\rangle$ stands for the usual scalar product in $L^2(I)$.

By subtracting (3.20) from (3.19) we obtain

$$Lu_\theta - \tilde{L}\tilde{u}_\theta + \tilde{L}u_\theta - \tilde{L}u_\theta = h_\theta - \tilde{h}_\theta + \sum_{i=1}^{n} D_i(f_\theta^i - \tilde{f}_\theta^i),$$

that is

$$\tilde{L}(u_\theta - \tilde{u}_\theta) = h_\theta - \tilde{h}_\theta + \sum_{i=1}^{n} D_i(f_\theta^i - \tilde{f}_\theta^i) - (L - \tilde{L})u_\theta.$$

Let $\varphi \in W_0^{1,2}$ be arbitrary. Denote by $\psi \in W_0^{1,2}$ the unique solution of the deterministic Dirichlet problem $\tilde{L}^*\psi(x) = \varphi(x)$, $\psi(x) \restriction_{\partial I}= 0$. This is well defined due to Theorem 3.1.1 since the adjoint operator $\tilde{L}^*$ inherits from $\tilde{L}$ all properties (3.4), (3.5), (3.6) and (3.9). Thus,

$$|\langle u_\theta - \tilde{u}_\theta, \varphi\rangle| = |\langle u_\theta - \tilde{u}_\theta, \tilde{L}^*\psi\rangle| = |\langle \tilde{L}(u_\theta - \tilde{u}_\theta), \psi\rangle|$$

$$= |\langle h_\theta - \tilde{h}_\theta + \sum_{i=1}^{n} D_i(f_\theta^i - \tilde{f}_\theta^i) - (L - \tilde{L})u_\theta, \psi\rangle|.$$

Now, by the Cauchy-Schwartz inequality and by continuity of L and $\tilde{L}$ on $W^{1,2}(I)$, we get

$$\begin{aligned} |\langle u_\theta - \tilde{u}_\theta, \varphi\rangle| &\le \|h_\theta - \tilde{h}_\theta\|_2\|\psi\|_2 + \sum_{i=1}^{n} \|f_\theta^i - \tilde{f}_\theta^i\|_2\|D_i\psi\|_2 + \|L - \tilde{L}\|_2\|u_\theta\|_2\|\psi\|_2 \\ &\le \|\mathbf{h}_\theta - \tilde{\mathbf{h}}_\theta\|_2\|\psi\|_{W^{1,2}} + \|L - \tilde{L}\|_{W^{-1,2}}\|u_\theta\|_{W^{1,2}}\|\psi\|_{W^{1,2}}. \end{aligned} \tag{3.21}$$

From Theorem 3.1.1 we also know there exists $C > 0$ such that

$$\|\psi\|_{W^{1,2}} \le C\|\varphi\|_{W^{1,2}}.$$

Thus,

$$\begin{aligned} &|\langle u(x,\omega) - \tilde{u}(x,\omega), \varphi(x)\theta(\omega)\rangle| \le \\ &\qquad C\left(\|\mathbf{h} - \tilde{\mathbf{h}}\|_{L^2(I)\otimes(S)_{-1,-p}} + \|L - \tilde{L}\|_{W^{-1,2}}\|u\|_{W^{1,2}\otimes(S)_{-1,-p}}\right)\|\varphi\|_{W^{1,2}}\|\theta\|_{1,p} \end{aligned}$$

where $\mathbf{h} = (h, f^1, f^2, \dots f^n)$.

Now, consider the general Dirichlet problems (3.19), (3.20) with nonzero boundary conditions. We transform them into zero boundary conditions in the following way: u_θ is a weak solution of (3.19) if and only if $u_{0,\theta} = u_\theta - g_\theta$ is a weak solution of $Lu_{0,\theta} = h_{0,\theta} - \sum_{i=1}^n D_i f^i_{0,\theta}$, where $h_{0,\theta} = h_\theta - \sum_{i=1}^n c^i D_i g_\theta - dg_\theta$, $f^i_{0,\theta} = f^i_\theta - \sum_{i=1}^n a^{ij} D_j g_\theta - b^i g_\theta$. We do the same for (3.20) and apply the estimate obtained for $|\langle u_0 - \tilde{u}_0, \varphi\theta\rangle|$.

We summarize the stability result in the following proposition.

Proposition 3.1.1 *Let L be a strictly elliptic operator of the form (3.2) with coefficients a^{ij}, b^i, c^i, d, satisfying conditions (3.4), (3.5) and (3.6). Let $\tilde{L}$ be another strictly elliptic operator of the form (3.2) with coefficients $\tilde{a}^{ij}$, $\tilde{b}^i$, $\tilde{c}^i$, $\tilde{d}$, satisfying all given conditions. Let h, $\tilde{h}$, f^i and $\tilde{f}^i$, $i = 1, 2, \ldots, n$ be generalized random processes from $\mathcal{L}((S)_1, L^2(I))$. Let g and $\tilde{g}$ be generalized random processes from $\mathcal{WS}$. Let $u, \tilde{u} \in \mathcal{WS}^*$ be the generalized solutions of the Dirichlet problems*

$$L\, u(x,\omega) = h(x,\omega) + \sum_{i=1}^n D_i f^i(x,\omega), \qquad x \in I, \omega \in \Omega,$$
$$u(x,\omega) \restriction_{\partial I} = g(x,\omega)$$

and

$$\tilde{L}\, \tilde{u}(x,\omega) = \tilde{h}(x,\omega) + \sum_{i=1}^n D_i \tilde{f}^i(x,\omega), \qquad x \in I, \omega \in \Omega,$$
$$\tilde{u}(x,\omega) \restriction_{\partial I} = \tilde{g}(x,\omega)$$

respectively. There exist $C > 0$, $p \in \mathbb{N}_0$ such that for every $\theta \in (S)_{1,p}$:

$$|\langle u(x,\omega) - \tilde{u}(x,\omega), \varphi(x)\theta(\omega)\rangle| \leq C\Big(\|\mathbf{h} - \tilde{\mathbf{h}}\|_{L^2(I)\otimes(S)_{-1,-p}}$$
$$+ \|L - \tilde{L}\|_{W^{-1,2}} \|u - g\|_{W^{1,2}\otimes(S)_{-1,-p}} + \|L - \tilde{L}\|_{W^{-1,2}} \|\tilde{g}\|_{W^{1,2}\otimes(S)_{-1,-p}}$$
$$+ \|g - \tilde{g}\|_{W^{1,2}\otimes(S)_{-1,-p}} \|L\|_{W^{-1,2}}\Big) \|\varphi\|_{W^{1,2}} \|\theta\|_{1,p}$$

where $\mathbf{h} = (h, f^1, f^2, \ldots f^n)$.

Remark. In particular, to address a similar stability question as in [LØUZ], let us consider the net of operators L_ϵ, $\epsilon \in (0,1]$, given by

$$L_\epsilon\, u(x,\cdot) = \sum_{i=1}^n D_i\Big(\sum_{j=1}^n a^{ij}_\epsilon(x) D_j u(x,\cdot) + b^i_\epsilon(x) u(x,\cdot)\Big)$$
$$+ \sum_{i=1}^n c^i_\epsilon(x) D_i u(x,\cdot) + d_\epsilon(x) u(x,\cdot)$$

and a net of data $h_\epsilon(x,\omega)$, $f^i_\epsilon(x,\omega)$, $i=1,2,\ldots,n$, where $a^{ij}_\epsilon(x)=a^{ij}*\check{\rho}_\epsilon(x)$, $b^i_\epsilon(x)=b^i*\check{\rho}_\epsilon(x)$, $c^i_\epsilon(x)=c^i*\check{\rho}_\epsilon(x)$, $d_\epsilon(x)=d*\check{\rho}_\epsilon(x)$,$h_\epsilon(x,\omega)=h(\cdot,\omega)*\check{\rho}_\epsilon(x)$, $f^i_\epsilon(x,\omega)=f^i(\cdot,\omega)*\check{\rho}_\epsilon(x)$, $i,j=1,2,\ldots,n$. Here ρ is a mollifier, i.e. a positive smooth function such that $\int_I \rho(x)dx=1$, $\check{\rho}(x)=\rho(-x)$, $\rho_\epsilon(x)=\epsilon^{-n}\rho(x/\epsilon)$, and $*$ denotes convolution with respect to x. Denote by u_ϵ the solution of

$$L_\epsilon\, u_\epsilon = h_\epsilon + \sum_{i=1}^n D_i f^i_\epsilon, \qquad u_\epsilon\restriction_{\partial I}=0.$$

Now, from (3.21) we get that $\|u_{\theta,\epsilon}-u_\theta\|_{W^{-1,2}}$ is bounded by the sum of the operator norm $\|L-L_\epsilon\|_{W^{-1,2}}$ and of $\|\mathbf{h}_{\theta,\epsilon}-\mathbf{h}_\theta\|_{L^2}$. It can be shown (see [NP]) that L_ϵ inherits the strict ellipticity and boundedness properties from L. Also, it is a well-known property of regularization in Sobolev spaces that $\|p*\rho_\epsilon(x)-p(x)\|_{W^{1,2}}\to 0$, $\epsilon\to 0$, for $p\in W^{1,2}(I)$.

Thus,

$$\|u_{\theta,\epsilon}-u_\theta\|_{W^{-1,2}}\to 0, \quad \epsilon\to 0, \tag{3.22}$$

which establishes the stability result.

This motivates us to consider the stochastic Dirichlet problem in the Colombeau setting, which will enable us to solve problem (3.1) also with more singular coefficients and data, i.e. with Colombeau generalized deterministic functions. In order to do this, we first have to introduce appropriate spaces of Colombeau algebras of generalized random processes, which will be done in the next section.

Proposition 3.1.1 and the Remark after it also relate our approach with those in [HØUZ] and [LØUZ]. Processes involved therein as data in SPDEs are first smoothed out in order to get an appropriate form for applying the S-transform and its inverse. For example, singular white noise given by

$$W(x,\omega)=\sum_{j=1}^\infty \eta_j(x)H_{\varepsilon_j}(\omega), \quad x\in\mathbb{R}^n, \omega\in\Omega,$$

where $\eta_j, j\in\mathbb{N}$ is the Hermite function basis of $\mathbb{R}^n$ and $\varepsilon_j=(0,0\ldots,1,0,\ldots)$ is a multiindex, is an element of $(S)_{-1}$ for fixed x. In [HØUZ] the "noise" part in SPDEs is modeled with smoothed white noise given by

$$W(\phi,x,\omega)=\langle\omega,\phi_x\rangle=\omega*\check{\phi}(x)=\sum_{j=1}^\infty(\eta_j,\phi_x)_{L^2(I)}H_{\varepsilon_j}(\omega),$$

where $\phi\in\mathcal{S}(\mathbb{R}^n)$, $\phi_x=\phi(\cdot-x)$, $x\in\mathbb{R}^n$ and $\omega\in\mathcal{S}'(\mathbb{R}^n)$. Formally, one can think of $W(x,\omega)$ as of $W(\delta,x,\omega)$, where δ denotes the Dirac delta distribution. Smoothed white noise is an element of $(L)^2$ for fixed ϕ and x. SPDEs involving smoothed white noise are then solved by standard methods, and then letting $\varphi \to \delta$ in $\mathcal{S}'(\mathbb{R}^n)$ one obtains a solution of the SPDE involving singular white noise. Due to Theorem 3.1.6 we can directly solve the Dirichlet problem (3.1) where h, f or g is singular white noise, since singular white noise belongs to $\mathcal{L}((S)_1, L^2(I))$.

For example, consider the Dirichlet problems $Lu = W(\rho_\epsilon, x, \omega)$ and $Lu = W(x,\omega)$, $u \restriction_{\partial I} = 0$, with generalized solutions u_ϵ, u, respectively. Now, using the fact that $W(\rho_\epsilon, x, \omega) = W(\cdot,\omega) * \rho_\epsilon(x)$, we get from the Remark after Proposition 3.1.1

$$\|u_{\theta,\epsilon} - u_\theta\|_{W^{-1,2}} \leq C\|W_{\theta,\epsilon} - W_\theta\|_{L^2} \to 0, \quad \epsilon \to 0.$$

This stability result shows that the solution of the Dirichlet problem where the right hand side is smoothed white noise, in fact is an adequate approximation of the solution where the data is singular white noise.

Regularity properties of the Dirichlet problem

One can also show regularity properties (in the x variable) of the solution obtained in Theorem 3.1.6. So far, we have found generalized solutions as elements of $\mathcal{WS}$, i.e. solutions which are in the x-variable elements of $W^{1,2}(I)$. Now we can state (for the proof see [GT]) that this generalized solution is in the x-variable twice weakly differentiable, i.e. $u \in W^{2,2}(I) \otimes (S)_{-1}$, provided the domain I and the data in the equation are sufficiently smooth.

Proposition 3.1.2 *Let $u \in W^{1,2}(I) \otimes (S)_{-1}$ be the generalized solution of*

$$Lu = h, \qquad u \restriction_{\partial I} = g.$$

Assume the coefficients a^{ij}, b^i, $i,j = 1,2,\ldots n$, are uniformly Lipschitz continuous in I, $c^i, d \in L^\infty(I)$, $i = 1,2,\ldots n$, and $h \in L^2(I) \otimes (S)_{-1}$. Then, for arbitrary I' such that $\overline{I'} \subset I$, it follows that $u \in W^{2,2}(I') \otimes (S)_{-1}$ and there exists $C(n,\lambda,K,d') > 0$ such that for each $\theta \in (S)_1$

$$\|u_\theta\|_{W^{2,2}(I')} \leq C(\|u_\theta\|_{W^{1,2}(I)} + \|h_\theta\|_{L^2(I)})$$

where $K = max\{\|a^{ij}, b^i\|_{C^{0,1}(\bar{I})},\ \|c^i, d\|_{L^\infty(I)}\}$ and $d' = dist(\partial I, I')$. Additionally, u satisfies the equation

$$L\,u = \sum_{i,j=1}^n a^{ij} D_{ij} u + \sum_{i=1}^n (\sum_{j=1}^n (D_j a^{ij} + b^i + c^i) D_i u + (\sum_{i=1}^n D_i b^i + d) u = h$$

for a.e. $x \in I$.

The next theorem (in deterministic case) was proved in [MP], but we may state it also in our setting. The polynomial growth rate estimated in (3.23) will later be crucial for the Colombeau solutions.

Proposition 3.1.3 *Assume additionally to the assumptions of Proposition 3.1.2 that I is of C^2–class and $g \in W^{2,2}(I) \otimes (S)_{-1}$. There exist $s \in \mathbb{R}$ and $C > 0$ such that for each $\theta \in (S)_1$*

$$\|u_\theta\|_{W^{2,2}} \leq C \left(\frac{\Lambda}{\lambda}\right)^s (\sup_{x \in \partial I} |u_\theta(x)|\ |I| + \|h_\theta\|_{L^2} + \|g_\theta\|_{W^{2,2}}). \tag{3.23}$$

3.1.3 Colombeau–type solutions of the stochastic Dirichlet problem

Now we solve problem (3.1) for more singular coefficients and data – we assume they are Colombeau generalized functions in the x-variable. The deterministic Dirichlet problem for linear elliptic operators in Colombeau setting has been studied in [MP] and [NP]; now we construct stochastic Colombeau–type algebras which allow us to solve the stochastic Dirichlet problem.

Colombeau algebras of Colombeau generalized functions are constructed in such a way that $\mathcal{S}'(\mathbb{R}^n)$ can be embedded into them – thus, they contain singularities like the Dirac delta distribution. Moreover, one can deal with multiplication of generalized functions and other nonlinearities.

This approach is motivated by the stability result we obtained in Proposition 3.1.1. If the right hand side of (3.1) and if the coefficients of L involve singularities in the x-variable, we consider a family of problems $L_\epsilon u_\epsilon = h_\epsilon$, $u_\epsilon \restriction_{\partial I} = g_\epsilon$, $\epsilon \in (0,1)$, where h_ϵ, g_ϵ are smooth enough (in our case twice weakly differentiable) approximations of h and g. Such nets of approximations are considered as elements of an appropriate quotient algebra where certain equivalence relations are introduced, in order to have weak equality between different approximations. Solving this family of problems, we obtain a family of solutions which represents a Colombeau–algebra solution of the original problem.

Throughout this section we assume that $n \leq 3$ and that ∂I is of C^2 class (in this case $W^{2,2}(I)$ is an algebra).

3.1.4 Colombeau–algebras $\mathcal{G}(W^{2,2})$ and $\mathcal{G}(W^{2,2}; (S)_{-1})$

First we define the Colombeau algebra for deterministic functions.

- Let $\mathcal{E}_M(W^{2,2})$ be the vector space of functions $r : (0,1) \to W^{2,2}(I)$, $\epsilon \mapsto r_\epsilon(x)$, such that there exists $a \in \mathbb{N}_0$ with the property that $\|r_\epsilon(x)\|_{W^{2,2}} = \mathcal{O}(\epsilon^{-a})$. Elements of $\mathcal{E}_M(W^{2,2})$ are called *moderate* functions.

- Let $\mathcal{N}(W^{2,2})$ denote the vector space of functions $r_\epsilon \in \mathcal{E}_M(W^{2,2})$ with the property that for every $a \in \mathbb{N}_0$, $\|r_\epsilon(x)\|_{W^{2,2}} = \mathcal{O}(\epsilon^{a})$ holds. Elements of $\mathcal{N}(W^{2,2})$ are called *negligible* functions.

- Since $\mathcal{N}(W^{2,2})$ is an ideal of the algebra $\mathcal{E}_M(W^{2,2})$, the quotient space

$$\mathcal{G}(W^{2,2}) = \mathcal{E}_M(W^{2,2}) \,/\, \mathcal{N}(W^{2,2})$$

  is also an algebra. It is called the Colombeau extension of $W^{2,2}(I)$, and its elements will be called *Colombeau generalized functions*. We denote the elements of $\mathcal{G}(W^{2,2})$ (equivalence classes) by $[r_\epsilon]$.

In a similar manner one can define also $\mathcal{G}(W^{1,2})$ and $\mathcal{G}(W^{0,2})$, but they are not algebras, only vector spaces (however, $\mathcal{G}(W^{1,2})$ is algebra for $n = 1$). For $f = [f_\epsilon] \in \mathcal{G}(W^{1,2})$ we define $\partial_{x_i} f = [\partial_{x_i} f_\epsilon]$, $i = 1, 2, \dots, n$, and note that $\partial_{x_i}^k f \in \mathcal{G}(W^{2-k,2})$, $k = 1, 2$.

The space $\mathcal{G}(W^{2,2})$ is constructed so that it involves singular data, for example distributions of the form $f_1 + Df_2 + D^2 f_3$, $f_i \in L^2(I)$, $i = 1, 2, 3$, are obviously embedded into $\mathcal{G}(W^{2,2})$. But $\mathcal{S}'(I)$ can also be embedded into $\mathcal{G}(W^{2,2})$ via convolution. Note that $\mathcal{S}(I) \subseteq W^{2,2}(I) \subseteq L^2(I)$, and for a fixed mollifier function ρ_ϵ, $\epsilon \in (0,1)$ we have that if $f \in \mathcal{S}(I)$, then $f_\epsilon = f * \rho_\epsilon \in \mathcal{S}(I) \subseteq W^{2,2}(I)$ and clearly, $f - f_\epsilon \in \mathcal{N}(W^{2,2})$. Thus, $[f_\epsilon] \in \mathcal{G}(W^{2,2})$. The embedding $\iota_\rho : \mathcal{S}(I) \to \mathcal{G}(W^{2,2})$, $f \mapsto [f_\epsilon]$, can be extended to an embedding $\iota_\rho : \mathcal{S}'(I) \to \mathcal{G}(W^{2,2})$, defined by

$$\mathcal{S}'(I) \ni F \mapsto [F * \rho_\epsilon] \in \mathcal{G}(W^{2,2}).$$

We continue with the construction of the appropriate Colombeau–type algebra for generalized random processes. The construction is similar as in the previous case, but now we consider $(S)_{-1}$–valued functions. The idea is to use the chaos expansion in $(S)_{-1}$: All coefficients in the chaos expansion will be deterministic Colombeau generalized functions i.e. elements of $\mathcal{G}(W^{2,2})$, but certain conditions must hold so that the result remains in $(S)_{-1}$.

- Let $\mathcal{E}_M(W^{2,2}; (S)_{-1})$ be the vector space of functions $R : (0,1) \to W^{2,2}(I) \otimes (S)_{-1}$, $\epsilon \mapsto R_\epsilon(x,\omega) = \sum_{\alpha \in \mathcal{I}} r_{\alpha,\epsilon}(x) \otimes H_\alpha(\omega)$, $r_{\alpha,\epsilon} \in W^{2,2}(I), x \in I, \omega \in \Omega$, such that there exist a sequence $\{C_\alpha\}_{\alpha \in \mathcal{I}}$ of positive numbers, $\epsilon_0 \in (0,1)$, $p \in \mathbb{N}_0$, and there exists a sequence

$\{a_\alpha\}_{\alpha\in\mathcal{I}}$ bounded from above (i.e. there exists $a \in \mathbb{R}$ such that $a_\alpha \leq a$, $\alpha \in \mathcal{I}$), with following properties:

$$\|r_{\alpha,\epsilon}\|_{W^{2,2}} \leq C_\alpha \epsilon^{-a_\alpha}, \quad \text{for all } \alpha \in \mathcal{I}, \epsilon < \epsilon_0, \tag{3.24}$$

$$\sum_{\alpha\in\mathcal{I}} C_\alpha^2 (2\mathbb{N})^{-p\alpha} < \infty. \tag{3.25}$$

Clearly, from (3.24) and (3.25) we have

$$\begin{aligned}\sum_{\alpha\in\mathcal{I}} \|r_{\alpha,\epsilon}\|^2_{W^{2,2}} (2\mathbb{N})^{-p\alpha} &\leq \sum_{\alpha\in\mathcal{I}} C_\alpha^2 \epsilon^{-2a_\alpha} (2\mathbb{N})^{-p\alpha} \leq \epsilon^{-2a} \sum_{\alpha\in\mathcal{I}} C_\alpha^2 (2\mathbb{N})^{-p\alpha} \\ &= K\epsilon^{-2a},\ \epsilon < \epsilon_0,\end{aligned}$$

where $K = \sum_{\alpha\in\mathcal{I}} C_\alpha^2 (2\mathbb{N})^{-p\alpha}$.

- Let $\mathcal{N}(W^{2,2};(S)_{-1})$ denote the vector space of functions $R_\epsilon \in \mathcal{E}_M(W^{2,2};(S)_{-1})$ with the property that there exist a sequence $\{C_\alpha\}_{\alpha\in\mathcal{I}}$ of positive numbers, $\epsilon_0 \in (0,1)$, $p \in \mathbb{N}_0$, and for all sequences $\{a_\alpha\}_{\alpha\in\mathcal{I}}$ bounded from below (i.e. there exists $a \in \mathbb{R}$ such that $a_\alpha \geq a$, $\alpha \in \mathcal{I}$), following hold:

$$\|r_{\alpha,\epsilon}\|_{W^{2,2}} \leq C_\alpha \epsilon^{a_\alpha}, \quad \text{for all } \alpha \in \mathcal{I}, \epsilon < \epsilon_0, \tag{3.26}$$

$$\sum_{\alpha\in\mathcal{I}} C_\alpha^2 (2\mathbb{N})^{-p\alpha} < \infty. \tag{3.27}$$

Clearly, from (3.26) and (3.27) we have

$$\begin{aligned}\sum_{\alpha\in\mathcal{I}} \|r_{\alpha,\epsilon}\|^2_{W^{2,2}} (2\mathbb{N})^{-p\alpha} &\leq \sum_{\alpha\in\mathcal{I}} C_\alpha^2 \epsilon^{2a_\alpha} (2\mathbb{N})^{-p\alpha} \leq \epsilon^{2a} \sum_{\alpha\in\mathcal{I}} C_\alpha^2 (2\mathbb{N})^{-p\alpha} \\ &= K\epsilon^{2a},\ \epsilon < \epsilon_0,\end{aligned}$$

where $K = \sum_{\alpha\in\mathcal{I}} C_\alpha^2 (2\mathbb{N})^{-p\alpha}$.

- Define the multiplication in $\mathcal{E}_M(W^{2,2};(S)_{-1})$ and $\mathcal{N}(W^{2,2};(S)_{-1})$ in the following way: For $F_\epsilon(x,\omega) = \sum_{\alpha\in\mathcal{I}} f_{\alpha,\epsilon}(x) \otimes H_\alpha(\omega)$, $G_\epsilon(x,\omega) = \sum_{\alpha\in\mathcal{I}} g_{\alpha,\epsilon}(x) \otimes H_\alpha(\omega)$ let

$$F_\epsilon \lozenge G_\epsilon(x,\omega) = \sum_{\gamma\in\mathcal{I}} \left(\sum_{\alpha+\beta=\gamma} f_{\alpha,\epsilon}(x) \cdot g_{\alpha,\epsilon}(x) \right) \otimes H_\gamma(\omega), \tag{3.28}$$

i.e. we use the Wick product for multiplication in $(S)_{-1}$ and the ordinary product in $W^{2,2}(I)$.

Lemma 3.1.1 *(i)* $\mathcal{E}_M(W^{2,2};(S)_{-1})$ *is an algebra under the multiplication rule given by* (3.28).

(ii) $\mathcal{N}(W^{2,2};(S)_{-1})$ *is an ideal of* $\mathcal{E}_M(W^{2,2};(S)_{-1})$.

Proof. (i) Let $F_\epsilon, G_\epsilon \in \mathcal{E}_M(W^{2,2};(S)_{-1})$ and prove that $F_\epsilon \Diamond G_\epsilon \in \mathcal{E}_M(W^{2,2};(S)_{-1})$. Since $F_\epsilon = \sum_{\alpha\in\mathcal{I}} f_{\alpha,\epsilon}(x) \otimes H_\alpha(\omega) \in \mathcal{E}_M(W^{2,2};(S)_{-1})$ and $G_\epsilon = \sum_{\beta\in\mathcal{I}} g_{\beta,\epsilon}(x) \otimes H_\beta(\omega) \in \mathcal{E}_M(W^{2,2};(S)_{-1})$, there exist $a, b \in \mathbb{R}$, $p, q \in \mathbb{N}_0$, $\epsilon_1, \epsilon_2 \in (0,1)$, and there exist sequences $\{C_\alpha\}_{\alpha\in\mathcal{I}}$, $\{D_\alpha\}_{\alpha\in\mathcal{I}}$, $\{a_\alpha\}_{\alpha\in\mathcal{I}}$, $\{b_\alpha\}_{\alpha\in\mathcal{I}}$, such that $a_\alpha \le a$, $b_\alpha \le b$ for all $\alpha \in \mathcal{I}$, $\|f_{\alpha,\epsilon}\|_{W^{2,2}} \le C_\alpha \epsilon^{-a_\alpha}$ for $\epsilon < \epsilon_1$, $\|g_{\alpha,\epsilon}\|_{W^{2,2}} \le D_\alpha \epsilon^{-b_\alpha}$ for $\epsilon < \epsilon_2$, and $\sum_{\alpha\in\mathcal{I}} C_\alpha^2 (2\mathbb{N})^{-p\alpha} < \infty$, $\sum_{\alpha\in\mathcal{I}} D_\alpha^2 (2\mathbb{N})^{-q\alpha} < \infty$. We will prove that for $\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$ there exist $r > 0$, a sequence $\{M_\gamma\}_{\gamma\in\mathcal{I}}$ and a sequence $\{m_\gamma\}_{\gamma\in\mathcal{I}}$ bounded from above such that $\|\sum_{\alpha+\beta=\gamma} f_{\alpha,\epsilon}(x) g_{\alpha,\epsilon}(x)\|_{W^{2,2}} \le M_\gamma \epsilon^{-m_\gamma}$ for $\epsilon < \epsilon_0$ and $\sum_{\gamma\in\mathcal{I}} M_\gamma^2 (2\mathbb{N})^{-r\alpha} < \infty$.

For a fixed multiindex $\gamma \in \mathcal{I}$ put $M_\gamma = \sum_{\alpha+\beta=\gamma} C_\alpha D_\beta$ and $m_\gamma = \max\{a_\alpha + b_\beta : \ \alpha, \beta \in \mathcal{I}, \ \alpha + \beta = \gamma\}$ (note, for γ fixed, there are only finite many α and β which give the sum γ). Now using the fact that $W^{2,2}(I)$ is an algebra for $n \le 3$, we get

$$\|\sum_{\alpha+\beta=\gamma} f_{\alpha,\epsilon}(x) g_{\alpha,\epsilon}(x)\|_{W^{2,2}} \le \sum_{\alpha+\beta=\gamma} \|f_{\alpha,\epsilon}(x)\|_{W^{2,2}} \|g_{\alpha,\epsilon}(x)\|_{W^{2,2}} \le$$

$$\sum_{\alpha+\beta=\gamma} C_\alpha D_\beta \epsilon^{-(a_\alpha+b_\beta)} \le \epsilon^{-m_\gamma} \sum_{\alpha+\beta=\gamma} C_\alpha D_\beta = \epsilon^{-m_\gamma} M_\gamma.$$

Clearly, $m_\gamma \le a+b$, $\gamma \in \mathcal{I}$, thus the sequence $\{m_\gamma\}_{\gamma\in\mathcal{I}}$ is bounded from above. Let $r = p+q+2$. Then, using the nuclearity of $(S)_{-1}$, we obtain

$$\sum_{\gamma\in\mathcal{I}} M_\gamma^2 (2\mathbb{N})^{-r\alpha} \le \sum_{\gamma\in\mathcal{I}} \left(\sum_{\alpha+\beta=\gamma} C_\alpha D_\beta \right)^2 (2\mathbb{N})^{-(p+q+2)\gamma}$$

$$\le \sum_{\gamma\in\mathcal{I}} (2\mathbb{N})^{-2\gamma} \sum_{\alpha\in\mathcal{I}} C_\alpha^2 (2\mathbb{N})^{-p\alpha} \sum_{\beta\in\mathcal{I}} D_\beta^2 (2\mathbb{N})^{-q\beta} < \infty.$$

(ii) Let us check first that $\mathcal{N}(W^{2,2};(S)_{-1})$ is a subalgebra of $\mathcal{E}_M(W^{2,2};(S)_{-1})$. Let $F_\epsilon, G_\epsilon \in \mathcal{N}(W^{2,2};(S)_{-1})$ be of the form as in (i). Let $\{m_\gamma\}_{\gamma\in\mathcal{I}}$ be an arbitrary sequence bounded from below. Put $a_\gamma = b_\gamma = \frac{m_\gamma}{2}$, $\gamma \in \mathcal{I}$. Now for the sequences $\{a_\gamma\}_{\gamma\in\mathcal{I}}, \{b_\gamma\}_{\gamma\in\mathcal{I}}$ also bounded from below, since $F_\epsilon, G_\epsilon \in \mathcal{N}(W^{2,2};(S)_{-1})$, there must exist $p, q \in \mathbb{N}_0$, $\epsilon_1, \epsilon_2 \in (0,1)$, and $\{C_\alpha\}_{\alpha\in\mathcal{I}}$, $\{D_\alpha\}_{\alpha\in\mathcal{I}}$ such that $\|f_{\alpha,\epsilon}\|_{W^{2,2}} \le C_\alpha \epsilon^{m_\alpha/2}$ for $\epsilon < \epsilon_1$, $\|g_{\alpha,\epsilon}\|_{W^{2,2}} \le D_\alpha \epsilon^{m_\alpha/2}$ for $\epsilon < \epsilon_2$, and $\sum_{\alpha\in\mathcal{I}} C_\alpha^2 (2\mathbb{N})^{-p\alpha} < \infty$,

$\sum_{\alpha\in\mathcal{I}} D_\alpha^2 (2\mathbb{N})^{-q\alpha} < \infty$. Let $\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$, $r = p + q + 2$ and $M_\gamma = \sum_{\alpha+\beta=\gamma} C_\alpha D_\beta$, $\gamma \in \mathcal{I}$. Now we may proceed as in (i) to get $\sum_{\alpha+\beta=\gamma} \|f_{\alpha,\epsilon}\|_{W^{2,2}} \|g_{\alpha,\epsilon}\|_{W^{2,2}} \le M_\gamma \epsilon^{m_\gamma}$ and $\sum_{\gamma\in\mathcal{I}} M_\gamma^2 (2\mathbb{N})^{-r\alpha} < \infty$.

In order to prove that $\mathcal{N}(W^{2,2};(S)_{-1})$ is an ideal of $\mathcal{E}_M(W^{2,2};(S)_{-1})$, we must check that for all $G_\epsilon \in \mathcal{N}(W^{2,2};(S)_{-1})$ and all $F_\epsilon \in \mathcal{E}_M(W^{2,2};(S)_{-1})$, we have $G_\epsilon \lozenge F_\epsilon = F_\epsilon \lozenge G_\epsilon \in \mathcal{N}(W^{2,2};(S)_{-1})$. Let $\{n_\gamma\}_{\gamma\in\mathcal{I}}$ be an arbitrary sequence bounded from below (i.e. $n_\gamma \ge n$, $\gamma \in \mathcal{I}$). Since $F_\epsilon \in \mathcal{E}_M(W^{2,2};(S)_{-1})$, there exist $p \in \mathbb{N}_0$, $\epsilon_1 \in (0,1)$, $\{D_\beta\}_{\beta\in\mathcal{I}}$, $b \in \mathbb{R}$ and a sequence $\{b_\beta\}_{\beta\in\mathcal{I}}$ such that $b_\beta \le b$, $\beta \in \mathcal{I}$, $\|f_{\beta,\epsilon}\|_{W^{2,2}} \le D_\beta \epsilon^{-b_\beta}$, $\epsilon < \epsilon_1$, and $\sum_\beta D_\beta^2 (2\mathbb{N})^{-p\beta} < \infty$.

For a fixed multiindex $\alpha \in \mathcal{I}$, let $a_\alpha = b + n_\alpha$. Clearly, the sequence $\{a_\alpha\}_{\alpha\in\mathcal{I}}$ is bounded from below by $a = b + n$, thus since $G_\epsilon \in \mathcal{N}(W^{2,2};(S)_{-1})$, there exist $q \in \mathbb{N}_0$, $\epsilon_2 \in (0,1)$, $\{C_\alpha\}_{\alpha\in\mathcal{I}}$, such that $\|g_{\alpha,\epsilon}\|_{W^{2,2}} \le C_\alpha \epsilon^{-a_\alpha}$, $\epsilon < \epsilon_2$, and $\sum_\alpha C_\alpha^2 (2\mathbb{N})^{-q\alpha} < \infty$. Let $\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$ and $N_\gamma = \sum_{\alpha+\beta=\gamma} C_\alpha D_\beta$, $\gamma \in \mathcal{I}$.

Now for $G_\epsilon \lozenge F_\epsilon = \sum_{\gamma\in\mathcal{I}} \sum_{\alpha+\beta=\gamma} g_{\alpha,\epsilon} f_{\alpha,\epsilon} \otimes H_\gamma$ we have that

$$\sum_{\alpha+\beta=\gamma} \|g_{\alpha,\epsilon}\|_{W^{2,2}} \|f_{\beta,\epsilon}\|_{W^{2,2}} \le \sum_{\alpha+\beta=\gamma} C_\alpha D_\beta \epsilon^{a_\alpha - b_\beta}$$

$$\le \sum_{\alpha+\beta=\gamma} C_\alpha D_\beta \epsilon^{b+n_\alpha-b_\beta} \le \sum_{\alpha+\beta=\gamma} C_\alpha D_\beta \epsilon^{b_\beta+n_\alpha-b_\beta} \le N_\gamma \epsilon^{n_\gamma}, \quad \epsilon < \epsilon_0.$$

It is clear that $\sum_{\gamma\in\mathcal{I}} N_\gamma^2 (2\mathbb{N})^{-r\gamma} < \infty$ for $r = p + q + 2$. Thus, $G_\epsilon \lozenge F_\epsilon \in \mathcal{N}(W^{2,2};(S)_{-1})$.

$\square$

- Since $\mathcal{N}(W^{2,2};(S)_{-1})$ is an ideal of the algebra $\mathcal{E}_M(W^{2,2};(S)_{-1})$, the quotient space

$$\mathcal{G}(W^{2,2};(S)_{-1}) = \mathcal{E}_M(W^{2,2};(S)_{-1}) \;/\; \mathcal{N}(W^{2,2};(S)_{-1})$$

  is also an algebra. It is called the $(S)_{-1}$–valued Colombeau extension of $W^{2,2}(I)$, and its elements will be called *Colombeau generalized random processes.* We denote the elements of $\mathcal{G}(W^{2,2};(S)_{-1})$ (equivalence classes) by $[R_\epsilon]$.

Note that $(S)_{-1}$ can be embedded into $\mathcal{G}(W^{2,2};(S)_{-1})$ by

$$(S)_{-1} \ni F(\omega) = \sum_{\alpha\in\mathcal{I}} f_\alpha H_\alpha(\omega) \rightsquigarrow \sum_{\alpha\in\mathcal{I}} f_\alpha(x) \otimes H_\alpha(\omega) \in \mathcal{G}(W^{2,2};(S)_{-1}),$$

where $f_\alpha(x) = f_\alpha$ is the constant mapping.

Also, $\mathcal{G}(W^{2,2})$ can be embedded into $\mathcal{G}(W^{2,2};(S)_{-1})$ by

$$\mathcal{G}(W^{2,2}) \ni f = [f_\epsilon(x)] \rightsquigarrow \sum_{\alpha=(0,0,0,\dots)} [f_\epsilon(x)]H_\alpha(\omega) \in \mathcal{G}(W^{2,2};(S)_{-1}).$$

The following proposition ensures that it is sufficient to consider only the cases when the polynomial ϵ-growth rate is uniform in $\alpha \in \mathcal{I}$.

Proposition 3.1.4 *(i) $R_\epsilon(x,\omega)$, $x \in I, \omega \in \Omega$, belongs to $\mathcal{E}_M(W^{2,2};(S)_{-1})$ if and only if there exist $\epsilon_0 \in (0,1)$, $C > 0$, $a \in \mathbb{R}$ and $q \geq 0$ such that*

$$\sum_{0\leq|\beta|\leq 2} \int_I \|D^\beta R_\epsilon(x,\cdot)\|^2_{-1,-q}\, dx \leq C\epsilon^{-a}, \quad \epsilon < \epsilon_0. \tag{3.29}$$

(ii) $R_\epsilon(x,\omega)$, $x \in I, \omega \in \Omega$, belongs to $\mathcal{E}_M(W^{2,2};(S)_{-1})$ if and only if there exist $\epsilon_0 \in (0,1)$, $C > 0$, $a \in \mathbb{R}$ and $q \geq 0$ such that

$$\|R_\epsilon(x,\omega)\|_{W^{2,2}(I)\otimes(S)_{-1,-q}} \leq C\epsilon^{-a}, \quad \epsilon < \epsilon_0. \tag{3.30}$$

(iii) $R_\epsilon(x,\omega)$, $x \in I, \omega \in \Omega$, belongs to $\mathcal{N}(W^{2,2};(S)_{-1})$ if and only if there exist $\epsilon_0 \in (0,1)$, $C > 0$ and $q \geq 0$ such that for every $a \in \mathbb{R}$,

$$\sum_{0\leq|\beta|\leq 2} \int_I \|D^\beta R_\epsilon(x,\cdot)\|^2_{-1,-q}\, dx \leq C\epsilon^{a}, \quad \epsilon < \epsilon_0. \tag{3.31}$$

(iv) $R_\epsilon(x,\omega)$, $x \in I, \omega \in \Omega$, belongs to $\mathcal{N}(W^{2,2};(S)_{-1})$ if and only if there exist $\epsilon_0 \in (0,1)$, $C > 0$ and $q \geq 0$ such that for every $a \in \mathbb{R}$,

$$\|R_\epsilon(x,\omega)\|_{W^{2,2}(I)\otimes(S)_{-1,-q}} \leq C\epsilon^{a}, \quad \epsilon < \epsilon_0. \tag{3.32}$$

Proof. (i) Let $R_\epsilon \in \mathcal{E}_M(W^{2,2};(S)_{-1})$. Then, there exist $b \in \mathbb{R}$, $p \geq 0$, $\{C_\alpha\}_{\alpha\in\mathcal{I}}$, $\{b_\alpha\}_{\alpha\in\mathcal{I}}$, such that $b_\alpha \leq b$, $\alpha \in \mathcal{I}$, $\|r_{\alpha,\epsilon}\|_{W^{2,2}} \leq C_\alpha \epsilon^{-b_\alpha}$, $\sum_{\alpha\in\mathcal{I}} C_\alpha^2(2\mathbb{N})^{-p\alpha} < \infty$. Let $C = \sum_{\alpha\in\mathcal{I}} C_\alpha^2 (2\mathbb{N})^{-p\alpha}$ and $a = 2b$. Then,

$$\sum_{0\leq|\beta|\leq 2} \int_I \|D^\beta R_\epsilon(x,\cdot)\|^2_{-1,-p}\, dx = \sum_{0\leq|\beta|\leq 2} \int_I \sum_{\alpha\in\mathcal{I}} |D^\beta r_{\alpha,\epsilon}(x)|^2 (2\mathbb{N})^{-p\alpha} dx$$

$$= \sum_{\alpha\in\mathcal{I}} \int_I \sum_{0\leq|\beta|\leq 2} |D^\beta r_{\alpha,\epsilon}(x)|^2 dx (2\mathbb{N})^{-p\alpha} = \sum_{\alpha\in\mathcal{I}} \|r_{\alpha,\epsilon}\|^2_{W^{2,2}} (2\mathbb{N})^{-p\alpha}$$

$$\leq \sum_{\alpha\in\mathcal{I}} C_\alpha^2 \epsilon^{-2b_\alpha} (2\mathbb{N})^{-p\alpha} \leq \epsilon^{-2b} \sum_{\alpha\in\mathcal{I}} C_\alpha^2 (2\mathbb{N})^{-p\alpha} = C\epsilon^{-a}.$$

Conversely, let (3.29) hold and let us show that $R_\epsilon \in \mathcal{E}_M(W^{2,2};(S)_{-1})$. Denote $K := \sum_{\alpha\in\mathcal{I}}(2\mathbb{N})^{-q\alpha} < \infty$ for an appropriate $q > 1$. Let $R_\epsilon(x,\omega) = \sum_{\alpha\in\mathcal{I}} r_{\alpha,\epsilon}(x) \otimes H_\alpha(\omega)$ as in construction (3.25). Then,

$$\left(\sum_{\alpha\in\mathcal{I}} \|r_{\alpha,\epsilon}\|_{W^{2,2}}(2\mathbb{N})^{-q\alpha}\right)^4 \leq \left(K\sum_{\alpha\in\mathcal{I}} \|r_{\alpha,\epsilon}\|^2_{W^{2,2}}(2\mathbb{N})^{-q\alpha}\right)^2$$

$$\leq K^2\left(\int_I \sum_{\alpha\in\mathcal{I}}\sum_{0\leq|\beta|\leq 2} |D^\beta r_{\alpha,\epsilon}(x)|^2(2\mathbb{N})^{-q\alpha}\,dx\right)^2$$

$$\leq K^2|I|\int_I \left(\sum_{\alpha\in\mathcal{I}}\sum_{0\leq|\beta|\leq 2} |D^\beta r_{\alpha,\epsilon}(x)|^2(2\mathbb{N})^{-q\alpha}\right)^2 dx$$

$$\leq 2K^2|I|\int_I \sum_{0\leq|\beta|\leq 2}\left(\sum_{\alpha\in\mathcal{I}} |D^\beta r_{\alpha,\epsilon}(x)|^2(2\mathbb{N})^{-q\alpha}\right)^2 dx$$

$$= \tilde{K}\int_I \sum_{0\leq|\beta|\leq 2} \|D^\beta R_\epsilon(x,\cdot)\|^2_{-1,-q}\,dx \leq \tilde{K}C\epsilon^{-a}.$$

Thus, $\sum_{\alpha\in\mathcal{I}}\|r_{\alpha,\epsilon}\|_{W^{2,2}}(2\mathbb{N})^{-q\alpha} \leq \sqrt[4]{\tilde{K}C}\epsilon^{-\frac{a}{4}}$, which implies that for each $\alpha \in \mathcal{I}$

$$\|r_{\alpha,\epsilon}\|_{W^{2,2}} \leq C_\alpha\epsilon^{-\frac{a}{4}}, \qquad \text{where} \qquad C_\alpha = (2\mathbb{N})^{q\alpha}\sqrt[4]{K^2|I|C},$$

and

$$\sum_{\alpha\in\mathcal{I}} C_\alpha^2(2\mathbb{N})^{-p\alpha} < \infty, \qquad \text{for} \qquad p > 1+2q.$$

(ii) Since

$$\|R_\epsilon(x,\omega)\|^2_{W^{2,2}(I)\otimes(S)_{-1,-q}} = \langle R_\epsilon(x,\omega), R_\epsilon(x,\omega)\rangle$$

$$= \sum_{\alpha\in\mathcal{I}}\langle r_{\alpha,\epsilon}, r_{\alpha,\epsilon}\rangle_{W^{2,2}}\langle H_\alpha, H_\alpha\rangle_{-1,-q} = \sum_{\alpha\in\mathcal{I}}\|r_{\alpha,\epsilon}\|^2_{W^{2,2}}(2\mathbb{N})^{-q\alpha} \leq C\epsilon^{-a},$$

it follows similarly as in (i) that $\|r_{\alpha,\epsilon}\|^2_{W^{2,2}} \leq C(2\mathbb{N})^{q\alpha}\epsilon^{-a}$ for each $\alpha \in \mathcal{I}$.

(iii) and (iv) follow similarly as (i) and (ii). $\square$

Elliptic differential operators on $\mathcal{G}(W^{2,2};(S)_{-1})$

We consider a net of linear differential operators

$$
\begin{aligned}
L_\epsilon\, u_\epsilon(x,\cdot) = \sum_{i=1}^n D_i\Big(\sum_{j=1}^n a_\epsilon^{ij}(x) D_j u_\epsilon(x,\cdot) + b_\epsilon^i(x) u_\epsilon(x,\cdot)\Big) \\
+ \sum_{i=1}^n c_\epsilon^i(x) D_i u_\epsilon(x,\cdot) + d_\epsilon(x) u_\epsilon(x,\cdot), \quad \epsilon \in (0,1),
\end{aligned}
\tag{3.33}
$$

where the nets of coefficients a_ϵ^{ij}, b_ϵ^i, c_ϵ^i, d_ϵ belong to $\mathcal{E}_M(W^{2,2})$. Define that two nets of operators are related: $L_\epsilon \sim \tilde{L}_\epsilon$ if and only if $(L_\epsilon u_\epsilon - \tilde{L}_\epsilon u_\epsilon) \in \mathcal{N}(W^{0,2};(S)_{-1})$ for all $u_\epsilon \in \mathcal{E}_M(W^{2,2};(S)_{-1})$. Clearly, $\sim$ is an equivalence relation, and following holds (refer to [MP]):

Proposition 3.1.5 *If $u_\epsilon \in \mathcal{N}(W^{2,2};(S)_{-1})$, then $L_\epsilon u_\epsilon \in \mathcal{N}(W^{0,2};(S)_{-1})$.*

Denote by $\mathcal{L}$ the family of all nets of differential operators of the form (3.33) and let $\mathcal{L}_0 = \mathcal{L}/\sim$. For $L \in \mathcal{L}_0$ we define $L : \mathcal{G}(W^{2,2};(S)_{-1}) \to \mathcal{G}(W^{0,2};(S)_{-1})$ by

$$
\begin{aligned}
L\,[u_\epsilon(x,\cdot)] = \sum_{i=1}^n D_i\Big(\sum_{j=1}^n [a_\epsilon^{ij}(x)][D_j u_\epsilon(x,\cdot)] + [b_\epsilon^i(x)][u_\epsilon(x,\cdot)\Big)] \\
+ \sum_{i=1}^n [c_\epsilon^i(x)][D_i u_\epsilon(x,\cdot)] + [d_\epsilon(x)][u_\epsilon(x,\cdot)].
\end{aligned}
\tag{3.34}
$$

The operator $L = [L_\epsilon]$ given by (3.34) is *strictly elliptic*, if there exist representatives a_ϵ^{ij}, b_ϵ^i, c_ϵ^i, $d_\epsilon \in \mathcal{E}_M(W^{2,2})$ of its coefficients, such that

$$
\sum_{i,j=1}^n a_\epsilon^{ij}(x)\xi_i\xi_j \geq \lambda_\epsilon|\xi|^2 \geq K\epsilon^a|\xi|^2, \quad x \in I, \xi \in \mathbb{R}^n, \tag{3.35}
$$

where K is a constant independent of ϵ.

Boundary conditions in $\mathcal{G}(W^{2,2};(S)_{-1})$

Let $u, g \in \mathcal{G}(W^{2,2};(S)_{-1})$. It is known that $W^{2,2}(I) \subseteq C(I_1)$, where I_1 is open and $\overline{I} \subseteq I_1$. We define $u\restriction_{\partial I} = g\restriction_{\partial I}$ if there exist representatives u_ϵ and g_ϵ of u and g, respectively, such that

$$
u_\epsilon\restriction_{\partial I} = g_\epsilon\restriction_{\partial I} + n_\epsilon\restriction_{\partial I}, \quad \epsilon \in (0,1)
$$

where n_ϵ is a net of continuous GRPs, defined in a neighborhood of ∂I with the property that there exists $p > 0$ such that for all $a > 0$, $\sup_{x\in\partial I} \|n_\epsilon(x,\cdot)\|_{-1,-p} = o(\epsilon^a)$.

It is easy to check that this definition does not depend on the choice of representatives: Let $\tilde{u}_\epsilon$ and $\tilde{g}_\epsilon$ be another choice of representatives for u and g, respectively. Then $\tilde{u}_\epsilon \restriction_{\partial I} = \tilde{g}_\epsilon \restriction_{\partial I} + \tilde{n}_\epsilon \restriction_{\partial I}$, where $\tilde{n}_\epsilon \restriction_{\partial I} = (h_\epsilon - \tilde{h}_\epsilon) \restriction_{\partial I} + n_\epsilon \restriction_{\partial I} + (u_\epsilon - \tilde{u}_\epsilon) \restriction_{\partial I}$. This implies $sup_{x\in\partial I} \|n_\epsilon(x,\cdot)\|_{-1,-p} = o(\epsilon^a)$, for all $a > 0$.

The Dirichlet problem in $\mathcal{G}(W^{2,2}; (S)_{-1})$

Consider now the stochastic Dirichlet problem

$$\begin{aligned} L\, u(x,\omega) &= h(x,\omega) \qquad \text{in } \mathcal{G}(W^{2,2}; (S)_{-1}) \\ u(x,\omega) \restriction_{\partial I} &= g(x,\omega) \restriction_{\partial I}, \end{aligned} \tag{3.36}$$

where L is defined in (3.34). In order to solve (3.36) in the Colombeau setting, one has to solve a family of problems:

$$\begin{aligned} L_\epsilon\, u_\epsilon(x,\omega) &= h_\epsilon(x,\omega) \qquad \text{in } W^{2,2}(I) \otimes (S)_{-1} \\ u_\epsilon(x,\omega) \restriction_{\partial I} &= g_\epsilon(x,\omega) \restriction_{\partial I}, \quad \epsilon \in (0,1), \end{aligned} \tag{3.37}$$

then to check whether the net of solutions u_ϵ belongs to $\mathcal{E}_M(W^{2,2}; (S)_{-1})$, and finally to check whether equation (3.37) holds with other representatives of L, h, f, g and u. Uniqueness of a solution means that if $u = [u_\epsilon]$ and $v = [v_\epsilon]$ satisfy (3.36), then $[u_\epsilon] = [v_\epsilon]$.

Theorem 3.1.7 *Let $h = [h_\epsilon] = \sum_{\alpha\in\mathcal{J}} [h_{\alpha,\epsilon}] \otimes H_\alpha$ and $g = [g_\epsilon] = \sum_{\alpha\in\mathcal{J}} [g_{\alpha,\epsilon}] \otimes H_\alpha$ belong to $\mathcal{G}(W^{2,2}; (S)_{-1})$. Let the operator $L = [L_\epsilon]$ be given by (3.34) with coefficients $[a_\epsilon^{ij}]$, $[b_\epsilon^i]$, $[c_\epsilon^i]$, $[d_\epsilon]$ in $\mathcal{G}(W^{2,2})$. Assume that following conditions hold:*

1. *L is strictly elliptic, i.e. (3.35) holds,*
2. *there exist $M > 0$ and $b > 0$ such that for all $\epsilon \in (0,1)$ and $\alpha \in \mathcal{J}$:*

$$\|a_\epsilon^{ij}\|_{W^{2,2}},\ \|b_\epsilon^i\|_{W^{2,2}},\ \|c_\epsilon^i\|_{W^{2,2}},\ \|d_\epsilon\|_{W^{2,2}} \le \Lambda_\epsilon;$$

$$\|h_{\alpha,\epsilon}\|_{W^{2,2}},\ \|g_{\alpha,\epsilon}\|_{W^{2,2}} \le \Lambda_\epsilon;$$

$$\Lambda_\epsilon \le M\epsilon^b.$$

3. for every $\epsilon \in (0,1)$ *and* $v \geq 0, v \in W_0^{1,2}(I)$,

$$\int_I (d_\epsilon(x)v(x) - \sum_{i=1}^n b_\epsilon^i(x) D_i v(x))\, dx \leq 0.$$

Then the stochastic Dirichlet problem (3.36) has a unique solution in $\mathcal{G}(W^{2,2}; (S)_{-1})$.

Proof. For $\epsilon \in (0,1)$ fixed, there exists a unique generalized solution $u_\epsilon \in W^{2,2}(I) \otimes (S)_{-1}$ of problem (3.37). This follows from Theorem 3.1.6, and moreover we know that u_ϵ is given by the chaos expansion $u_\epsilon(x,\omega) = \sum_{\alpha\in\mathcal{J}} u_{\alpha,\epsilon}(x) \otimes H_\alpha(\omega)$, $u_{\alpha,\epsilon} \in W^{2,2}(I)$, $x \in I$, $\omega \in \Omega$, where $u_{\alpha,\epsilon}$ is the solution of the deterministic Dirichlet problem

$$\begin{aligned} L_\epsilon\, u_{\alpha,\epsilon}(x) &= h_{\alpha,\epsilon}(x) \qquad \text{in } W^{2,2}(I) \\ u_{\alpha,\epsilon}(x) \restriction_{\partial I} &= g_{\alpha,\epsilon}(x) \restriction_{\partial I} . \end{aligned} \tag{3.38}$$

Using the estimate derived in (3.23) we obtain that there exist $C > 0$, $s > 0$ (from the proof in [MP] one can see that C and s may depend on n, λ, $|I|$, ∂I, and the coefficients of L, but they are uniform in ϵ and α) such that:

$$\|u_{\alpha,\epsilon}\|_{W^{2,2}} \leq C \left(\frac{\Lambda_\epsilon}{\lambda_\epsilon}\right)^s \Big(\sup_{x\in\partial I} |g_{\alpha,\epsilon}(x)| \cdot |I| + \|g_{\alpha,\epsilon}\|_{W^{2,2}} + \|h_{\alpha,\epsilon}\|_{W^{0,2}} \Big). \tag{3.39}$$

Now, since $g_{\alpha,\epsilon}$ and $h_{\alpha,\epsilon}$, $\alpha \in \mathcal{J}$, are all bounded by Λ_ϵ, which has polynomial growth with respect to ϵ, we obtain

$$\|u_{\alpha,\epsilon}\|_{W^{2,2}} \leq \tilde{C}\epsilon^a, \quad \alpha \in \mathcal{J},$$

for an appropriate $a \in \mathbb{R}$, $\tilde{C} > 0$. Thus, there exists $p > 1$ such that

$$\sum_{\alpha\in\mathcal{J}} \|u_{\alpha,\epsilon}(x)\|_{W^{2,2}}^2 (2\mathbb{N})^{-p\alpha} \leq \tilde{C}^2 \epsilon^{2a} \sum_{\alpha\in\mathcal{J}} (2\mathbb{N})^{-p\alpha} < \infty.$$

This proves that $[u_\epsilon(x,\omega)] \in \mathcal{G}(W^{2,2}; (S)_{-1})$.

The definition of operators in $\mathcal{L}_0$ implies that with some other representatives of h, g and the coefficients of L in (3.37), another representative of $u = [u_\epsilon]$ also satisfies (3.36).

Now, we prove uniqueness of the solution: Let $u_{1,\epsilon}$ and $u_{2,\epsilon}$ satisfy

$$\begin{aligned} L_{1,\epsilon}\, u_\epsilon(x,\omega) &= h_{1,\epsilon}(x,\omega) \qquad \text{in } W^{2,2}(I) \otimes (S)_{-1} \\ u_\epsilon(x,\omega) \restriction_{\partial I} &= g_{1,\epsilon}(x,\omega) \restriction_{\partial I}, \quad \epsilon \in (0,1), \end{aligned} \tag{3.40}$$

and

$$\begin{aligned} L_{2,\epsilon}\, u_\epsilon(x,\omega) &= h_{2,\epsilon}(x,\omega) && \text{in } W^{2,2}(I)\otimes(S)_{-1} \\ u_\epsilon(x,\omega)\restriction_{\partial I} &= g_{2,\epsilon}(x,\omega)\restriction_{\partial I}, && \epsilon\in(0,1), \end{aligned} \tag{3.41}$$

respectively, where the operators $L_{k,\epsilon}$, $k=1,2$, are of the form (3.33), if we replace the coefficients with $a^{ij}_{k,\epsilon}$, $b^i_{k,\epsilon}$, $c^i_{k,\epsilon}$, $d_{k,\epsilon}$, $k=1,2$, respectively, and $(a^{ij}_{1,\epsilon}-a^{ij}_{2,\epsilon})$, $(b^i_{1,\epsilon}-b^i_{2,\epsilon})$, $(c^i_{1,\epsilon}-c^i_{2,\epsilon})$, $(d_{1,\epsilon}-d_{2,\epsilon})\in\mathcal{N}(W^{2,2})$, and $(h_{1,\epsilon}-h_{2,\epsilon})$ $(g_{1,\epsilon}-g_{2,\epsilon})\in\mathcal{N}(W^{2,2};(S)_{-1})$ holds.

We will prove $(u_{1,\epsilon}-u_{2,\epsilon})\in\mathcal{N}(W^{2,2};(S)_{-1})$, which establishes the assertion $[u_{1,\epsilon}]=[u_{2,\epsilon}]$.

For technical simplicity, for $k=1,2$, let us denote by $A_{k,\epsilon}$ the matrix $[a^{ij}_{k,\epsilon}]_{i,j=n\times n}$, and by $b_{k,\epsilon}$, $c_{k,\epsilon}$ the n-dimensional vectors $(b^1_{k,\epsilon},\dots,b^n_{k,\epsilon})$, $(c^1_{k,\epsilon},\dots,c^n_{k,\epsilon})$, respectively. Then (3.33) can be written as

$$L_{k,\epsilon}u_\epsilon = \nabla\cdot(A_{k,\epsilon}\cdot\nabla u_\epsilon + b_{k,\epsilon}\cdot u_\epsilon) + c_{k,\epsilon}\cdot\nabla u_\epsilon + d_{k,\epsilon}u_\epsilon,$$

where $\cdot$ denotes the standard scalar product in $\mathbb{R}^n$.

If we insert $u_{1,\epsilon}$ into (3.40) and $u_{2,\epsilon}$ into (3.41), then subtract (3.40) from (3.41), we obtain

$$\begin{aligned} &\nabla\cdot((A_{2,\epsilon}-A_{1,\epsilon})\cdot\nabla u_{1,\epsilon} + (b_{2,\epsilon}-b_{1,\epsilon})\cdot u_{1,\epsilon}) + (c_{2,\epsilon}-c_{1,\epsilon})\cdot\nabla u_{1,\epsilon} \\ &\quad + (d_{2,\epsilon}-d_{1,\epsilon})u_{1,\epsilon} + \nabla\cdot(A_{1,\epsilon}\cdot\nabla(u_{2,\epsilon}-u_{1,\epsilon}) + b_{1,\epsilon}\cdot(u_{2,\epsilon}-u_{1,\epsilon})) \\ &\quad + c_{1,\epsilon}\cdot\nabla(u_{2,\epsilon}-u_{1,\epsilon}) + d_{1,\epsilon}(u_{2,\epsilon}-u_{1,\epsilon}) \\ &\quad = h_{2,\epsilon}-h_{1,\epsilon} \\ (u_{2,\epsilon}-u_{1,\epsilon})\restriction_{\partial I} &= (g_{2,\epsilon}-g_{1,\epsilon})\restriction_{\partial I}, \quad \epsilon\in(0,1). \end{aligned}$$

Thus,

$$\begin{aligned} \nabla\cdot(A_{1,\epsilon}\cdot\nabla(u_{2,\epsilon}-u_{1,\epsilon}) + b_{1,\epsilon}\cdot(u_{2,\epsilon}-u_{1,\epsilon})) + c_{1,\epsilon}\cdot\nabla(u_{2,\epsilon}-u_{1,\epsilon}) + d_{1,\epsilon}(u_{2,\epsilon}-u_{1,\epsilon}) \\ = h_{2,\epsilon}-h_{1,\epsilon} - (\nabla\cdot((A_{2,\epsilon}-A_{1,\epsilon})\cdot\nabla u_{1,\epsilon} \\ + (b_{2,\epsilon}-b_{1,\epsilon})\cdot u_{1,\epsilon}) + (c_{2,\epsilon}-c_{1,\epsilon})\cdot\nabla u_{1,\epsilon} + (d_{2,\epsilon}-d_{1,\epsilon})u_{1,\epsilon}) \\ = H_\epsilon \\ (u_{2,\epsilon}-u_{1,\epsilon})\restriction_{\partial I} = (g_{2,\epsilon}-g_{1,\epsilon})\restriction_{\partial I}, \quad \epsilon\in(0,1), \end{aligned}$$

where $H_\epsilon\in\mathcal{N}(W^{2,2};(S)_{-1})$. Using the estimate as in (3.39) we finally obtain that there exist $p>1$ and $C>0$ such that for all $a\in\mathbb{R}$

$$\sum_{\alpha\in\mathcal{J}}\|u_{\alpha,2,\epsilon}(x)-u_{\alpha,1,\epsilon}(x)\|^2_{W^{2,2}}(2\mathbb{N})^{-p\alpha} \le C^2\epsilon^{2a}\sum_{\alpha\in\mathcal{J}}(2\mathbb{N})^{-p\alpha} < \infty.$$

Thus, $(u_{2,\epsilon} - u_{1,\epsilon}) \in \mathcal{N}(W^{2,2}; (S)_{-1})$. □

Remark. Considering only the term for $\alpha = (0,0,0,\dots)$ in the expansion we get that the generalized expectation $E([u_\epsilon(x,\cdot)])$ coincides with the Colombeau solution $[v_\epsilon(x)]$ of the deterministic Dirichlet problem

$$E([L_\epsilon])\,[v_\epsilon] = E([h_{\alpha,\epsilon}]), \qquad [v_\epsilon] \restriction_{\partial I} = E([g_{\alpha,\epsilon}]) \restriction_{\partial I} .$$

3.2 A Linear Elliptic Dirichlet Problem with Stochastic Coefficients and Stochastic Data

This section is devoted to the Dirichlet problem (3.1), when the coefficients of the operator L are also generalized random processes. We will give an interpretation of the operator itself, of the equation and its solution in terms of Wick products. In praxis, the Wick product showed not just as typographical phantasy, but also as a good model in physical interpretation. Historically, the Wick product first arised in quantum physics, later it was introduced in the framework of white noise analysis. It is closely related to the notion of renormalization and to the S-transform, which converts SPDEs into PDEs and converts the Wick product into ordinary products. From probabilistic viewpoint, the solutions of an SPDE may show different properties depending on the type of product (Wick or ordinary) that was used for modeling. But in order to investigate these probabilistic properties, one has to consider each SPDE separately (for examples see [HØUZ]).

The plan of exposition is following: At the very beginning we develop all necessary tools for the Hilbert space methods to be used. The solution of the stochastic Dirichlet problem, similarly as in the case of deterministic coefficients, exists and is unique. In Theorem 3.2.4 we investigate the stability and regularity properties of this solution. The aim of Theorem 3.2.8 is to extend the approach for even more singular stochastic coefficients, namely Colombeau generalized stochastic processes. We prove existence and uniqueness of the solution for the Dirichlet problem in this setting, too.

In order to prove existence and uniqueness of the solution of (3.1) provided the coefficients of the operator L are also generalized random processes, we have to go deeper into the Hilbert space structure of $\mathcal{L}(W^{1,2}, (S)_{-1,-p})$. The main reference for this remain [GT] and [Ev]. Throughout this part of the paper $\langle\cdot,\cdot\rangle$ will denote (unless otherwise stated) the dual pairing of $L^2(I) \otimes (S)_{-1,-p}$ and $L^2(I) \otimes (S)_{1,p}$. That is, for $F(x,\omega) = \sum_{\alpha\in\mathcal{J}} F_\alpha(x) \otimes H_\alpha(\omega)$, $F_\alpha \in W^{-1,2}(I)$, such that $\sum_{\alpha\in\mathcal{J}} \|F_\alpha\|^2_{W^{-1,2}} (2\mathbb{N})^{-p\alpha} < \infty$ and $f(x,\omega) =$

$\sum_{\alpha\in\mathcal{I}} f_\alpha(x)H_\alpha(\omega)$, $f_\alpha \in W_0^{1,2}(I)$ such that $\sum_{\alpha\in\mathcal{I}} \|f_\alpha\|^2_{W^{1,2}}(2\mathbb{N})^{p\alpha} < \infty$, the dual action is:

$$\langle F, f\rangle = \sum_{\alpha\in\mathcal{I}} \alpha!\langle F_\alpha, f_\alpha\rangle_{L^2(I)}.$$

Clearly, the norm corresponding to $\langle\cdot,\cdot\rangle$ is $\|F\| = \sqrt{\sum_{\alpha\in\mathcal{I}} \alpha!\|F_\alpha\|^2_{L^2(I)}}$.

3.2.1 Wick products

The product between two generalized random processes in (3.2) will be interpreted as the Wick product (recall Definition 2.3.1).

We will now define the Wick product of two generalized random processes (recall Theorem 3.1.5) in an analogous way. First we introduce the class of *essentially bounded* (in the x variable) generalized random processes, for which this product will be well defined.

Definition 3.2.1 *Let $F \in \mathcal{WS}^*$ be a generalized random process given by chaos expansion $F(x,\omega) = \sum_{\alpha\in\mathcal{I}} f_\alpha(x)\otimes H_\alpha(\omega)$. We will call F an* essentially bounded generalized random process*, if $f_\alpha \in L^\infty(I)$ for all $\alpha \in \mathcal{I}$ and if there exists $p > 0$ such that*

$$\sum_{\alpha\in\mathcal{I}} \|f_\alpha\|^2_{L^\infty(I)}(2\mathbb{N})^{-p\alpha} < \infty.$$

Note that we may consider essentially bounded GRPs as elements of the tensor product space $L^\infty(I)\otimes(S)_{-1}$. This follows from the fact that $L^1(I)$ is complete and $(S)_1$ is nuclear, thus $L^\infty(I)\otimes(S)_{-1} \cong \mathcal{L}(L^1(I);(S)_{-1})$. Clearly, for fixed $p > 0$ we also have $L^\infty(I)\otimes(S)_{-1,-p} \cong \mathcal{L}(L^1(I);(S)_{-1,-p})$.

Lemma 3.2.1 *F is an essentially bounded GRP if and only if there exists $C \in (S)_{-1}$ such that*

$$\operatorname*{esp}_{x\in I} F(x,\omega) = C(\omega) \quad \textit{for a.e. } \omega\in\Omega,$$

where $\operatorname{esp}_{x\in I} F(x,\omega) = \sum_{\alpha\in\mathcal{I}} \operatorname{essup}_{x\in I}|f_\alpha(x)|H_\alpha(\omega) = \sum_{\alpha\in\mathcal{I}} \|f_\alpha\|_{L^\infty(I)}H_\alpha(\omega)$, *provided that* $\operatorname{essup}_{x\in I}|f_\alpha(x)|$ *exists for all $\alpha\in\mathcal{I}$.*

Proof. Put $C(\omega) = \sum_{\alpha\in\mathcal{I}} \|f_\alpha\|_{L^\infty(I)}H_\alpha(\omega)$. □

Definition 3.2.2 *Let F be an essentially bounded GRP given by $F(x,\omega) = \sum_{\alpha\in\mathcal{I}} f_\alpha(x)\otimes H_\alpha(\omega)$, $f_\alpha \in L^\infty(I)$. Let $G \in \mathcal{WS}^*$ be given by its chaos* *expansion* $G(\omega) = \sum_{\beta\in\mathcal{J}} g_\beta(x)\otimes H_\beta(\omega)$, $g_\beta \in W^{-1,2}(I)$. *The Wick product of* F *and* G *is the unique element in* $\mathcal{WS}^*$ *defined by:*

$$F\lozenge G(x,\omega) = \sum_{\gamma\in\mathcal{J}}\left(\sum_{\alpha+\beta=\gamma} f_\alpha(x)g_\beta(x)\right)\otimes H_\gamma(\omega).$$

The next lemma shows that the Wick product is well defined, and that for fixed F the mapping $G\mapsto F\lozenge G$ is continuous.

Lemma 3.2.2 *If* F *is essentially bounded, and if* $G\in\mathcal{WS}^*$, *then* $F\lozenge G\in\mathcal{WS}^*$. *Moreover, there exist* $C,p,q,r>0$ *such that:*

$$\|F\lozenge G\|_{W^{-1,2}\otimes(S)_{-1,-p}} \le C\|F\|_{L^\infty\otimes(S)_{-1,-q}}\|G\|_{W^{-1,2}\otimes(S)_{-1,-r}}.$$

Proof.

Since F is essentially bounded and G is an element of $\mathcal{WS}^*$, there exist $q>0$ and $r>0$ such that

$$\sum_\alpha \|f_\alpha\|^2_{L^\infty}(2\mathbb{N})^{-q\alpha}<\infty \quad\text{and}\quad \sum_\beta \|g_\beta\|^2_{W^{-1,2}}(2\mathbb{N})^{-r\alpha}<\infty.$$

By definition of the Wick product and using the Cauchy-Schwartz inequality, we have for $C=\sum_\gamma(2\mathbb{N})^{-k\gamma}$ and $p=q+r+k$, for arbitrary $k>1$

$$\sum_\gamma\left\|\sum_{\alpha+\beta=\gamma} f_\alpha(x)g_\beta(x)\right\|^2_{W^{-1,2}}(2\mathbb{N})^{-p\gamma}$$

$$\le \sum_\gamma(2\mathbb{N})^{-k\gamma}(2\mathbb{N})^{-(q+r)\gamma}\left(\sum_{\alpha+\beta=\gamma}\|f_\alpha\|^2_{L^\infty(I)}\right)\left(\sum_{\alpha+\beta=\gamma}\|g_\beta\|^2_{W^{-1,2}}\right)$$

$$\le \sum_\gamma(2\mathbb{N})^{-k\gamma}\left(\sum_\alpha\|f_\alpha\|^2_{L^\infty(I)}(2\mathbb{N})^{-q\alpha}\right)\left(\sum_\beta\|g_\beta\|^2_{W^{-1,2}}(2\mathbb{N})^{-r\beta}\right). \qquad \square$$

3.2.2 Interpretation of the operator L

According to Definition 3.2.2, we will assume that the coefficients a^{ij}, b^i, c^i, d, for $i,j=1,2\dots,n$, of the operator L are essentially bounded GRPs, and will thus interpret the action of L onto a GRP $u\in\mathcal{WS}^*$ as:

$$
\begin{aligned}
L\lozenge u(x,\omega) = \sum_{i=1}^{n} D_i\Big(\sum_{j=1}^{n} a^{ij}(x,\omega)\lozenge D_j u(x,\omega) + b^i(x,\omega)\lozenge u(x,\omega)\Big)& \\
+ \sum_{i=1}^{n} c^i(x,\omega)\lozenge D_i u(x,\omega) + d(x,\omega)\lozenge u(x,\omega).&
\end{aligned}
\tag{3.42}
$$

Now the operator L acts as a differential operator (in the x variable) and as a Wick–multiplication operator (in the ω variable) as well. Thus, it is natural to consider the "package" $L\lozenge$ as a whole unit. But now we have to take into account that for Wick multiplication in general $\langle f\lozenge g, h\rangle \neq \langle f, g\lozenge h\rangle$, while in the deterministic case we used $\langle fg, h\rangle = \langle f, gh\rangle$ for ordinary multiplication of functions. Nevertheless, we will develop the necessary tools for Wick calculations.

First we note that Wick multiplication satisfies the chain rule:

Lemma 3.2.3 *For an arbitrary GRP f such that $D_i f$, $i = 1,2,\ldots,n$, are essentially bounded and arbitrary $g \in \mathcal{WS}^*$ we have*

$$
D_i(f\lozenge g) = (D_i f)\lozenge g + f\lozenge(D_i g), \qquad i = 1,2,\ldots,n.
$$

Proof. Let $f(x,\omega) = \sum_\alpha f_\alpha(x)\otimes H_\alpha(\omega)$, $g(x,\omega) = \sum_\beta g_\beta(x)\otimes H_\beta(\omega)$. Then, for arbitrary $i = 1,2,\ldots,n$ (recall, D_i denotes the weak derivative with respect to x_i), we have

$$
D_i f(x,\omega) = \sum_\alpha D_i f_\alpha(x)\otimes H_\alpha(\omega), \quad \text{and} \quad D_i g(x,\omega) = \sum_\beta D_i g_\beta(x)\otimes H_\beta(\omega).
$$

Due to definition of Wick multiplication, $f\lozenge g(x,\omega) = \sum_\gamma \left(\sum_{\alpha+\beta=\gamma} f_\alpha(x) g_\beta(x)\right)\otimes H_\gamma(\omega)$, and thus

$$
\begin{aligned}
D_i f\lozenge g(x,\omega) &= \sum_\gamma D_i\left(\sum_{\alpha+\beta=\gamma} f_\alpha(x) g_\beta(x)\right)\otimes H_\gamma(\omega) \\
&= \sum_\gamma \left(\sum_{\alpha+\beta=\gamma} D_i(f_\alpha(x) g_\beta(x))\right)\otimes H_\gamma(\omega) \\
&= \sum_\gamma \left(\sum_{\alpha+\beta=\gamma} D_i f_\alpha(x) g_\beta(x) + f_\alpha(x) D_i g_\beta(x)\right)\otimes H_\gamma(\omega) \\
&= D_i f(x,\omega)\lozenge g(x,\omega) + f(x,\omega)\lozenge D_i g(x,\omega).
\end{aligned}
$$

□

Fix now an essentially bounded GRP $a \in L^\infty(I) \otimes (S)_{-1,-p}$. Due to Lemma 3.2.2 we have that the operator $A : \mathcal{W}\mathcal{S}^* \to \mathcal{W}\mathcal{S}^*$ defined by $g \mapsto a\Diamond g$ is a continuous linear operator from the Hilbert space $W^{-1,2} \otimes (S)_{-1,-p}$ into the Hilbert space $W^{-1,2} \otimes (S)_{-1,-2(p+1)}$. (Note, we fixed $k = 2$ from the proof of Lemma 3.2.2). For a fixed $c \in W_0^{1,2} \otimes (S)_{1,2(p+1)}$ consider the operator $F_c : W^{-1,2} \otimes (S)_{-1,-2(p+1)} \to \mathbb{R}$, given by $F_c(b) = \langle a\Diamond b, c\rangle$. According to Lemma 3.2.2 and the Cauchy-Schwartz inequality we have

$$|F_c(b)| = |\langle a\Diamond b, c\rangle| \leq$$

$$\|a\|_{L^\infty(I)\otimes(S)_{-1,-p}} \|b\|_{W^{-1,2}\otimes(S)_{-1,-2(p+1)}} \|c\|_{W^{1,2}\otimes(S)_{1,2(p+1)}}$$

i.e. F_c is a linear continuous operator on the Hilbert space $W^{-1,2} \otimes (S)_{-1,-2(p+1)}$. Due to the Riesz representation theorem there exists a unique $f_c \in W_0^{1,2} \otimes (S)_{1,2(p+1)}$ such that $F_c(b) = \langle b, f_c\rangle$.

This defines a mapping $c \mapsto f_c$, which we will denote by $a^\circledast$. In fact, $a^\circledast : c \mapsto f_c$ is the adjoint operator of A.

Definition 3.2.3 *The unique linear continuous mapping* $a^\circledast : W_0^{1,2} \otimes (S)_{1,2(p+1)} \to W_0^{1,2} \otimes (S)_{1,2(p+1)}$ *such that for each* $b \in W^{-1,2} \otimes (S)_{-1,-p}$ *and* $c \in W_0^{1,2} \otimes (S)_{1,2(p+1)}$

$$\langle a\Diamond b, c\rangle = \langle b, a^\circledast(c)\rangle \tag{3.43}$$

holds, is called the Wick–adjoint multiplication operator *of the generalized random process* $a \in L^\infty(I) \otimes (S)_{-1,-p}$.

In further notation we will suppress the indexes, and simply denote $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$. From the context it is clear whether the index is $L^\infty(I) \otimes (S)_{-1,-p}$ or $L^2(I) \otimes (S)_{\pm1,\pm p}$ or $L^2(I) \otimes (S)_{\pm1,\pm2(p+1)}$.

Due to the previous definition we are able to develop something that might be called a *weak Wick* calculus: The Wick-adjoint operator inherits its properties from the classical Wick multiplication – most important, the chain rule holds in weak sense.

Lemma 3.2.4 *For an arbitrary GRP* f *such that* $D_i f$, $i = 1, 2, \ldots, n$, *are essentially bounded and arbitrary* $g \in \mathcal{W}\mathcal{S}^*$ *we have*

$$\langle D_i(f^\circledast g), v\rangle = \langle (D_i f^\circledast)g + f^\circledast(D_i g), v\rangle,$$

for all $v \in W_0^{1,2} \otimes (S)_{1,p}$ *and* $i = 1, 2, \ldots, n$.

Proof. From the definition of the Wick–adjoint multiplication we have

$$\langle D_i(f^{\circledast}g), v\rangle = -\langle f^{\circledast}g, D_i v\rangle = -\langle g, f\Diamond D_i v\rangle$$
$$= -\langle g, D_i(f\Diamond v) - (D_i f)\Diamond v\rangle = \langle D_i g, f\Diamond v\rangle + \langle g, (D_i f)\Diamond v\rangle$$
$$= \langle f^{\circledast}(D_i g), v\rangle + \langle (D_i f^{\circledast})g\rangle.$$

where we used the chain rule for Wick multiplication (Lema 3.2.3). □

Later we will see that all properties of the operator L (ellipticity and other assumptions on its coefficients) can be carried over to the formal Wick–adjoint of L.

Assumptions on L. The concept of generalized weak solutions.

The assumptions on L must also be modified to be compatible with the Wick calculus. Thus, we imply following assumptions on L: There exists $\lambda > 0$ such that for all $u, v_1, v_2, \ldots v_n \in \mathcal{W}_0\mathcal{S}$ and all $v \geq 0 \in \mathcal{W}_0\mathcal{S}$ following conditions hold:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\langle a^{ij}\Diamond v_j, v_i\rangle \geq \lambda \sum_{i=1}^{n}\|v_i\|^2 \qquad \text{(ellipticity)}, \tag{3.44}$$

$$a^{ij}, b^i, c^i, d, \ (i, j, = 1, 2, \ldots, n) \quad \text{are essentially bounded GRPs }, \tag{3.45}$$

$$\langle d\Diamond u, v\rangle - \sum_{i=1}^{n}\langle b^i\Diamond u, D_i v\rangle \leq 0. \tag{3.46}$$

Conditions (3.45) and (3.46) will be adequate to prove existence and uniqueness of the generalized weak solution of (3.47). Later on we will imply a stronger condition to (3.45) and a weaker condition to (3.46) in order to prove stability and regularity properties.

We turn now back to our stochastic Dirichlet problem

$$\begin{aligned} &L\Diamond u(x,\omega) = h(x,\omega) + \sum_{i=1}^{n} D_i f^i(x,\omega), \qquad x \in I, \omega \in \Omega, \\ &u(x,\omega)\restriction_{\partial I} = g(x,\omega) \end{aligned} \tag{3.47}$$

where the action $L\Diamond u$ is defined as in (3.42). Applying partial integration we obtain

$$\langle L\Diamond u, v\rangle = \sum_{i=1}^{n}\langle D_i(\sum_{j=1}^{n} a^{ij}\Diamond D_j u + b^i\Diamond u), v\rangle + \sum_{i=1}^{n}\langle c^i\Diamond D_i u, v\rangle + \langle d\Diamond u, v\rangle$$

$$= -\sum_{i=1}^{n}\sum_{j=1}^{n}\langle a^{ij}\lozenge D_j u, D_i v\rangle - \sum_{i=1}^{n}\langle b^i\lozenge u, D_i v\rangle + \sum_{i=1}^{n}\langle c^i\lozenge D_i u, v\rangle + \langle d\lozenge u, v\rangle.$$

for $u \in \mathcal{W}\mathcal{S}^*, v \in \mathcal{W}_0\mathcal{S}$.

As in [GT], we associate a bilinear form $B : \mathcal{W}\mathcal{S}^* \times W_0 S \to \mathbb{R}$ associated with L, defined by

$$B(u,v) = -\langle L\lozenge u, v\rangle$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\langle a^{ij}\lozenge D_j u, D_i v\rangle + \sum_{i=1}^{n}\langle b^i\lozenge u, D_i v\rangle - \sum_{i=1}^{n}\langle c^i\lozenge D_i u, v\rangle - \langle d\lozenge u, v\rangle.$$

We call $u \in \mathcal{W}\mathcal{S}^*$ a *generalized weak solution* of (3.47) if $\langle L\lozenge u, v\rangle = \langle h + \sum_{i=1}^{n} D_i f^i, v\rangle = \langle h, v\rangle - \sum_{i=1}^{n}\langle f^i, D_i v\rangle$ for all $v \in W_0 S$, i.e. if

$$B(u,v) = -\langle h, v\rangle + \sum_{i=1}^{n}\langle f^i, D_i v\rangle$$

for all $v \in \mathcal{W}_0\mathcal{S}$.

Lemma 3.2.5 *The bilinear form $B(\cdot,\cdot)$ is continuous.*

Proof. Since the Wick product is distributive with respect to addition, B is indeed bilinear. Continuity (boundedness) follows from (3.45) and the Cauchy–Schwartz inequality:

$$|B(u,v)| \le \sum_{i=1}^{n}\sum_{j=1}^{n}|\langle a^{ij}\lozenge D_j u + b^i\lozenge u, D_i v\rangle| + \sum_{i=1}^{n}|\langle c^i\lozenge D_i u + d\lozenge u, v|\rangle$$

$$\le \sum_{i=1}^{n}\sum_{j=1}^{n}\|a^{ij}\|\|D_j u\|\|D_i v\| + \sum_{i=1}^{n}\|b^i\|\|u\|\|D_i v\| + \sum_{i=1}^{n}\|c^i\|\|D_i u\|\|v\| + \|d\|\|u\|\|v\|$$

$$\le C\left(\sum_{i=0}^{n}\|D_i u\|\right)\left(\sum_{i=0}^{n}\|D_i v\|\right) = C\|u\|_{\mathcal{W}\mathcal{S}^*}\|v\|_{\mathcal{W}_0\mathcal{S}},$$

where $C = \max_{1\le i,j\le n}\{\|a^{ij}\|, \|b^i\|, \|c^i\|, \|d\|\}$. Thus, B is indeed a bilinear continuous mapping $B : \mathcal{W}\mathcal{S}^* \times \mathcal{W}_0\mathcal{S} \to \mathbb{R}$. □

Now, similarly as in the deterministic case, we can identify the operator $L\lozenge$ with its unique extension $L\lozenge : \mathcal{W}\mathcal{S} \to \mathcal{W}\mathcal{S}^*$ defined via the bilinear form B. Existence of a weak generalized solution is equivalent to surjectivity, while uniqueness is equivalent to injectivity of the mapping $L\lozenge$.

3.2.3 The stochastic weak maximum principle

First recall that $v \in \mathcal{W}_0\mathcal{S}$ is called positive, denoted by $v \geq 0$, if it has expansion $v(x,\omega) = \sum_{\alpha\in\mathcal{J}} v_\alpha(x) \otimes H_\alpha(\omega)$, $v_\alpha \in W_0^{1,2}$, and $v_\alpha(x) \geq 0$ for all $x \in I$, $\alpha \in \mathcal{J}$. An element $u \in \mathcal{W}\mathcal{S}^*$ is positive in weak sense, denoted by $u \geq 0$ if $\langle u, v\rangle \geq 0$ for all $v \in \mathcal{W}_0\mathcal{S}$, $v \geq 0$. Note that if $u \in \mathcal{W}\mathcal{S}$ has expansion $u(x,\omega) = \sum_{\alpha\in\mathcal{J}} u_\alpha(x) \otimes H_\alpha(\omega)$, $u_\alpha \in W^{1,2}$, and $u_\alpha(x) \geq 0$ for all $x \in I$, $\alpha \in \mathcal{J}$, then u is also positive in weak sense (this follows from the fact that for each $v \in \mathcal{W}_0\mathcal{S}$, $v \geq 0$, we have $\langle u, v\rangle = \sum_{\alpha\in\mathcal{J}} \alpha! \int_I u_\alpha(x) v_\alpha(x) dx \geq 0$).

Now we introduce some terminology convention. In a weak sense, $u \in \mathcal{W}\mathcal{S}^*$ is said to satisfy $L\Diamond u \geq 0$ in I, if $B(u,v) \leq 0$ for all $v \geq 0$. Respectively, $L\Diamond u \leq 0$, if $B(u,v) \geq 0$ for all $v \geq 0$.

For $u \in \mathcal{W}\mathcal{S}$, $u(x,\omega) = \sum_{\alpha\in\mathcal{J}} u_\alpha(x) \otimes H_\alpha(\omega)$, $u_\alpha \in W^{1,2}$, define $u^+(x,\omega) = \sum_{\alpha\in\mathcal{J}} u_\alpha^+(x) \otimes H_\alpha(\omega) = \sum_{\alpha\in\mathcal{J}} \max\{u_\alpha(x), 0\} \otimes H_\alpha(\omega)$. Let $u(x,\omega) \leq 0$ on ∂I if $u^+(x,\omega) \in \mathcal{W}_0\mathcal{S}$ i.e. if $u_\alpha^+(x) \in W_0^{1,2}$ for all $\alpha \in \mathcal{J}$.

Also, for $u \in \mathcal{W}\mathcal{S}^*$, $u(x,\omega) = \sum_{\alpha\in\mathcal{J}} u_\alpha(x) \otimes H_\alpha(\omega)$, $u_\alpha \in W^{-1,2}$, we define:

$$\begin{aligned}
\operatorname*{Sup}_{x\in I} u(x,\omega) &= \sum_{\alpha\in\mathcal{J}} \sup_{x\in I} u_\alpha(x) H_\alpha(\omega) \\
&= \sum_{\alpha\in\mathcal{J}} \inf\{k \in \mathbb{R} | \forall x \in I, \quad u_\alpha(x) \leq k\} H_\alpha(\omega) \\
&= \sum_{\alpha\in\mathcal{J}} k_\alpha H_\alpha(\omega) = K(\omega) \in (S)_{-1}.
\end{aligned}$$

Clearly, $k_\alpha \leq \|u_\alpha\|_{W^{1,2}}$, which implies $\sum_{\alpha\in\mathcal{J}} k_\alpha^2 (2\mathbb{N})^{-p\alpha} < \infty$ for some $p > 0$ and thus $\operatorname{Sup}_{x\in I} u(x,\omega) \in (S)_{-1}$. Let $\operatorname{Inf}_{x\in I} u(x,\omega) = -\operatorname{Sup}_{x\in I}(-u(x,\omega))$. In a similar manner on can define $\operatorname{Sup}_{x\in\partial I} u(x,\omega)$ and $\operatorname{Inf}_{x\in\partial I} u(x,\omega)$.

Note that *Inf* and *Sup* are only notations, they do not mean a classical infimum or supremum, since for fixed $x \in I$, $u(x,\cdot)$ is an element of $(S)_{-1}$ which has no partial ordering.

We use the same procedure to define $\operatorname{Spt} u(x,\omega) = \bigcup_{\alpha\in\mathcal{J}} \operatorname{supp} u_\alpha(x)$, and note that it is not a support in classical sense since it is not necessarily a closed set.

Theorem 3.2.1 *(i) Let $u \in \mathcal{W}\mathcal{S}^*$ satisfy $L\Diamond u \geq 0$ in I. Then*

$$\operatorname*{Sup}_{x\in I} u(x,\omega) \leq \operatorname*{Sup}_{x\in\partial I} u^+(x,\omega)$$

(i.e. $\operatorname{Sup}_{x\in\partial I} u^+(x,\omega) - \operatorname{Sup}_{x\in I} u(x,\omega)$ is positive in weak sense).

(ii) Let $u \in \mathcal{W}\mathcal{S}^*$ satisfy $L\Diamond u \leq 0$ in I. Then

$$\operatorname*{Inf}_{x\in I} u(x,\omega) \geq \operatorname*{Inf}_{x\in\partial I} u^-(x,\omega).$$

Proof. (i) Let $B(u,v) \le 0$. Then, using (3.46) and boundedness of the coefficients c^i we obtain

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n}\langle a^{ij}\lozenge D_j u, D_i v\rangle &\le \sum_{i=1}^{n}\langle c^i\lozenge D_i u, v\rangle \\ &\le C\sum_{i=1}^{n}\|D_i u\|\,\|v\|\end{aligned}\tag{3.48}$$

for some constant $C > 0$.

If $c^i = 0$, $i = 1,2,\dots,n$, then put

$$\begin{aligned}v(x,\omega) &= \max\{u(x,\omega) - \operatorname*{Sup}_{x\in\partial I} u^+(x,\omega),\ 0\}\\ &= \sum_{\alpha\in\mathcal{I}}\max\{u_\alpha(x) - \sup_{x\in\partial I} u^+_\alpha(x), 0\}\otimes H_\alpha(\omega)\end{aligned}$$

and note that $v \ge 0$ (since $v_\alpha(x) = \max\{u_\alpha(x) - \sup_{x\in\partial I} u^+_\alpha(x), 0\} \ge 0$ for each $\alpha \in \mathcal{I}$), and $D_i v = D_i u$, $i = 1,2,\dots,n$. Now from (3.48) and the ellipticity condition (3.44) we retain that

$$\lambda\sum_{i=1}^{n}\|D_i v\|^2 \le \sum_{i=1}^{n}\sum_{j=1}^{n}\langle a^{ij}\lozenge D_j v, D_i v\rangle \le 0,$$

i.e. $\|D_i v\| = 0$, for all $i = 1,2,\dots,n$. Thus, $v(x,\omega)$ is constant in the x variable, i.e. $v(x,\omega) = V(\omega) \in (S)_{-1}$. Now using the Poincaré inequality $\|v\| \le K\|Dv\|$ we obtain $\|v\| = 0$ i.e. $v_\alpha(x) = 0$ for a.e. $x \in I$ and all $\alpha \in \mathcal{I}$. Thus, for every $\alpha \in \mathcal{I}$ we have $\sup_{x\in I} u_\alpha(x) - \sup_{x\in\partial I} u^+_\alpha(x) \le 0$. From this follows that we have also in weak sense $\operatorname{Sup}_{x\in I} u(x,\omega) - \operatorname{Sup}_{x\in\partial I} u^+(x,\omega) \le 0$.

If there exists $i = 1,2,\dots,n$ such that $c^i \ne 0$, we follow a similar idea. Assume, there exists $K(\omega) = \sum_{\alpha\in\mathcal{I}} k_\alpha H_\alpha(\omega) \in (S)_{-1}$ such that $\sup_{x\in\partial I} u^+_\alpha(x) \le k_\alpha < \sup_{x\in I} u_\alpha(x)$, $\alpha \in \mathcal{I}$, and put $v(x,\omega) = \sum_{\alpha\in\mathcal{I}} v_\alpha(x)\otimes H_\alpha(\omega)$, where $v_\alpha(x) = \max\{u_\alpha(x) - k_\alpha,\ 0\}$, $\alpha \in \mathcal{I}$.

Now for each $\alpha \in \mathcal{I}$ we have $D_i v_\alpha = D_i u_\alpha$ for $u_\alpha > k_\alpha$ (i.e. for $v_\alpha \ne 0$) and $D_i v_\alpha = 0$ for $u_\alpha \le k_\alpha$ (i.e. for $v_\alpha = 0$). Now from (3.48) and the ellipticity condition we retain that

$$\lambda\sum_{i=1}^{n}\|D_i v\|^2 \le \sum_{i=1}^{n}\sum_{j=1}^{n}\langle a^{ij}\lozenge D_j v, D_i v\rangle \le C\sum_{i=1}^{n}\|D_i v\|\,\|v\|,$$

and consequently $\sum_{i=1}^{n}\|D_i v\| \le \frac{2C}{\lambda}\|v\|$. Now

$$\sum_{i=1}^{n}\left(\sum_{\alpha\in\mathcal{I}}\alpha!\|D_i v_\alpha\|^2_{L^2(I)}\right)^{\frac{1}{2}} \le \frac{2C}{\lambda}\left(\sum_{\alpha\in\mathcal{I}}\alpha!\|v_\alpha\|^2_{L^2(I)}\right)^{\frac{1}{2}}$$

implies

$$\sum_{i=1}^{n} \|D_i v_\alpha\|_{L^2(I)} \leq \tilde{C}\|v_\alpha\|_{L^2(I)}, \quad \alpha \in \mathcal{J}, \tag{3.49}$$

for an appropriate $\tilde{C} > 0$. But now, using Sobolev type inequalities one can prove that (3.49) implies that $\operatorname{supp} Dv_\alpha$ is a set of strictly positive measure. This is a contradiction with $Du_\alpha = 0$. Thus, $\sup_{x\in I} u_\alpha \leq \sup_{x\in\partial I} u_\alpha^+$ for all $\alpha \in \mathcal{J}$. This proves the assertion. □

Remark. Instead of condition (3.46) we could have used (similarly as it is done in [GT]) the following equivalent condition:

$$\langle d\lozenge u, v\rangle + \sum_{i=1}^{n} \langle c^i \lozenge u, D_i v\rangle \leq 0. \tag{3.50}$$

Theorem 3.2.1 remains valid also if we replace (3.46) by a weaker condition

$$\langle d, v\rangle - \sum_{i=1}^{n} \langle b^i, D_i v\rangle \leq 0 \tag{3.51}$$

and at the same time replace (3.45) by a stronger condition: Let there exist $p, \Lambda, \nu > 0$ such that for all $x \in I$,

$$\sum_{i,j=1}^{n} \|a^{ij}(x,\cdot)\|_{-1,-p}^2 \leq \Lambda^2 \qquad \text{and}$$

$$\frac{1}{\lambda^2}\sum_{i=1}^{n} \left(\|b^i(x,\cdot)\|_{-1,-p}^2 + \|c^i(x,\cdot)\|_{-1,-p}^2\right) + \frac{1}{\lambda}\|d(x,\cdot)\|_{-1,-p} \leq \nu^2. \tag{3.52}$$

The uniqueness of the generalized weak solution of the homogeneous Dirichlet problem now follows directly from the maximum principle:

Corollary 3.2.1 *Let $u \in \mathcal{WS}^*$ satisfy $L\lozenge u(x,\omega) = 0$ in $I \times \Omega$. Then $u = 0$.*

3.2.4 Solvability of the stochastic Dirichlet problem

First we note that it suffices to solve the Dirichlet problem (3.47) for zero boundary values. Namely, for $\tilde{u}(x,\omega) = u(x,\omega) - g(x,\omega)$ we have by linearity of the $L\Diamond$ operator:

$$\begin{aligned}
L\Diamond\tilde{u} &= L\Diamond u - L\Diamond g \\
&= h + \sum_{i=1}^{n} D_i f^i - \left(\sum_{i=1}^{n} D_i (\sum_{j=1}^{n} a^{ij}\Diamond D_j g + b^i\Diamond g) + \sum_{i=1}^{n} c^i\Diamond D_i g + d\Diamond g \right) \\
&= h - \sum_{i=1}^{n} c^i\Diamond D_i g - d\Diamond g + \sum_{i=1}^{n} D_i \left(f^i - \sum_{j=1}^{n} a^{ij}\Diamond D_j g - b^i\Diamond g \right) \\
&= \tilde{h} + \sum_{i=1}^{n} D_i \tilde{f}^i,
\end{aligned}$$

where $\tilde{h} = h - \sum_{i=1}^{n} c^i\Diamond D_i g - d\Diamond g$ and $\tilde{f}^i = f^i - \sum_{j=1}^{n} a^{ij}\Diamond D_j g - b^i\Diamond g$, $i = 1, 2, \ldots, n$. Clearly, $\tilde{u} \restriction_{\partial I} = 0$. Thus, any stochastic Dirichlet problem of the form (3.47) can be reduced to the zero boundary condition. Moreover, if $h, f^i \in L^2(I) \otimes (S)_{-1}$ and $g \in W^{1,2}(I) \otimes (S)_{-1}$, then $\tilde{h}, \tilde{f}^i \in L^2(I) \otimes (S)_{-1}$ and $\tilde{u} \in W_0^{1,2}(I) \otimes (S)_{-1}$.

The following two lemmas state that the bilinear form B associated with the operator L can be made coercive by adding a sufficiently large multiple of the identity operator to it. (Recall, Du denotes the total differential $Du = \sum_{i=1}^{n} D_i u$.)

Lemma 3.2.6 *Let L satisfy conditions (3.44) and (3.45). There exist constants $K_1, K_2 > 0$ such that*

$$B(u, u) \geq K_1 \|Du\|^2 - K_2 \|u\|^2. \tag{3.53}$$

Proof. Clearly, $\|Du\|^2 \leq 2 \sum_{i=1}^{n} \|D_i u\|^2$. Now, using the assumptions on L we get

$$\begin{aligned}
\lambda \sum_{i=1}^{n} \|D_i u\|^2 &\leq \sum_{i,j=1}^{n} \langle a^{ij}\Diamond D_j u, D_i u\rangle \\
&= B(u, u) - \sum_{i=1}^{n} \langle b^i\Diamond u, D_i u\rangle + \sum_{i=1}^{n} \langle c^i\Diamond D_i u, u\rangle + \langle d\Diamond u, u\rangle \\
&\leq B(u, u) + \sum_{i=1}^{n} \langle c^i\Diamond D_i u, u\rangle \\
&\leq B(u, u) + \sum_{i=1}^{n} \|c^i\| \langle D_i u, u\rangle \leq B(u, u) + C\langle Du, u\rangle,
\end{aligned} \tag{3.54}$$

where $C = \max\{\|c^i\|,\ i = 1, 2, \ldots, n\}$. Now by the Schwartz inequality,

$$\langle Du, u\rangle \leq \varepsilon\|Du\|^2 + \frac{1}{4\varepsilon}\|u\|^2.$$

Choose ε such that $\varepsilon C \leq \frac{\lambda}{4}$. Then from (3.54) we get

$$B(u, u) \geq \frac{\lambda}{4}\|Du\|^2 - \frac{C}{4\varepsilon}\|u\|^2. \qquad \square$$

Lemma 3.2.7 *There exists $\sigma > 0$ such that the operator*

$$L_\sigma \lozenge u = L\lozenge u - \sigma u$$

has a coercive bilinear form B_σ associated with it.

Proof. Using the result of the previous lemma we have

$$B_\sigma(u, u) = -\langle L_\sigma\lozenge u, u\rangle = -\langle L\lozenge u, u\rangle + \sigma\langle u, u\rangle = B(u, u) + \sigma\|u\|^2$$

$$\geq K_1\|Du\|^2 - K_2\|u\|^2 + \sigma\|u\|^2 \geq K_3\left(\|Du\|^2 + \|u\|^2\right) = K_3\|u\|^2_{\mathcal{WS}^*}$$

for $\sigma > K_2$. Thus, B_σ is coercive on $\mathcal{WS}^*$. $\square$

Lemma 3.2.8 *The embedding $I : \mathcal{WS} \to \mathcal{WS}^*$ defined by*

$$u \mapsto \Big(v \mapsto \langle I(u), v\rangle\Big)$$

is compact.

Proof. From [GT] we know that the embedding $W^{1,2}(I) \to L^2(I)$ is compact. From [HØUZ] we have that the embeddings $(S)_{1,p} \to (L)^2$ are of Hilbert-Schmidt type for all $p > 0$. Thus, the embedding of their projective limit $(S)_1 \to (L)^2$ is compact. It can be extended to a compact embedding $(S)_1 \to (S)_{-1}$. From these we get that the embedding $W^{1,2}(I) \otimes (S)_1 \to L^2(I) \otimes (S)_{-1}$ is also compact. $\square$

Now we are ready to state the main theorem about the existence and uniqueness of generalized weak solutions of the stochastic Dirichlet problem.

Theorem 3.2.2 *Let the operator L satisfy conditions (3.44), (3.45) and (3.46). Then for $h, f^i \in L^2(I) \otimes (S)_{-1}$, $i = 1, 2, \ldots, n$ and for $g \in W^{1,2}(I) \otimes (S)_{-1}$ the stochastic Dirichlet problem (3.47) has a unique generalized weak solution $u \in \mathcal{WS}^*$.*

Proof. Choose $\sigma > 0$ such that the bilinear form B_σ is coercive (this can be done due to Lemma 3.2.7). From Lemma 3.2.5 we know that B_σ is also continuous. According to the Lax–Milgram theorem, for L_σ there exists the inverse operator

$$(L_\sigma \Diamond)^{-1} : \mathcal{WS}^* \to \mathcal{WS}$$

and it is a continuous, injective mapping.

The mapping $F : \mathcal{W}_0\mathcal{S} \to \mathbb{R}$, $v \mapsto \langle F, v\rangle = \langle f^i, D_i v\rangle - \langle h, v\rangle$ is linear and continuous, i.e. $F \in \mathcal{WS}^*$.

The equation $L\Diamond u = F$, $u \in \mathcal{W}_0\mathcal{S}$, $F \in \mathcal{WS}^*$ is equivalent to $L_\sigma \Diamond u + \sigma I u = F$, i.e. to

$$u - (-\sigma(L_\sigma\Diamond)^{-1} I)u = (L_\sigma\Diamond)^{-1} F.$$

Due to Lemma 3.2.8 the operator $T = -\sigma(L_\sigma\Diamond)^{-1} I$ is compact. Thus, by the Fredholm alternative,

(i) either the equation $u - Tu = 0$ has a nontrivial solution $u \neq 0$,

(ii) or the equation $u - Tu = (L_\sigma\Diamond)^{-1}F$ has a unique solution u.

Case (i) is impossible, since the homogeneous equation has a unique trivial solution according to Theorem 3.2.1. Thus, case (ii) must hold true. □

Similarly as in Corollary 3.1.1, we know that the generalized expectation of the solution u can be obtained as the weak solution of the deterministic Dirichlet problem, of the form (3.47) where all the stochastic processes (coefficients of L, data and boundary value) are replaced by their generalized expectations.

Corollary 3.2.2 *Let $u \in \mathcal{WS}^*$ be the generalized solution of (3.47). Then its generalized expectation $E(u)$ coincides with the weak solution of the deterministic Dirichlet problem*

$$\begin{aligned} \tilde{L}\, v(x) &= \tilde{h}(x) + \sum_{i=1}^n D_i \tilde{f}^i(x), \qquad x \in I, \\ v(x) \restriction_{\partial I} &= \tilde{g}(x), \end{aligned} \tag{3.55}$$

where $\tilde{h} = E(h)$, $\tilde{g} = E(g)$, $\tilde{f}^i = E(f^i)$, $i = 1, 2, \ldots, n$ and

$$\tilde{L}\, v = \sum_{i=1}^n D_i(\sum_{j=1}^n E(a^{ij})D_j v + E(b^i)v) + \sum_{i=1}^n E(c^i)D_i v + E(d)v. \tag{3.56}$$

Proof. The assertion follows from the construction of a generalized weak solution if we choose a test function $v \in \mathcal{W}_0\mathcal{S}$ of the form $v(x,\omega) = w(x)\theta(\omega)$ and then put $\theta = 1$. $\square$

Remark. Since the stochastic Dirichlet problem has a unique solution, it follows from the Fredholm alternative theorem, that the operator $L^{-1} : \mathcal{WS}^* \to \mathcal{WS}$ is a bounded linear operator on $\mathcal{WS}^*$. Consequently, we have the following *apriori* estimate: Let u be the generalized weak solution of (3.47). Then there exists $p > 0$ and a constant $C > 0$ depending only on L and I such that

$$\|u\|_{W^{1,2}\otimes(S)_{-1,-p}} \leq C\left(\|\mathbf{h}\|_{L^2\otimes(S)_{-1,-p}} + \|g\|_{W^{1,2}\otimes(S)_{-1,-p}}\right) \tag{3.57}$$

where $\mathbf{h} = (h, f^1, f^2, \ldots, f^n)$.

3.2.5 The Wick–adjoint of L

Define the formal Wick–adjoint of L, denoted by $L^{\circledast}$, as

$$L^{\circledast}u = \sum_{i=1}^{n} D_i(\sum_{j=1}^{n} a^{ji\circledast}D_j u - c^{i\circledast}u) - \sum_{i=1}^{n} b^{i\circledast}D_i u + d^{\circledast}u, \tag{3.58}$$

for $u \in \mathcal{W}_0\mathcal{S}$, where $a^{ji\circledast}$, $b^{i\circledast}$, $c^{i\circledast}$ and $d^{\circledast}$ are the Wick–adjoint multiplication operators (recall Definition 3.2.3) of the coefficients a^{ji}, b^i, c^i and d.

Lemma 3.2.9 *For arbitrary $u \in \mathcal{WS}^*$ and $v \in \mathcal{W}_0\mathcal{S}$*

$$\langle u, L^{\circledast}v\rangle = \langle L\lozenge u, v\rangle,$$

i.e. $L^{\circledast} : \mathcal{W}_0\mathcal{S} \to \mathcal{W}_0\mathcal{S}$ is the Hilbert space adjoint of $L\lozenge : \mathcal{WS}^* \to \mathcal{WS}^*$.

Proof.

$$\begin{aligned}
\langle u, L^{\circledast}v\rangle &= \sum_{i=1}^{n}\langle u, D_i(\sum_{j=1}^{n} a^{ji\circledast}D_j v - c^{i\circledast}v)\rangle - \sum_{i=1}^{n}\langle u, b^{i\circledast}D_i v\rangle + \langle u, d^{\circledast}v\rangle \\
&= -\sum_{i=1}^{n}\langle D_i u, \sum_{j=1}^{n} a^{ji\circledast}D_j v - c^{i\circledast}v\rangle - \sum_{i=1}^{n}\langle u, b^{i\circledast}D_i v\rangle + \langle u, d^{\circledast}v\rangle \\
&= -\sum_{i=1}^{n}\langle\sum_{j=1}^{n} a^{ji}\lozenge D_i u, D_j v\rangle + \sum_{i=1}^{n}\langle c^i\lozenge D_i u, v\rangle - \sum_{i=1}^{n}\langle b^i\lozenge u, D_i v\rangle + \langle d\lozenge u, v\rangle \\
&= \langle L\lozenge u, v\rangle.
\end{aligned}$$

$\square$

It is a routine calculation to show that $L^{\circledast}$ echoes properties (3.44), (3.45) and (3.46) of the operator L in following form:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\langle a^{ji\circledast}v_i, v_j\rangle \geq \lambda \sum_{i=1}^{n}\|v_i\|^2 \tag{3.59}$$

$$a^{ij\circledast}, b^{i\circledast}, c^{i\circledast}, d^{\circledast},\ (i,j,=1,2,\ldots,n) \quad \text{are bounded operators on } \mathcal{WS}^*, \tag{3.60}$$

$$\langle d^{\circledast}u, v\rangle + \sum_{i=1}^{n}\langle c^{i\circledast}u, D_i v\rangle \leq 0, \quad v \geq 0. \tag{3.61}$$

Let us check (3.59). From the ellipticity property of L we get

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\langle a^{ji\circledast}v_i, v_j\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n}\langle v_i, a^{ji}\lozenge v_j\rangle \geq \lambda \sum_{i=1}^{n}\|v_i\|^2.$$

Concerning (3.60), let us prove e.g. that $d^{\circledast}$ is bounded. For $u, v \in \mathcal{WS}$ we have $|\langle d^{\circledast}u, v\rangle| = |\langle u, d\lozenge v\rangle| \leq \|d\|\ \|u\|\ \|v\|$. Thus, $\|d^{\circledast}\| \leq \|d\|$. And finally, (3.61) can also be reduced to $\langle u, d\lozenge v\rangle + \sum_{i=1}^{n}\langle u, c^i\lozenge D_i v\rangle \leq 0$. But as already stated in (3.50), this is also a sufficient condition for the weak maximum principle.

Consequently, Theorem 3.2.2 can be applied to get a unique solution of the Dirichlet problem $L^{\circledast}u = h$.

The following theorem describes the spectral behavior of L and is a consequence of the previous considerations and the Fredholm alternative.

Theorem 3.2.3 *Let the operator L satisfy conditions (3.44) and (3.45). There exists a countable, discrete set $\Sigma \subset \mathbb{R}$ with following properties:*

(i) If $\sigma \notin \Sigma$, the Dirichlet problems $L_\sigma\lozenge u = h + \sum_{i=1}^{n} D_i f^i$ and $L_\sigma^{\circledast}u = h + \sum_{i=1}^{n} D_i f^i$, $u\restriction_{\partial I}= g$, are uniquely solvable in $\mathcal{WS}$ for arbitrary $h, f^i \in L^2(I)\otimes(S)_{-1}$ and $g \in \mathcal{WS}$. Moreover, there exists $C > 0$ (depending on L, I and σ) such that $\|u\| \leq C(\|\mathbf{h}\| + \|g\|)$.

(ii) If $\sigma \in \Sigma$, then the subspaces of solutions of the homogeneous problems $L_\sigma\lozenge u = 0$, $L_\sigma^{\circledast}u = 0$, $u\restriction_{\partial I}= 0$, are of positive, finite dimension and the problem $L_\sigma\lozenge u = h + \sum_{i=1}^{n} D_i f^i$, $u\restriction_{\partial I}= g$ is solvable in $\mathcal{WS}$ if and only if $\langle L_\sigma\lozenge g - h - \sum_{i=1}^{n} D_i f^i, v\rangle = 0$ for all v satisfying $L_\sigma^{\circledast}v = 0$, $v\restriction_{\partial I}= 0$.

Moreover, if condition (3.46) holds, then $\Sigma \subset (-\infty, 0)$.

3.2.6 Stability and regularity properties

Concerning stability properties of the generalized weak solution of (3.47) we can prove the same results as we did for the deterministic-coefficients case in the previous section: The generalized weak solution is continuously dependent on the data and the coefficients of L.

Theorem 3.2.4 *Let L be an operator of the form (3.42) with coefficients a^{ij}, b^i, c^i, d, satisfying conditions (3.44), (3.45) and (3.46). Let $\tilde{L}$ be another operator of the form (3.42) with coefficients $\tilde{a}^{ij}$, $\tilde{b}^i$, $\tilde{c}^i$, $\tilde{d}$, satisfying all given conditions. Let h, $\tilde{h}$, f^i and $\tilde{f}^i$, $i = 1, 2, \ldots, n$ be generalized random processes from $L^2(I) \otimes (S)_{-1}$. Let g and $\tilde{g}$ be generalized random processes from $\mathcal{WS}$. Let $u, \tilde{u} \in \mathcal{WS}^*$ be the generalized weak solutions of the Dirichlet problems*

$$L\lozenge u = h + \sum_{i=1}^{n} D_i f^i, \qquad u \restriction_{\partial I} = g, \tag{3.62}$$

$$\tilde{L}\lozenge \tilde{u} = \tilde{h} + \sum_{i=1}^{n} D_i \tilde{f}^i, \qquad \tilde{u} \restriction_{\partial I} = \tilde{g}, \tag{3.63}$$

respectively. There exist $C > 0$, $p \in \mathbb{N}_0$ such that for every $v \in W_0^{1,2}(I) \otimes (S)_{1,p}$ following estimate holds:

$$|\langle u-\tilde{u}, v\rangle| \le C\Big(\|\mathbf{h}-\tilde{\mathbf{h}}\|_{L^2(I)\otimes(S)_{-1,-p}} + \|L-\tilde{L}\|_{W^{-1,2}\otimes(S)_{-1,-p}}\|u-g\|_{W^{1,2}\otimes(S)_{-1,-p}}$$

$$+\|L-\tilde{L}\|_{W^{-1,2}\otimes(S)_{-1,-p}}\|\tilde{g}\|_{W^{1,2}\otimes(S)_{-1,-p}}$$

$$+\|g-\tilde{g}\|_{W^{1,2}\otimes(S)_{-1,-p}}\|L\|_{W^{-1,2}\otimes(S)_{-1,-p}}\Big)\|v\|_{W^{1,2}\otimes(S)_{1,p}}$$

where $\mathbf{h} = (h, f^1, f^2, \ldots f^n)$.

Proof. To keep technicalities to a minimum, we consider the Dirichlet problems with zero boundary conditions. Let $v \in \mathcal{W}_0\mathcal{S}$ be arbitrary. According to Theorem 3.2.3 there exists a unique solution $w \in \mathcal{W}_0\mathcal{S}$ of the equation $\tilde{L}^{\circledast}w = v$, $w \restriction_{\partial I} = 0$, and there exists $K > 0$ such that $\|w\| \le K\|v\|$. From (3.62) and (3.63) we obtain

$$\tilde{L}\lozenge(u-\tilde{u}) = h - \tilde{h} + \sum_{i=1}^{n}(f^i - \tilde{f}^i) - (L-\tilde{L})\lozenge u.$$

Thus, using the Cauchy–Schwartz inequality, continuity of h, $\tilde{h}$, f, $\tilde{f}$ and continuity of L and $\tilde{L}$ (Lemma 3.2.2) we get

$$\begin{aligned}
|\langle u-\tilde{u}, v\rangle| &= \langle u-\tilde{u}, \tilde{L}^{\circledast} w\rangle| \\
&= |\langle \tilde{L}\Diamond(u-\tilde{u}), w\rangle| \\
&\leq |\langle h-\tilde{h}, w\rangle| + \sum_{i=1}^{n} |\langle f^i - \tilde{f}^i, w\rangle| + |\langle (L-\tilde{L})\Diamond u, w\rangle| \\
&\leq \left(\|h-\tilde{h}\| + \sum_{i=1}^{n} \|f^i - \tilde{f}^i\| + \|L-\tilde{L}\| \, \|u\| \right) \|w\| \\
&\leq K \left(\|h-\tilde{h}\| + \sum_{i=1}^{n} \|f^i - \tilde{f}^i\| + \|L-\tilde{L}\| \, \|u\| \right) \|v\|
\end{aligned}$$

Note that $\|L-\tilde{L}\| = \max_{1\leq i,j\leq n}\{\|a^{ij}-\tilde{a}^{ij}\|, \|b^i-\tilde{b}^i\|, \|c^i-\tilde{c}^i\|, \|d-\tilde{d}\|\}$. Since v is arbitrary, we finally obtain

$$\|u-\tilde{u}\| \leq K\left(\|\mathbf{h}-\tilde{\mathbf{h}}\| + \|L-\tilde{L}\|\|u\|\right). \qquad \square$$

In particular, let us consider again a net of operators L_ϵ, $\epsilon \in (0,1]$, given by

$$L_\epsilon\Diamond u = \sum_{i=1}^{n} D_i(\sum_{j=1}^{n} a_\epsilon^{ij}\Diamond D_j u + b_\epsilon^i\Diamond u) + \sum_{i=1}^{n} c_\epsilon^i\Diamond D_i u + d_\epsilon\Diamond u$$

and a net of data h_ϵ, f_ϵ^i, $i=1,2,\ldots,n$, where $a_\epsilon^{ij}(x,\omega) = a^{ij}(\cdot,\omega) * \check{\rho}_\epsilon(x)$, $b_\epsilon^i(x,\omega) = b^i(\cdot,\omega) * \check{\rho}_\epsilon(x)$, $c_\epsilon^i(x,\omega) = c^i(\cdot,\omega) * \check{\rho}_\epsilon(x)$, $d_\epsilon(x,\omega) = d(\cdot,\omega) * \check{\rho}_\epsilon(x)$,$h_\epsilon(x,\omega) = h(\cdot,\omega) * \check{\rho}_\epsilon(x)$, $f_\epsilon^i(x,\omega) = f^i(\cdot,\omega) * \check{\rho}_\epsilon(x)$, $i,j = 1,2,\ldots,n$. Denote by u_ϵ the solution of

$$L_\epsilon\Diamond u_\epsilon = h_\epsilon + \sum_{i=1}^{n} D_i f_\epsilon^i, \quad u_\epsilon \restriction_{\partial I} = 0.$$

Now, from Theorem 3.2.4 we get that $\|u_\epsilon - u\|_{\mathcal{W}\mathbb{S}^*}$ is bounded by the sum of the operator norm $\|L-L_\epsilon\|_{\mathcal{W}\mathbb{S}^*}$ and of $\|\mathbf{h}_\epsilon - \mathbf{h}\|_{L^2(I)\otimes(S)_{-1}}$. Thus,

$$\|u_\epsilon - u\|_{\mathcal{W}\mathbb{S}^*} \to 0, \quad \epsilon \to 0.$$

This leads us to consider problem (3.47) in the Colombeau setting, just as we did in the previous section with deterministic coefficients. In Theorem 3.2.8 we will prove existence and uniqueness of generalized weak solutions of (3.47) when the coefficients of the operator L are Colombeau generalized processes i.e. elements of the algebra $\mathcal{G}(W^{2,2};(S)_{-1})$. But in order to do this, first we have to establish some necessary polynomial growth rate estimates for the generalized weak solution u of (3.47).

In order to get these polynomial growth rates we need stricter assumptions on L: Assume that beside the ellipticity condition (3.44), conditions (3.51) and (3.52) hold. By careful investigation of the proofs in [GT], we see that one may carry over the regularity properties also in our setting to the Hilbert space $\mathcal{WS}^*$. One only needs to consider $(S)_{-1}$–valued Sobolev functions $u(x,\omega) \in W^{2,2}(I) \otimes (S)_{-1}$, but this is no problem since the chain rule and all other necessary tools hold (one can define differential quotients and carry out the calculations as in [GT]). We state now our analogies of [GT, Theorem 8.8.] and [GT, Theorem 8.12.] without proof.

Definition 3.2.4 *Let $F \in \mathcal{WS}^*$ be a generalized random process given by chaos expansion $F(x,\omega) = \sum_{\alpha\in\mathcal{J}} f_\alpha(x) \otimes H_\alpha(\omega)$. We will call F a uniformly Lipschitz continuous generalized random process, if $f_\alpha \in C^{0,1}(I)$ for all $\alpha \in \mathcal{J}$ and if there exists $p > 0$ such that*

$$\sum_{\alpha\in\mathcal{J}} \|f_\alpha\|^2_{C^{0,1}(I)} (2\mathbb{N})^{-p\alpha} < \infty.$$

We may consider uniformly Lipschitz continuous GRPs as elements of the tensor product space $C^{0,1}(I) \otimes (S)_{-1}$.

Theorem 3.2.5 *Let the operator L satisfy conditions (3.44),(3.51) and (3.52). Let $u \in W^{1,2}(I) \otimes (S)_{-1}$ be the generalized weak solution of*

$$L\lozenge u = h, \qquad u \restriction_{\partial I} = g.$$

Assume the coefficients a^{ij}, b^i, $i,j = 1,2,\dots n$, are uniformly Lipschitz continuous GRPs, c^i, d, $i = 1,2,\dots n$ are essentially bounded GRPs, and $h \in L^2(I) \otimes (S)_{-1}$. Then, for arbitrary I' such that $\overline{I'} \subset I$, it follows that $u \in W^{2,2}(I') \otimes (S)_{-1}$ and there exist $p > 0$ and $C(n,\lambda,K,d') > 0$ such that

$$\|u\|_{W^{2,2}(I')\otimes(S)_{-1,-p}} \leq C(\|u\|_{W^{1,2}(I)\otimes(S)_{-1,-p}} + \|h\|_{L^2(I)\otimes(S)_{-1,-p}})$$

where $K = \max_{1\leq i,j\leq n}\{\|a^{ij}, b^i\|_{C^{0,1}(\bar{I})\otimes(S)_{-1,-p}},\ \|c^i, d\|_{L^\infty(I)\otimes(S)_{-1,-p}}\}$ and $d' = dist(\partial I, I')$. Additionally, u satisfies the equation

$$L\lozenge u = \sum_{i,j=1}^n a^{ij}\lozenge D_{ij}u + \sum_{i=1}^n(\sum_{j=1}^n(D_j a^{ij} + b^i + c^i)\lozenge D_i u + (\sum_{i=1}^n D_i b^i + d)\lozenge u = h$$

for a.e. $x \in I$ and $\omega \in \Omega$.

Theorem 3.2.6 *Assume in addition to the hypothesis of Theorem 3.2.5 that* ∂I *is of* C^2*–class and that there exists* $g \in W^{2,2}(I) \otimes (S)_{-1}$ *such that* $u - g \in W_0^{1,2}(I) \otimes (S)_{-1}$. *Then* $u \in W^{2,2}(I) \otimes (S)_{-1}$ *and there exist* $p > 0$ *and* $C(n, \partial I) > 0$ *such that*

$$\|u\|_{W^{2,2}(I)\otimes(S)_{-1,-p}} \leq C\frac{K}{\lambda^2}\Big(\|u\|_{L^2(I)\otimes(S)_{-1,-p}} + \|h\|_{L^2(I)\otimes(S)_{-1,-p}} + \|g\|_{W^{2,2}(I)\otimes(S)_{-1,-p}}\Big). \tag{3.64}$$

The deterministic case of the following theorem was proved in [MP] as an improvement of [GT, Theorem 8.16.]. By careful investigation of the proof in [MP], we see that one may carry it over also to $\mathcal{WS}^*$. Since lack of space, we will give only a sketch of the proof.

Theorem 3.2.7 *Suppose the operator* L *satisfies (3.44),(3.51) and (3.52). Assume that* $f^i \in L^q(I) \otimes (S)_{-1}$, $i = 1, 2, \ldots, n$, *and* $h \in L^{q/2}(I) \otimes (S)_{-1}$ *for some* $q > n$. *If* $u \in W^{1,2}(I) \otimes (S)_{-1}$ *satisfies* $L\Diamond u - (h + \sum_{i=1}^n f^i) \geq 0$ *in weak sense, then there exist* $p > 0$, $s > 0$ *and* $C(n, q, \nu, |I|) > 0$ *such that*

$$\sup_{x\in I}\|u(x,\cdot)\|_{-1,-p} \leq \sup_{x\in\partial I}\|u^+(x,\cdot)\|_{-1,-p} + \Upsilon + C\lambda^{-s}\Big(\Upsilon + \sum_{i=1}^n \|b^i + c^i\|^2_{L^2(I)\otimes(S)_{-1,-p}} + 1\Big)^2, \tag{3.65}$$

where $\Upsilon = \lambda^{-1}(\sum_{i=1}^n \|f^i\|_{L^q(I)\otimes(S)_{-1,-p}} + \|g\|_{L^{q/2}(I)\otimes(S)_{-1,-p}})$.

Proof. From $B(u, v) \leq \langle h + \sum_{i=1}^n f^i, v\rangle$, $v \geq 0$, $v \in \mathcal{W}_0\mathcal{S}$, using the chain rule and (3.51) we get

$$\sum_{i,j=1}^n \langle a^{ij}\Diamond D_j u, D_i\rangle \leq \sum_{i=1}^n \langle (b^i + c^i)\Diamond D_i u, v\rangle + \sum_{i=1}^n \langle f^i, D_i v\rangle - \langle h, v\rangle,$$

for $v \in \mathcal{W}_0\mathcal{S}$, $u \in \mathcal{WS}^*$ such that $u^{\circledast}v \geq 0$. Let $M = sup_{x\in I}\|u^+(x,\cdot)\|_{-1,-p}$. Define

$$\varphi(x) = \frac{\|u^+(x,\cdot)\|_{-1,-p}}{\sqrt{2M + \Upsilon - \|u^+(x,\cdot)\|_{-1,-p}}} \in W_0^{1,2}(I)$$

and consider the test function $v(x, \omega) = \varphi(x)\theta(\omega)$, for arbitrary $\theta \in (S)_{1,p}$. Now we can apply the ellipticity condition etc. and proceed by the same pattern as in [MP, Theorem 6.] to prove the assertion. □

Now it is easy to get an estimate as in Proposition 3.1.3 for the Dirichlet problem $L\Diamond u = h$, $u\restriction_{\partial I} = g$. By the Sobolev lemma we have $W^{2,2}(I) \subset C^\beta(I)$

for $n \leq 3$ and $\beta < \frac{1}{2}$. Thus, $\|u\|_{L^2(I)\otimes(S)_{-1,-p}} \leq \sup_{x\in I} \|u(x,\cdot)\|_{-1,-p}|I|$. Combining the results of Theorem 3.2.6 and Theorem 3.2.7 we get the existence of $s > 0$ and $C > 0$ such that

$$\|u\|_{W^{2,2}(I)\otimes(S)_{-1,-p}} \leq C\Big(\frac{\Lambda}{\lambda}\Big)^s\Big(\sup_{x\in I}\|u(x,\cdot)\|_{-1,-p}|I| + \|g\|_{W^{2,2}(I)\otimes(S)_{-1,-p}} + \|h\|_{L^2(I)\otimes(S)_{-1,-p}} + \|h\|_{L^q(I)\otimes(S)_{-1,-p}}\Big). \tag{3.66}$$

3.2.7 Colombeau–solutions of the Dirichlet problem

We assume that I is of C^2–class and that $n \leq 3$. Clearly, then Theorem 3.2.6 holds for $q = 4$, but as we already assumed in Theorem 3.2.2 that $h \in L^2(I) \otimes (S)_{-1}$, we always have condition $h \in L^{q/2}$ satisfied. Thus, the last term in 3.66 vanishes.

We consider again a net of linear differential operators

$$L_\epsilon \lozenge u_\epsilon(x,\omega) = \sum_{i=1}^n D_i\Big(\sum_{j=1}^n a_\epsilon^{ij} \lozenge D_j u_\epsilon(x,\omega) + b_\epsilon^i \lozenge u_\epsilon(x,\omega)\Big) + \sum_{i=1}^n c_\epsilon^i \lozenge D_i u_\epsilon(x,\omega) + d_\epsilon \lozenge u_\epsilon(x,\omega), \quad \epsilon \in (0,1), \tag{3.67}$$

where the nets of coefficients a_ϵ^{ij}, b_ϵ^i, c_ϵ^i, d_ϵ belong to $\mathcal{E}_M(W^{2,2};(S)_{-1})$. Define that two nets of operators are related: $L_\epsilon \sim \tilde{L}_\epsilon$ if and only if $(L_\epsilon - \tilde{L}_\epsilon)\lozenge u_\epsilon \in \mathcal{N}(W^{0,2};(S)_{-1})$ for all $u_\epsilon \in \mathcal{E}_M(W^{2,2};(S)_{-1})$. Clearly, $\sim$ is an equivalence relation, and following holds:

Proposition 3.2.1 *If $u_\epsilon \in \mathcal{N}(W^{2,2};(S)_{-1})$, then $L_\epsilon \lozenge u_\epsilon \in \mathcal{N}(W^{0,2};(S)_{-1})$.*

Denote by $\mathcal{L}$ the family of all nets of differential operators of the form (3.67) and let $\mathcal{L}_0 = \mathcal{L}/\sim$. For $L \in \mathcal{L}_0$ we define $L : \mathcal{G}(W^{2,2};(S)_{-1}) \to \mathcal{G}(W^{0,2};(S)_{-1})$ by

$$L\lozenge[u_\epsilon(x,\cdot)] = \sum_{i=1}^n D_i\Big(\sum_{j=1}^n [a_\epsilon^{ij}(x,\omega)]\lozenge[D_j u_\epsilon(x,\omega)] + [b_\epsilon^i(x,\omega)]\lozenge[u_\epsilon(x,\omega)\Big)] + \sum_{i=1}^n [c_\epsilon^i(x,\omega)]\lozenge[D_i u_\epsilon(x,\omega)] + [d_\epsilon(x,\omega)]\lozenge[u_\epsilon(x,\omega)]. \tag{3.68}$$

The operator $L = [L_\epsilon]$ given by (3.68) is *strictly elliptic*, if there exist representatives a_ϵ^{ij}, b_ϵ^i, c_ϵ^i, $d_\epsilon \in \mathcal{E}_M(W^{2,2};(S)_{-1})$ of its coefficients, such that

$$\sum_{i,j=1}^{n} \langle a_\epsilon^{ij} \Diamond v_j(x,\omega), v_i(x,\omega)\rangle \geq \lambda_\epsilon \sum_{i=1}^{n} \|v_i\|^2 \geq K\epsilon^a \sum_{i=1}^{n} \|v_i\|^2, \tag{3.69}$$

for all $v_i \in \mathcal{W}_0\mathcal{S}$, $i = 1, 2, \ldots, n$, and K is a constant independent of ϵ.

The boundary condition is defined in the same way as in Section 3.1.4.

Consider now the stochastic Dirichlet problem

$$\begin{aligned} L\Diamond u(x,\omega) &= h(x,\omega) \qquad \text{in } \mathcal{G}(W^{2,2};(S)_{-1}) \\ u(x,\omega)\restriction_{\partial I} &= g(x,\omega), \end{aligned} \tag{3.70}$$

where L is defined in (3.68). In order to solve (3.70) in the Colombeau setting, one has to solve a family of problems:

$$\begin{aligned} L_\epsilon\Diamond u_\epsilon(x,\omega) &= h_\epsilon(x,\omega) \qquad \text{in } W^{2,2}(I)\otimes(S)_{-1} \\ u_\epsilon(x,\omega)\restriction_{\partial I} &= g_\epsilon(x,\omega), \quad \epsilon\in(0,1), \end{aligned} \tag{3.71}$$

then to check whether the net of solutions u_ϵ belongs to $\mathcal{E}_M(W^{2,2};(S)_{-1})$, and finally to check whether equation (3.71) holds with other representatives of L, h, f, g and u. Uniqueness of a solution means that if $u = [u_\epsilon]$ and $v = [v_\epsilon]$ satisfy (3.70), then $[u_\epsilon] = [v_\epsilon]$.

Theorem 3.2.8 *Let h and g belong to $\mathcal{G}(W^{2,2};(S)_{-1})$. Let the operator $L = [L_\epsilon]$ be given by (3.68) with coefficients $[a_\epsilon^{ij}]$, $[b_\epsilon^i]$, $[c_\epsilon^i]$, $[d_\epsilon]$ in $\mathcal{G}(W^{2,2};(S)_{-1})$. Assume that following conditions hold:*

1. *L is strictly elliptic, i.e. (3.69) holds,*

2. *there exist $M > 0$ and $b > 0$ such that for all $\epsilon \in (0,1)$:*

$$\|a_\epsilon^{ij}\|,\ \|b_\epsilon^i\|,\ \|c_\epsilon^i\|,\ \|d_\epsilon\|,\ \|h_\epsilon\|,\ \|g_\epsilon\| \leq \Lambda_\epsilon \ \leq M\epsilon^b$$

   where $\|\cdot\|$ denotes the norm in $W^{2,2}\otimes(S)_{-1,-p}$ for some $p > 0$ fixed.

3. *for every $\epsilon \in (0,1)$ and $v \geq 0, v \in \mathcal{W}_0\mathcal{S}$,*

$$\langle d_\epsilon, v\rangle - \sum_{i=1}^{n} \langle b_\epsilon^i, D_i v\rangle \leq 0.$$

Then the stochastic Dirichlet problem (3.36) has a unique solution in $\mathcal{G}(W^{2,2};(S)_{-1})$.

Proof. For $\epsilon \in (0,1)$ fixed, there exists a unique generalized weak solution $u_\epsilon \in W^{2,2}(I) \otimes (S)_{-1}$ of problem (3.71). This follows from Theorem 3.2.2 and Theorem 3.2.6.

Using the estimate derived in (3.66) we obtain that there exist $C > 0$, $s > 0$ (independent of ϵ) such that:

$$\|u_\epsilon\|_{W^{2,2}\otimes(S)_{-1,-p}} \leq C \left(\frac{\Lambda_\epsilon}{\lambda_\epsilon}\right)^s \Big(\sup_{x\in\partial I} \|g_\epsilon(x,\cdot)\|_{-1,-p} \, |I| + \|g_\epsilon\|_{W^{2,2}\otimes(S)_{-1,-p}} + \|h_\epsilon\|_{L^2\otimes(S)_{-1,-p}} \Big). \tag{3.72}$$

Now, since g_ϵ and h_ϵ, are all bounded by Λ_ϵ, which has polynomial growth with respect to ϵ, we obtain

$$\|u_\epsilon\|_{W^{2,2}\otimes(S)_{-1,-p}} \leq \tilde{C}\epsilon^a,$$

for an appropriate $a \in \mathbb{R}$, $\tilde{C} > 0$.

Thus, by Proposition 3.1.4 (ii), $[u_\epsilon(x,\omega)] \in \mathcal{G}(W^{2,2}; (S)_{-1})$.

The definition of operators $\mathcal{L}_0$ implies that with some other representatives of h, g and the coefficients of L in (3.71), another representative of $u = [u_\epsilon]$ also satisfies (3.70).

Now, we prove uniqueness of the solution: Let $u_{1,\epsilon}$ and $u_{2,\epsilon}$ satisfy

$$\begin{aligned} L_{1,\epsilon}\Diamond u_\epsilon(x,\omega) &= h_{1,\epsilon}(x,\omega) \qquad \text{in } W^{2,2}(I)\otimes(S)_{-1} \\ u_\epsilon(x,\omega)\restriction_{\partial I} &= g_{1,\epsilon}(x,\omega), \quad \epsilon \in (0,1), \end{aligned} \tag{3.73}$$

and

$$\begin{aligned} L_{2,\epsilon}\Diamond u_\epsilon(x,\omega) &= h_{2,\epsilon}(x,\omega) \qquad \text{in } W^{2,2}(I)\otimes(S)_{-1} \\ u_\epsilon(x,\omega)\restriction_{\partial I} &= g_{2,\epsilon}(x,\omega), \quad \epsilon \in (0,1), \end{aligned} \tag{3.74}$$

respectively, where the operators $L_{k,\epsilon}$, $k = 1,2$, are of the form (3.67), if we replace the coefficients with $a^{ij}_{k,\epsilon}$, $b^i_{k,\epsilon}$, $c^i_{k,\epsilon}$, $d_{k,\epsilon}$, $k = 1,2$, respectively, and $(a^{ij}_{1,\epsilon} - a^{ij}_{2,\epsilon})$, $(b^i_{1,\epsilon} - b^i_{2,\epsilon})$, $(c^i_{1,\epsilon} - c^i_{2,\epsilon})$, $(d_{1,\epsilon} - d_{2,\epsilon})$, $(h_{1,\epsilon} - h_{2,\epsilon})$ and $(g_{1,\epsilon} - g_{2,\epsilon}) \in \mathcal{N}(W^{2,2}; (S)_{-1})$ holds.

We will prove $(u_{1,\epsilon} - u_{2,\epsilon}) \in \mathcal{N}(W^{2,2}; (S)_{-1})$, which establishes the assertion $[u_{1,\epsilon}] = [u_{2,\epsilon}]$.

If we insert $u_{1,\epsilon}$ into (3.73) and $u_{2,\epsilon}$ into (3.74), then subtract (3.73) from (3.74), we obtain

$$\begin{aligned}
&\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}(a_{1,\epsilon}^{ij}\Diamond D_j(u_{2,\epsilon}-u_{1,\epsilon})+b_{1,\epsilon}^{i}\Diamond(u_{2,\epsilon}-u_{1,\epsilon})\Big)\\
&\quad+\sum_{i=1}^{n}c_{1,\epsilon}^{i}\Diamond D_i(u_{2,\epsilon}-u_{1,\epsilon})+d_{1,\epsilon}\Diamond(u_{2,\epsilon}-u_{1,\epsilon})\\
&\quad=h_{2,\epsilon}-h_{1,\epsilon}-\Big(\sum_{i=1}^{n}\big(\sum_{j=1}^{n}(a_{2,\epsilon}^{ij}-a_{1,\epsilon}^{ij})\Diamond D_j u_{1,\epsilon}+(b_{2,\epsilon}^{i}-b_{1,\epsilon}^{i})\Diamond u_{1,\epsilon}\big)\\
&\quad+\sum_{i=1}^{n}(c_{2,\epsilon}^{i}-c_{1,\epsilon}^{i})\Diamond D_i u_{1,\epsilon}+(d_{2,\epsilon}-d_{1,\epsilon})\Diamond u_{1,\epsilon}\Big)\\
&\quad=H_\epsilon\\
&(u_{2,\epsilon}-u_{1,\epsilon})\restriction_{\partial I}=(g_{2,\epsilon}-g_{1,\epsilon}),\quad \epsilon\in(0,1),
\end{aligned}$$

where $H_\epsilon \in \mathcal{N}(W^{2,2};(S)_{-1})$. Using the estimate (3.66) we finally obtain that there exist $p>0$ and $C>0$ such that for all $a\in\mathbb{R}$

$$\|u_{2,\epsilon}-u_{1,\epsilon}\|_{W^{2,2}\otimes(S)_{-1,-p}}\le C\epsilon^a.$$

Thus, by Proposition 3.1.4 (iv), $(u_{2,\epsilon}-u_{1,\epsilon})\in\mathcal{N}(W^{2,2};(S)_{-1})$. □

The Hilbert space valued stochastic Dirichlet problem

Let H be a separable Hilbert space with orthonormal base $\{e_i : i\in\mathbb{N}\}$. Consider now a Dirichlet problem of the form (3.47) where the coefficients a, b, c, d and f, g, h are Hilbert space valued GRPs i.e. for fixed $x\in I$ they take value in $S(H)_{-1}$. The operator L is interpreted as in (3.42), the Wick product now taken in $S(H)_{-1}$. With largely cosmetic changes, one can carry out all calculations also in this setting; just interpret $\langle\cdot,\cdot\rangle$ as the dual pairing and $\|\cdot\|$ as the norm in $L^2(I)\otimes S(H)_{-1,-p}$ for fixed $p>0$. For example, $u(x,\omega)=\sum_{i=1}^{\infty}\sum_{\alpha\in\mathcal{J}}u_{i,\alpha}(x)\otimes H_\alpha(\omega)e_i\in W^{-1,2}(I)\otimes S(H)_{-1}$ and $v(x,\omega)=\sum_{i=1}^{\infty}\sum_{\alpha\in\mathcal{J}}v_{i,\alpha}(x)\otimes H_\alpha(\omega)e_i\in W_0^{1,2}(I)\otimes S(H)_{-1}$ act as $\langle u,v\rangle=\sum_{i=1}^{\infty}\sum_{\alpha\in\mathcal{J}}\alpha!\int_I u_{i,\alpha}(x)v_{i,\alpha}(x)dx$.

For example we state the H–valued version of Theorem 3.2.2

Theorem 3.2.9 *Let the operator L satisfy conditions (3.44), (3.45) and (3.46). Then for $h, f^i\in L^2(I)\otimes S(H)_{-1}$, $i=1,2,\dots,n$ and for $g\in W^{1,2}(I)\otimes S(H)_{-1}$ the stochastic Dirichlet problem (3.47) has a unique generalized weak solution in $W^{1,2}(I)\otimes S(H)_{-1}$.*

Analogue H–valued versions of all the theorems concerning regularity, stability properties and Colombeau solutions hold.

3.3 Applications of the Fourier Transformation to Generalized Random Processes of type (I)

In this section we define the Fourier transformation for GRPs (I) and present its application to solve some SPDEs involving singular generalized stochastic processes. In particular, we solve the stochastic version of the Helmholtz equation.

The Fourier transformation of tempered distributions

The Fourier transformation $\mathcal{S}(\mathbb{R}) \ni f \to \hat{f} \in \mathcal{S}(\mathbb{R})$ is defined by

$$\hat{f}(y) = \mathcal{F}(f)(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ixy} dx, \quad y \in \mathbb{R}, \tag{3.75}$$

and the inverse Fourier transformation is given by the formula $\mathcal{F}^{-1}(f)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y) e^{ixy} dy$. The Fourier transformation of a tempered distribution $F \in \mathcal{S}'(\mathbb{R})$ is defined by the action

$$\langle \hat{F}, \varphi \rangle = \langle F, \hat{\varphi} \rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}).$$

Clearly, $\hat{F} \in \mathcal{S}'(\mathbb{R})$. The inverse Fourier transformation is defined analogously by $\langle \mathcal{F}^{-1}(F), \varphi \rangle = \langle F, \mathcal{F}^{-1}(\varphi) \rangle$. It is easy to check that the Fourier transformation of the Hermite functions is given by

$$\hat{\xi}_n(y) = (-i)^{n-1} \xi_n(y), \quad n \in \mathbb{N}. \tag{3.76}$$

Thus, by linearity of the Fourier transformation we obtain that if $F \in \mathcal{S}'(\mathbb{R})$ has the formal expansion $F(x) = \sum_{k=1}^{\infty} a_k \xi_k(x)$, then

$$\hat{F}(y) = \sum_{k=1}^{\infty} (-i)^{k-1} a_k \xi_k(y). \tag{3.77}$$

Moreover, if $\sum_{k=1}^{\infty} |a_k|^2 (2k)^{-p} < \infty$ for some $p \geq 0$, then also $\sum_{k=1}^{\infty} |(-i)^{k-1}|^2 |a_k|^2 (2k)^{-p} < \infty$. Thus, the Fourier transformation maps $\mathcal{S}_{-p}(\mathbb{R})$ into itself and we have $\|F\|_{-p} = \|\hat{F}\|_{-p}$. The same considerations show that the Fourier transformation maps $\exp \mathcal{S}'(\mathbb{R})$ into itself.

In $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ the Fourier transformation is defined in a similar manner by $\hat{f}(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-ixy} dx$, where xy is interpreted as the standard inner product in $\mathbb{R}^n$. For example, in $\mathcal{S}'(\mathbb{R}^n)$ the Dirac delta distribution δ has Fourier transformation $\hat{\delta} = (2\pi)^{-n/2}$.

For convenience of the reader we list some crucial formulae related to the Fourier transformation. For further properties refer e.g. to [PS].

- $\widehat{\frac{\partial}{\partial x_j}} f(y) = \frac{1}{i} y_j \hat{f}(y), \quad j = 1, 2, \dots n,$

- $\widehat{x_j f}(y) = \frac{1}{i} \widehat{\frac{\partial}{\partial y_j}} f(y), \quad j = 1, 2, \dots n,$

- $\widehat{f * g}(y) = (2\pi)^{n/2} \hat{f}(y) \hat{g}(y), \quad$ (exchange formula)

- $\widehat{fg}(y) = (2\pi)^{-n/2} \hat{f}(y) * \hat{g}(y),$

- $f\widehat{(x - h)}(y) = e^{-ihy} \hat{f}(y), \quad h \in \mathbb{R}^n, \quad$ (translation formula)

- $\widehat{f(\lambda x)}(y) = |\lambda|^{-n} \hat{f}(\frac{y}{\lambda}), \quad \lambda \in \mathbb{C}. \quad$ (dilatation formula)

In the Hilbert space valued case, the Fourier transformation on $\mathcal{S}'(\mathbb{R}^n; H)$ is defined (see [Tr]) by

$$\mathcal{F} \otimes Id : \ \mathcal{S}'(\mathbb{R}^n) \otimes H \to \mathcal{S}'(\mathbb{R}^n) \otimes H$$

where $\mathcal{F}$ is the Fourier transformation on $\mathcal{S}'(\mathbb{R}^n)$ and Id is the identity mapping of H.

3.3.1 The Fourier transformation of GRPs (I)

The considerations given above allow us to define the Fourier transformation for GRPs (I) defined on the Schwartz space of tempered distributions. Thus, we restrict our attention to GRPs considered as elements of the spaces

$$\mathcal{L}(\mathcal{S}(\mathbb{R}^n), (S)_{-1}), \qquad \mathcal{L}(\exp \mathcal{S}(\mathbb{R}^n), (S)_{-1})$$

and $\mathcal{L}(\mathcal{S}(\mathbb{R}^n), S(H)_{-1})$, $\mathcal{L}(\exp \mathcal{S}(\mathbb{R}^n), S(H)_{-1})$, respectively in the H–valued case.

Definition 3.3.1 *Let Φ be a GRP (I) given by expansion $\Phi = \sum_{j=1}^{\infty} f_j \otimes H_{\alpha^j}$, $f_j \in \mathcal{S}_{-k}(\mathbb{R}^n)$, $j = 1, 2, \dots$ such that $\sum_{j=1}^{\infty} \|f_j\|_{-k}^2 (2\mathbb{N})^{-p\alpha^j} < \infty$ for some $p \geq 0$. The Fourier transformation of Φ, denoted by $\mathcal{F}(\Phi)$ is defined by the expansion*

$$\mathcal{F}(\Phi) = \sum_{j=1}^{\infty} \widehat{f_j} \otimes H_{\alpha^j}, \tag{3.78}$$

where $\widehat{f_j}$ is the Fourier transformation of f_j, $j = 1, 2, \dots$ in $\mathcal{S}'(\mathbb{R}^n)$.

Since for each $j \in \mathbb{N}$, we have $\|\widehat{f_j}\|_{-k} = \|f_j\|_k$ and consequently $\sum_{j=1}^{\infty} \|\widehat{f_j}\|_{-k}^2 (2\mathbb{N})^{-p\alpha^j} < \infty$, the Fourier transformation is well defined i.e. $\mathcal{F}(\Phi)$ is also a GRP (I). Moreover, we obtain that $\mathcal{F}$ maps $\mathcal{L}(\mathcal{S}_k(\mathbb{R}^n), (S)_{-1,-p})$ into itself.

Note that in [HØUZ] there is a Fourier transformation defined for generalized random processes, but it acts in the $\omega \in \Omega$ variable, while our Fourier transformation acts in the space variable x.

In the Hilbert space valued case, the definition is similar:

Definition 3.3.2 *Let Φ be a H–valued GRP (I) given by the expansion $\Phi = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f_{ij} \otimes H_{\alpha^j} e_i$, $f_{ij} \in \mathcal{S}_{-k}(\mathbb{R}^n)$, $i, j = 1, 2, \dots$ such that $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \|f_{ij}\|_{-k}^2 (2\mathbb{N})^{-p\alpha^j} < \infty$ for some $p \geq 0$. The Fourier transformation of Φ, denoted by $\mathcal{F}(\Phi)$ is the unique H–valued GRP (I) defined by the expansion*

$$\mathcal{F}(\Phi) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \widehat{f_{ij}} \otimes H_{\alpha^j} e_i. \tag{3.79}$$

A simple application

Let $A \in (S)_{-1}$, $\delta \in \mathcal{S}'(\mathbb{R})$ be the Dirac delta distirbution, and consider the SDE

$$\frac{d^2}{dx^2} u(x,\omega) - u(x,\omega) = \delta'(x) \otimes A(\omega), \quad x \in \mathbb{R}, \omega \in \Omega. \tag{3.80}$$

Proposition 3.3.1 *Equation* (3.80) *has a solution of the form*

$$u(x,\omega) = \frac{1}{2}\operatorname{sgn}(x) e^{-|x|} \otimes A(\omega).$$

Proof. Let the chaos expansion of A be $A(\omega) = \sum_{j=1}^{\infty} a_j H_{\alpha^j}(\omega)$, $\omega \in \Omega$. We seek for the solution in form of $u(x,\omega) = \sum_{j=1}^{\infty} u_j(x) \otimes H_{\alpha^j}(\omega)$. Now (3.80) obtains the form $\sum_{j=1}^{\infty}\left(\frac{d^2}{dx^2} u_j(x) - u_j(x)\right) \otimes H_{\alpha^j}(\omega) = \delta'(x) \otimes \sum_{j=1}^{\infty} a_j H_{\alpha^j}(\omega)$, from which we get the system of ODEs

$$\frac{d^2}{dx^2} u_j(x) - u_j(x) = a_j \delta'(x), \quad j \in \mathbb{N}.$$

Applying the Fourier transformation we get

$$-y^2 \hat{u}_j(y) - \hat{u}_j(y) = a_j iy \frac{1}{\sqrt{2\pi}},$$

i.e.

$$\hat{u}_j(y) = \frac{-iya_j}{\sqrt{2\pi}(1+y^2)}.$$

Now we apply the inverse Fourier transformation to get

$$u_j(x) = a_j \frac{1}{2}\operatorname{sgn}(x)e^{-|x|}, \quad j \in \mathbb{N},$$

where sgn $\in \mathcal{S}'(\mathbb{R})$ is understood in distributional sense. □

We can check that this solution is *bona fide*, also using the results from Example 2.2.3. The function

$$u_j(x) = \begin{cases} \frac{a_j}{2}e^{-x}, & x > 0 \\ -\frac{a_j}{2}e^{x}, & x < 0 \end{cases}$$

has a jump height a_j at point $x = 0$. Thus, its distributional derivative is

$$\frac{d}{dx}u_j(x) = \begin{cases} -\frac{a_j}{2}e^{-x}, & x > 0 \\ -\frac{a_j}{2}e^{x}, & x < 0 \end{cases} + a_j\delta(x)$$

and

$$\frac{d^2}{dx^2}u_j(x) = \begin{cases} \frac{a_j}{2}e^{-x}, & x > 0 \\ -\frac{a_j}{2}e^{x}, & x < 0 \end{cases} + a_j\delta'(x) = u_j(x) + a_j\delta'(x).$$

3.3.2 The one dimensional stochastic Helmholtz equation

Let $t_1 \leq t_2 \leq t_3 \cdots \to \infty$ and δ_{t_j} denote the Dirac delta distribution in t_j i.e. $\delta_{t_j}(x) = \delta(x - t_j)$, $j \in \mathbb{N}$. The formal sum

$$\Delta(x,\omega) = \sum_{j=1}^{\infty} \delta_{t_j}(x) \otimes H_{\alpha^j}(\omega) \tag{3.81}$$

defines a GRP (I). This process was introduced in [Se], where it was shown that it is an element of $\mathcal{L}(\mathcal{S}_k(\mathbb{R}), (S)_{-1})$ for $k > \frac{5}{12}$.

Consider now the one dimensional stochastic Helmholtz equation

$$\frac{d^2}{dx^2}\Phi(x,\omega) + k^2\Phi(x,\omega) = \Delta(x,\omega), \quad x \in \mathbb{R}, \omega \in \Omega, \tag{3.82}$$

where k is a constant.

The (deterministic) Helmholtz equation is closely related to the wave equation: Considering the wave equation $\frac{d^2}{dt^2}u(x,t) - c^2\frac{d^2}{dx^2}u(x,t) = 0$ and applying a separation of variables $u(x,t) = e^{-jvt}X(x)$ one gets that $X(x)$ satisfies $(\frac{d^2}{dx^2} + k^2)X(x) = 0$, $k = \frac{v}{c}$. Thus, the solution of the Helmholtz equation represents the spatial part of solution of the wave equation. If there is a source $f(x)e^{-jvt}$ (e.g. some source producing acoustic waves), then it appears as the right hand side of the Helmholtz equation $(\frac{d^2}{dx^2} + k^2)X(x) = f(x)$. Thus, one can interpret f as the wave source.

Because of its relationship to the wave equation, the Helmholtz equation arises in the study of electromagnetic radiation, seismology, acoustics, thermal and mechanical wave propagation etc.

The constant k, known as the "wave number", is the quotient of the angular frequency v and the wave velocity c. E.g. for electromagnetical waves in homogeneous conducting media, k is a function of magnetic permeability and electric conductivity. The right hand side of the Helmholtz equation describes the wave source; thus, the Dirac delta distribution δ on the right hand side is a model for waves propagating from a point source. If we now consider the right hand side of the Helmholtz equation to be the stochastic process Δ, then (3.82) is a model for waves propagating from point sources, which are randomly appearing (their presence or lack is due to some random impulses).

It is also known that any elliptic equation with constant coefficients can be reduced to the Helmholtz equation. Thus, the stochastic Helmholtz equation is just a special case of the elliptic problems considered in Section 3.1, but here we present a new solving technique involving the Fourier transformation. This will provide an explicit form of the solution (the Hilbert space methods in Section 3.1 guarantee existence of a solution but no explicit form). For technical simplicity we consider the one dimensional stochastic Helmholtz equation, but it is easy to carry over the results to $\Delta\Phi(x,\omega) + k^2\Phi(x,\omega) = \Delta(x,\omega), \quad x \in \mathbb{R}^n, \omega \in \Omega$, where Δ is the n–dimensional Laplace operator.

Proposition 3.3.2 *The stochastic Helmholtz equation* (3.82) *has a solution in* $\mathcal{L}(\exp \mathcal{S}(\mathbb{R}), (S)_{-1})$.

Proof. We seek for the solution in form of $\Phi(x,\omega) = \sum_{j=1}^{\infty} f_j(x) \otimes H_{\alpha^j}(\omega)$, where the coefficients f_j, $j \in \mathbb{N}$, are to be determined. Thus, (3.82) is equivalent to

$$\sum_{j=1}^{\infty} \left(\frac{d^2}{dx^2} f_j(x) + k^2 f_j(x) \right) \otimes H_{\alpha^j}(\omega) = \sum_{j=1}^{\infty} \delta_{t_j}(x) \otimes H_{\alpha^j}(\omega),$$

from which we obtain the system of equations

$$\frac{d^2}{dx^2} f_j(x) + k^2 f_j(x) = \delta_{t_j}(x), \quad j \in \mathbb{N}.$$

Applying the Fourier transformation we obtain $-y^2\hat{f}_j(y)+k^2\hat{f}_j(y)=\widehat{\delta(x-t_j)}(y)$. Since $\widehat{\delta(x-t_j)}(y)=e^{-iyt_j}\hat{\delta}(y)=e^{-iyt_j}\frac{1}{\sqrt{2\pi}}$, we obtain that

$$\widehat{f_j}(y)=\frac{1}{\sqrt{2\pi}}\frac{1}{k^2-y^2}e^{-iyt_j},\quad j\in\mathbb{N}.$$

First we note that $\widehat{f_j}\in\exp\mathcal{S}'(\mathbb{R})$, since it has expansion $\widehat{f_j}(y)=\sum_{n=1}^{\infty}\left(\frac{e^{-\frac{1}{2}t_j^2}\sqrt[4]{\pi}}{k^2-y^2}\,i^n\,h_n(t_j)\right)\xi_n(y)$, where h_n are the Hermite polynomials, ξ_n are the Hermite functions, and $\frac{e^{-t_j^2}\sqrt{\pi}}{(k^2-y^2)^2}\sum_{n=1}^{\infty}|h_n(t_j)|^2(e^{2n})^{-2k}<\infty$ for k large enough. Thus, we can apply the inverse Fourier transformation in $\exp\mathcal{S}'(\mathbb{R})$ to obtain

$$f_j(x)=\frac{1}{2k}\sin(k(x-t_j))\,\mathrm{sgn}(x-t_j),$$

where sgn is understood again in distributional sense as $\mathrm{sgn}(x)=2H(x)-1$, and H is the Heaviside function.

Thus, the solution of the Helmholtz equation (3.82) is given by

$$\Phi(x,\omega)=\frac{1}{2k}\sum_{j=1}^{\infty}\sin(k(x-t_j))\,\mathrm{sgn}(x-t_j)\otimes H_{\alpha^j}(\omega),\quad x\in\mathbb{R},\omega\in\Omega.\tag{3.83}$$

It remains to prove that (3.83) is a well-defined GRP (I) i.e. the series converges in $\mathcal{L}(\exp\mathcal{S}(\mathbb{R}),(S)_{-1})$.

Denote by $a_{n,j}=\langle\sin(k(x-t_j)),\xi_n\rangle$ the nth coefficient in the expansion $\sin(k(x-t_j))=\sum_{n=1}^{\infty}a_{n,j}\xi_n(x)$. By elementary calculus we get $a_{1,j}=-\frac{1}{\sqrt[4]{\pi}}e^{-\frac{k^2}{2}}\sqrt{2\pi}\sin(kt_j)$, $a_{2,j}=\frac{1}{\sqrt[4]{\pi}}e^{-\frac{k^2}{2}}k\sqrt{2\pi}\cos(kt_j)$, $a_{3,j}=\frac{1}{2}e^{-\frac{k^2}{2}}(2k^2-1)\sqrt[4]{\pi}\sin(kt_j)$, ..., $a_{n,j}=P_{n-1}(k)e^{-\frac{k^2}{2}}f(kt_j)$, where P_{n-1} is some polynomial of order $n-1$ and f is either the sine or the cosine function. By boundedness of f we get

$$\|\sin(k(x-t_j))\|^2_{-l,exp}=\sum_{n=1}^{\infty}|a_{n,j}|^2\left(e^{2n}\right)^{-2l}=e^{-k^2}\sum_{n=1}^{\infty}P_{2n-2}(k)e^{-4nl}<\infty$$

for $l>0$ large enough. Moreover, $\|\sin(k(x-t_j))\|^2_{-l,exp}$ does not depend on $j\in\mathbb{N}$. Denote $A=\|\sin(k(x-t_j))\|^2_{-l,exp}$. Thus,

$$\sum_{j=1}^{\infty}\|\sin(k(x-t_j))\,sgn(x-t_j)\|^2_{-l,exp}(2\mathbb{N})^{-p\alpha^j}=A\sum_{j=1}^{\infty}(2\mathbb{N})^{-p\alpha^j}<\infty,$$

for $p>1$. □

By considering only the zeroth terms in the expansion of the GRPs involved in the Helmholtz equation we obtain following consequence of Proposition 3.3.2.

Corollary 3.3.1 *Let Φ be the solution of (3.82). Its generalized expectation $E(\Phi)$ is the solution of the deterministic Helmholtz equation*

$$\frac{d^2}{dx^2}u(x) + k^2 u(x) = E(\Delta(x, \cdot)), \quad x \in \mathbb{R},$$

where $E(\Delta)$ is the generalized expectation of Δ.

Note that a similar consideration can be carried out for the Hilbert space valued case, if we consider the H–valued GRP (I) defined in Example 2.2.1 (ii) as the right hand side of equation (3.82).

3.3.3 The Helmholtz equation with stochastic wave number

Now we consider the Helmholtz equation, and assume the wave number k is also random. In physical interpretation this describes waves propagating with a random speed from randomly appearing point sources.

Assume $K \in (S)_{-1}$ has expansion $K(\omega) = \sum_{j=1}^{\infty} k_j H_{\alpha^j}(\omega)$, and $k_j \geq 0$ for all $j \in \mathbb{N}$. Consider the Helmholtz SDE

$$\frac{d^2}{dx^2}\Phi(x, \omega) + K(\omega) \Diamond \Phi(x, \omega) = \Delta(x, \omega), \quad x \in \mathbb{R}, \omega \in \Omega, \tag{3.84}$$

where Δ is the GRP defined in (3.81).

Proposition 3.3.3 *The stochastic Helmholtz equation* (3.84) *has a solution Φ in $\mathcal{L}(\exp \mathcal{S}(\mathbb{R}), (S)_{-1})$. Moreover, its generalized expectation $E(\Phi)$ is the solution of the deterministic Helmholtz equation*

$$\frac{d^2}{dx^2}u(x) + E(K(\cdot))u(x) = E(\Delta(x, \cdot)), \quad x \in \mathbb{R},$$

where $E(K)$ and $E(\Delta)$ are the generalized expectations of K and Δ respectively.

Proof. We seek for the solution in form of $\Phi(x, \omega) = \sum_{j=1}^{\infty} f_j(x) \otimes H_{\alpha^j}(\omega)$, where the coefficients f_j, $j \in \mathbb{N}$, are to be determined. Thus, (3.84) is equivalent to the system

$$\frac{d^2}{dx^2} f_n(x) + \sum_{\substack{i,j \in \mathbb{N} \\ i+j=n+1}} k_i f_j(x) = \delta_{t_n}(x), \quad n \in \mathbb{N}. \tag{3.85}$$

For $n=1$ we obtain $(i=j=1)$ the equation $f_1''+k_1f_1=\delta_{t_1}$ which has the solution

$$f_1(x)=\frac{1}{\sqrt{k_1}}\sin(\sqrt{k_1}(x-t_1))\,\mathrm{sgn}(x-t_1),\quad x\in\mathbb{R}.$$

For $n=2$ we get $(i=1,j=2$ and $i=2,j=1)$ the equation $f_2''+k_1f_2+k_2f_1=\delta_{t_2}$. This we solve using the Fourier transformation to obtain $\hat{f}_2(y)=\frac{1}{\sqrt{2\pi}}\frac{1}{k_1-y^2}\left(e^{-iyt_2}-\frac{k_2}{k_1-y^2}e^{-iyt_1}\right)$ and now we apply the inverse Fourier transformation to obtain

$$\begin{aligned}f_2(x)=&\frac{1}{2\sqrt{k_1}}\sin(\sqrt{k_1}(x-t_2))\,\mathrm{sgn}(x-t_2)\\&-\frac{k_2}{4\sqrt{k_1^3}}\Big(\sin(\sqrt{k_1}(x-t_1))-k_1(x-t_1)\cos(\sqrt{k_1}(x-t_1))\Big)\,\mathrm{sgn}(x-t_1).\end{aligned}$$

For $n=3$ we get $(i=1,j=3$ and $i=3,j=1$ and $i=j=2)$ the equation $f_3''+k_1f_3+k_3f_1+k_2f_2=\delta_{t_3}$. Since f_1 and f_2 are already known, we apply the Fourier and its inverse transformation to get $\hat{f}_3(y)=\frac{1}{\sqrt{2\pi}}\frac{1}{k_1-y^2}\left(e^{-iyt_3}-\frac{k_3}{k_1-y^2}e^{-iyt_1}-\frac{k_2}{k_2-y^2}e^{-iyt_2}\right)$ and

$$\begin{aligned}f_3(x)=&\frac{1}{2\sqrt{k_1}}\sin(\sqrt{k_1}(x-t_3))\,\mathrm{sgn}(x-t_3)\\&-\frac{k_3}{4\sqrt{k_1^3}}\Big(\sin(\sqrt{k_1}(x-t_1))+\sqrt{k_1}(t_1-x)\cos(\sqrt{k_1}(x-t_1))\Big)\,\mathrm{sgn}(x-t_1)\\&-\frac{k_2\,\mathrm{sgn}(x-t_2)}{2(k_1-k_2)\sqrt{k_1k_2}}\Big(\sqrt{k_1}\sin(\sqrt{k_2}(x-t_2))-\sqrt{k_2}\sin(\sqrt{k_1}(x-t_2))\Big).\end{aligned}$$

We proceed by the same procedure to calculate the coefficients $f_4(x),f_5(x),\ldots$ etc. By boundedness of the sine and cosine function, we obtain that $|f_n(x)|$ is bounded by a polynomial of order not greater than n. Thus, convergence in $\mathcal{L}(\exp\mathcal{S}(\mathbb{R}),(S)_{-1})$ follows by customary arguments, analogously as in Proposition 3.3.2. $\square$

Epilogue

White noise theory offers one of the most compelling instances of infinite dimensional analysis, leading to a clear understanding of many empirical phenomena. As a contribution to this noble theory, in Chapter 2 of the dissertation, fundamental theorems were obtained characterizing the structure of generalized stochastic processes. To begin the harvest of consequences, a few examples have found place in Chapter 3 to illustrate the applications to singular SPDEs. Further applications to SPDEs serving more concrete demands, and the modeling of probabilistical properties of their solutions (e.g. such as the martingale or the Markov property) are deferred but certainly not denied. They remain as enticing possibilities for future investigations.

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Curriculum Vitae

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Academic Degrees Awarded

- *Master of Sciences* (Mathematics), February 2004, Faculty of Science, Novi Sad, GPA: 10,00.
- *Bachelor of Sciences* (Mathematics), June 2000, Faculty of Science, Novi Sad, GPA: 9,92.

Education

2004 – Present	Doctoral studies
2000 – 2004	Master studies (Mathematics, Major: analysis and probability)
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Study Abroad Experience

February – July 2005, Department of Engineering Mathematics, Geometry and Computer Science (Institut für Technische Mathematik, Geometrie und Bauinformatik) in Innsbruck, Austria.

Academic Honors, Awards and Scholarships

- 2005 — *Ernst Mach* scholarship of the ÖAD for study stay in Austria
- 2000 — Scholarship of the Royal Norway Embassy
- 2000 — University award *Mileva Marić-Einstein* for best student paper
- 1995 – 2001 — Scholar of the Republic Foundation for Development of Scientific and Artistic Youth

Interests

Language Skills

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Membership

MENSA Yugoslavia, 1998 – 2005.

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2004 – Present	*Assistant*, Department of mathematics and informatics, Faculty of Science, Novi Sad.
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Teaching

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- *Metric and normed spaces* (for students of Mathematics)
- *Partial Differential Equations* (for students of Mathematics)
- *Financial Mathematics* II (for students of Mathematics)
- *Introduction to Calculus* (for students of Mathematics)
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Presentations at Conferences

- D. Seleši, *Stochastic Models of Financial Markets*, XV Conference on Applied Mathematics, Zlatibor, 2002.
- D. Seleši, *Expansion Theorems for Generalized Random Processes, Wick Products and Applications to Stochastic Differential Equations,* Generalized Functions 2004: Topics in PDE, Harmonic Analysis and Mathematical Physics, Novi Sad, 2004.

Publication List

- D. Seleši, *Expansion of Generalized Stochastic Processes with Applications to Equations,* Master Thesis, University of Novi Sad, 2004.
- S. Pilipović, D. Seleši, *Structure Theorems for Generalized Random Processes,* Acta Mathematica Hungarica, 2007, to appear.
- S. Pilipović, D. Seleši, *Expansion Theorems for Generalized Random Processes, Wick Products and Applications to Stochastic Differential Equations,* Infinite Dimensional Analysis, Quantum Probability and Related Topics, 2007, to appear in Vol.10.
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Biografija

Rođena sam 22. aprila 1978. godine u Zrenjaninu, gde sam završila osnovnu školu i gimnaziju sa prosekom 5,00 i Vukovom diplomom, kao i nižu muzičku školu. Od 1995. do 2001. bila sam stipendista Republičke fondacije za razvoj naučnog i umetničkog podmlatka. Studije matematike sam započela 1996. na Prirodno-matematičkom fakultetu u Novom Sadu na smeru *Diplomirani matematičar*. Diplomirala sam juna 2000. sa prosečnom ocenom 9,92 i nagradom *Mileva Marić-Einstein* za najbolji seminarski rad.

Od oktobra 2000. godine sam zaposlena kao asistent-pripravnik, a od juna 2004. kao asistent na Departmanu za matematiku i informatiku, PMF-a u Novom Sadu. Tokom proteklih godina držala sam vežbe studentima matematike, informatike i fizike iz predmeta: *Verovatnoća, Statistika, Statističko modeliranje, Funkcionalna analiza, Metrički i normirani prostori, Parcijalne diferencijalne jednačine, Finansijska matematika* II, *Uvod u analizu, Analiza* I, *Matematika* I, *Matematika* II. Od oktobra 2004. godine zaposlena sam kao asistent i na katedri za farmaciju Medicinskog fakulteta, gde izvodim vežbe iz predmeta *Matematika*, kao i vežbe iz predmeta *Mathematics* za studijsku grupu na engleskom jeziku.

Postdiplomske studije sam upisala novembra 2000. godine iz oblasti Analiza i verovatnoća. U toku studija do 2002. položila sam sve ispite sa prosečnom ocenom 10,00. Magistarsku tezu pod naslovom *Ekspanzija uopštenih stohastičkih procesa sa primenama u jednačinama*, odbranila sam 16. februara 2004. na Prirodno-matematičkom fakultetu u Novom Sadu i time stekla zvanje magistra matematičkih nauka. Od februara do jula 2005. godine boravila sam u Austriji kao stipendista ÖAD na Institutu za tehničku matematiku, geometriju i informatiku (Institut für Technische Mathematik, Geometrie und Bauinformatik), univerziteta u Innsbrucku.

U novembru 2005. položila sam sertifikacioni ispit Goethe Instituta *Grosses Deutsches Sprachdiplom* za poznavanje nemačkog jezika na nivou maternjeg jezika sa najvišom ocenom. Tečno govorim i engleski kao strani jezik, a maternji jezik mi je mađarski.

Oblasti mog naučnog interesovanja su stohastički procesi, teorija uopštenih funkcija, diferencijalne jednačine, finansijska matematika i statistika. Imala sam izlaganja na konferenciji PRIM XV (Zlatibor, 2002.) i na međunarodnoj konferenciji GF04 (Novi Sad, 2004.). Pored toga učestvovala sam još na više konferencija, seminara i letnjih/zimskih škola. Autor ili koautor sam tri naučna rada, kao i nekoliko skripti za studente. Od 2002. angažovana sam na projektu *Metode funkcionalne analize, ODJ i PDJ sa singularitetima*, koji finansira Ministarstvo nauke i zaštite životne sredine Republike Srbije.

Oblasti interesovanja su mi pored nauke i naučna fantastika, učenje stranih jezika i plivanje. Od 1998. do 2005. godine bila sam član MENSA Jugoslavije.

Novi Sad, 2007. *Dora Seleši*

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Predmetna odrednica/Ključne reči: Uopšteni stohastički procesi, beli šum, Braunovo kretanje, haos ekspanzija, Vikov proizvod, distribucije, Kolomboove algebre, regularizacija, stohastičke diferencijalne jednačine, prostori Soboljeva, prostori Kondratieva, prostori Zemaniana, Hilbertovi prostori, linearni eliptički diferencijalni operator, Dirihleov problem, Furijeova transformacija.
PO

UDK:
Čuva se: u biblioteci Departmana za matematiku i informatiku, Novi Sad
ČU
Važna napomena:
VN
Izvod: Doktorska disertacija je posvećena raznim klasama uopštenih stohastičkih procesa i njihovim primenama na rešavanje singularnih stohastičkih parcijalnih diferencijalnih jednačina. U osnovi, disertacija se može podeliti na dva dela. Prvi deo disertacije (Glava 2) je posvećen strukturnoj karakterizaciji uopštenih stohastičkih procesa u vidu haos ekspanzije i integralne reprezentacije. Drugi deo disertacije (Glava 3) čini primena dobijenih rezultata na rešavanje stohastičkog Dirihleovog problema u kojem se množenje modelira Vikovim proizvodom, a koeficijenti eliptičnog diferencijalnog operatora su Kolomboovi uopšteni stohastički procesi.
IZ

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