Applied Mathematics and Computation 208 (2009) 225–230

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

# Special *H*-matrices and their Schur and diagonal-Schur complements ☆

Ljiljana Cvetković [a,*], Maja Nedović [b]

[a] Faculty of Science, Mathematics and Informatics, University of Novi Sad, Trg D. Obradovica 4, Novi Sad 21000, Serbia
[b] Faculty of Technical Sciences, University of Novi Sad, Serbia

ARTICLE INFO

*Keywords:*
*H*-matrices
Schur complement
Diagonal scaling

ABSTRACT

It is well known, see [D. Carlson, T. Markham, Schur complements of diagonally dominant matrices, Czech. Math. J. 29 (104) (1979) 246–251 [2]; J. Liu, J. Li, Z. Huang, X. Kong, Some properties of Schur complements and diagonal-Schur complements of diagonally dominant matrices, Linear Alg. Appl. 428 (2008) 1009–1030] [14], that the Schur complement of a strictly diagonally dominant matrix is strictly diagonally dominant, as well as its diagonal-Schur complement. Also, if a matrix is an *H*-matrix, then its Schur complement and diagonal-Schur complement are *H*-matrices, too, see [J. Liu, Y. Huang, Some properties on Schur complements of *H*-matrices and diagonally dominant matrices, Linear Alg. Appl. 389 (2004) 365–380] [13]. Recent research, see [J. Liu, Y. Huang, F. Zhang, The Schur complements of generalized doubly diagonally dominant matrices, Linear Alg. Appl. 378 (2004) 231–244 [12]; J. Liu, J. Li, Z. Huang, X. Kong, Some properties of Schur complements and diagonal-Schur complements of diagonally dominant matrices, Linear Alg. Appl. 428 (2008) 1009–1030] [14], showed that the similar statements hold for some special subclasses of *H*-matrices. The aim of this paper is to give more invariance results of this type, and simplified proofs for some already known results, by using scaling approach.



## 1. Introduction

The main idea of the considerations that follow is the fact that a matrix $A$ is an $H$-matrix if and only if there exists a diagonal nonsingular matrix $W$ such that $AW$ is a strictly diagonally dominant (SDD) matrix. In other words, see [16], the class of $H$-matrices is diagonally derived from the class of SDD matrices. Some special subclasses of $H$-matrices could be characterized by the form of the corresponding scaling matrix $W$. These characterizations will be presented in short in the first section, as they have already been proven in [6], and some other subclasses of $H$-matrices will be recalled. In the second section simplified proofs of the statements from [14] will be presented, as well as another result of the same type concerning diagonal-Schur complement and Dashnic–Zusmanovich (DZ) matrices. The third section deals with another subclass of $H$-matrices, called S-Nekrasov matrices, for which we give some closure properties under taking the Schur complement and the diagonal-Schur complement in a similar way, i.e., by using scaling approach.

☆ This work is partly supported by the Ministry of Science of Serbia, grant 144025 and by the Provincial Secretariat of Science and Technological Development of Vojvodina, grant 0708.

* Corresponding author.
*E-mail address:* lila@im.ns.ac.yu (L. Cvetković).

0096-3003/$ - see front matter © 2008 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2008.11.040

Throughout the paper we will use the following notations:

$$N := \{1,2,\ldots,n\} \quad \text{for the set of indices},$$

$$S \quad \text{for any nonempty proper subset of } N,$$

$$\overline{S} := N \setminus S \quad \text{for the complement of } S,$$

$$r_i(A) := \sum_{k\in N, k\neq i} |a_{ik}| \quad \text{for } i \text{ th row sum, and}$$

$r_i^S(A) := \sum_{k\in S,k\neq i}|a_{ik}|$ for part of $i$th row sum, which corresponds to the subset $S$.

Obviously, for arbitrary subset $S$ and each index $i \in N$, $r_i(A) = r_i^S(A) + r_i^{\overline{S}}(A)$.

It is important to emphasize that all the time we are dealing with *nonsingular H*-matrices, calling them shortly *H*-matrices. To be precise, we recall the definition of *H*-matrices, as well as some more preliminaries.

**Definition 1.** A matrix $A = [a_{ij}] \in \mathbf{C}^{n,n}$ is called an *H*-matrix if its comparison matrix $\langle A\rangle = [m_{ij}]$ defined by

$$m_{ii} = |a_{ii}|, \quad m_{ij} = -|a_{ij}|, \quad i,j = 1,2,\ldots,n, \;\; i \neq j$$

is an *M*-matrix, i.e., $\langle A\rangle^{-1} \geqslant 0$.

**Definition 2.** A matrix $A = [a_{ij}] \in \mathbf{C}^{n,n}$ is called an SDD matrix if, for each $i \in N$, it holds that

$$|a_{ii}| > r_i(A).$$

**Theorem 1.** *If a matrix $A \in \mathbf{C}^{n,n}$ is an SDD matrix, then it is nonsingular, moreover it is an H-matrix.*

The above statement that SDD matrices are nonsingular is an old and recurring result in matrix theory, see [15]. This basic result can be traced back to at least Levy (1881), Desplanques (1887), Minkowski (1900) and Hadamard (1903).

The next theorem was formulated in the present form in [3], but it can be treated as the same result as ($M35$) of Theorem 2.3 in the chapter 6 of [1].

**Theorem 2.** *A matrix A is an H-matrix if and only if there exists a diagonal nonsingular matrix W such that AW is an SDD matrix. Moreover, we can always assume that W has only positive diagonal entries.*

The following subclass of *H*-matrices has been investigated in [8,9].

**Definition 3.** A matrix $A = [a_{ij}] \in \mathbf{C}^{n,n}$ is called a Dashnic–Zusmanovich matrix if there exists an index $i \in N$ such that

$$|a_{ii}| \cdot (|a_{jj}| - r_j(A) + |a_{ji}|) > r_i(A) \cdot |a_{ji}|, \quad \text{for all } j \neq i, j \in N.$$

**Theorem 3.** [8]. *If a matrix $A \in \mathbf{C}^{n,n}$ is a Dashnic–Zusmanovich matrix, then it is nonsingular, moreover it is an H-matrix.*

Class of $\mathcal{S}$-SDD matrices was defined in the present form in [4,15]. It is easy to see that this class (which is also the subclass of *H*-matrices) is the same one defined in [14] under the name *strictly generalized doubly diagonally dominant* matrices. Here we will recall one of several equivalent definitions of the $\mathcal{S}$-SDD class, for more details see [5].

**Definition 4.** Given any matrix $A = [a_{i,j}] \in \mathbf{C}^{n,n}, n \geqslant 2$, and given any nonempty proper subset $S$ of $N$, then $A$ is an $S$-strictly diagonally dominant ($S$-SDD) matrix if

$$|a_{ii}| > r_i^S(A) \quad \text{for all } i \in S \text{ and},$$

$$(|a_{ii}| - r_i^S(A))(|a_{jj}| - r_j^{\overline{S}}(A)) > r_i^{\overline{S}}(A) r_j^S(A) \quad \text{for all } i \in S, j \in \overline{S}.$$

**Definition 5.** If there exists a nonempty proper subset $S$ of $N$, such that $A = [a_{i,j}] \in \mathbf{C}^{n,n}, n \geqslant 2$ is an $S$-SDD matrix, then we will say that $A$ belongs to class of $\mathcal{S}$-SDD matrices.

The following classes have been investigated under different names, see, for example, [11]. In order to be precise, we will recall all definitions we need.

First of all, we define $h_i(A)$ recursively:

$$h_1(A) := \sum_{j\neq 1} |a_{1j}|,$$

$$h_i(A) := \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=i+1}^{n} |a_{ij}|,$$

and $h_i^S(A)$:

$$h_1^S(A) := r_1^S(A),$$
$$h_i^S(A) := \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j^S(A)}{|a_{jj}|} + \sum_{j=i+1, j\in S}^{n} |a_{ij}|.$$

Obviously, for arbitrary subset $S$ and each index $i \in N$,

$$h_i(A) = h_i^S(A) + h_i^{\overline{S}}(A).$$

**Definition 6.** A matrix $A = [a_{ij}] \in \mathbf{C}^{n,n}$, $n \geqslant 2$ is called Nekrasov matrix if, for each $i \in N$, it holds that

$$|a_{ii}| > h_i(A).$$

**Definition 7.** Given any matrix $A = [a_{i,j}] \in \mathbf{C}^{n,n}$, $n \geqslant 2$, and given any nonempty proper subset $S$ of $N$, then $A$ is an $S$-Nekrasov matrix if

$$|a_{ii}| > h_i^S(A) \quad \text{for all } i \in S,$$
$$|a_{jj}| > h_j^{\overline{S}}(A) \quad \text{for all } j \in \overline{S} \text{ and,}$$
$$(|a_{ii}| - h_i^S(A))(|a_{jj}| - h_j^{\overline{S}}(A)) > h_i^{\overline{S}}(A) h_j^S(A) \quad \text{for all } i \in S, j \in \overline{S}.$$

**Definition 8.** If there exists a nonempty proper subset $S$ of $N$, such that $A = [a_{i,j}] \in \mathbf{C}^{n,n}$, $n \geqslant 2$ is an $S$-Nekrasov matrix, then we will say that $A$ belongs to class of $\mathscr{S}$-Nekrasov matrices.

## 2. Scaling matrices in characterization of some subclasses of *H*-matrices

According to Theorem 2, a matrix $A \in \mathbf{C}^{n,n}$ is an $H$-matrix if and only if there exists a nonsingular diagonal matrix $W$ such that $AW$ is an SDD matrix. But, such a matrix $W$ could be found in a very few special cases. Up to now, we are aware of two such cases: Dashnic–Zusmanovich matrices and $\mathscr{S}$-SDD matrices.

Namely, Dashnic–Zusmanovich class can be characterized as a subclass of $H$-matrices for which the corresponding scaling matrix $W$ belongs to the set $\mathscr{F}$, defined as the set of diagonal matrices, whose diagonal entries are equal to 1, all except one, which is an arbitrary positive number, i.e.,

$$\mathscr{F} = \{W = diag(w_1, w_2, \ldots, w_n) : w_i = \gamma > 0 \text{ for one } i \in N, \text{and } w_j = 1 \text{ for } j \neq i\}. \tag{1}$$

From the other hand, the $\mathscr{S}$-SDD class can be characterized as a subclass of $H$-matrices for which the corresponding scaling matrix $W$ belongs to the set $\mathscr{W}$, defined as the set of all diagonal matrices whose diagonal entries are either 1 or $\gamma$, where $\gamma$ is an arbitrary positive number, i.e.,

$$\begin{aligned} \mathscr{W} &= \bigcup_{S\subset N} \mathscr{W}^S, \\ \mathscr{W}^S &= \{W = diag(w_1, w_2, \ldots, w_n) : w_i = \gamma > 0 \text{ for } i \in S \text{ and } w_i = 1 \text{ otherwise}\}. \end{aligned} \tag{2}$$

In the next section we will use the following theorems, proved in [6]:

**Theorem 4.** *A matrix A is an $\mathscr{S}$-SDD matrix if and only if there exists a matrix $W \in \mathscr{W}$ such that AW is an SDD matrix.*

**Theorem 5.** *A matrix A is a Dashnic–Zusmanovich matrix if and only if there exists a matrix $W \in \mathscr{F}$ such that AW is an SDD matrix.*

Concerning the class of S-Nekrasov matrices, at this point we just want to emphasize that it can be characterized as a subclass of $H$-matrices for which the corresponding scaling matrix $W$, which scales it into the class of Nekrasov matrix, belongs to the set $\mathscr{W}$ defined above.

## 3. Diagonal-Schur complement of S-SDD matrices and DZ matrices

The diagonal-Schur complement of $A$ with respect to a proper subset of $N, \alpha$, is denoted by $A/_\circ\alpha$ and defined to be

$$A(\overline{\alpha}) - \{A(\overline{\alpha}, \alpha)(A(\alpha))^{-1} A(\alpha, \overline{\alpha})\} \circ I$$

where $A(\alpha, \beta)$ stands for the submatrix of $A \in \mathbf{C}^{n,n}$ lying in the rows indexed by $\alpha$ and the columns indexed by $\beta$, while $A(\alpha, \alpha)$ is abbreviated to $A(\alpha)$. For $A = (a_{ij}) \in \mathbf{C}^{m,n}$ and $B = (b_{ij}) \in \mathbf{C}^{m,n}$, the Hadamard product of $A$ and $B$ is the matrix $(a_{ij}b_{ij})$, which we denote by $A \circ B$. Throughout the paper we assume that $A(\alpha)$ is a nonsingular matrix.

In [14] the following theorem has been proven.

**Theorem 6.** *Let $A \in SGDD_n^{N_1,N_2}, \alpha \subset N$. If $N_1 \subseteq \alpha$ or $N_2 \subseteq \alpha$, then*

$$A/_\circ\alpha \in SD_{n-|\alpha|}.$$

*If $N_1 \not\subseteq \alpha$ and $N_2 \not\subseteq \alpha$, then*

$$A/_\circ\alpha \in SGDD_{n-|\alpha|}^{N_1-\alpha,N_2-\alpha}.$$

First, let us explain the above notation. A matrix $A$ from $\mathbf{C}^{n,n}$ is called a *strictly generalized doubly diagonally dominant* matrix in $\mathbf{C}^{n,n}$ if there exist proper subsets $N_1, N_2$ of $N$ such that $N_1 \cap N_2 = \emptyset, N_1 \cup N_2 = N$ and

$$(|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) > \beta_i\alpha_j$$

for all $i \in N_1$ and $j \in N_2$, where, with $s = i$ or $j$,

$$\alpha_s = \sum_{t\in N_1, t\neq s} |a_{st}|,$$
$$\beta_s = \sum_{t\in N_2, t\neq s} |a_{st}|.$$

For this choice of $N_1, N_2$, we write $A \in SGDD_n^{N_1,N_2}$. But, obviously, $SGDD_n^{N_1,N_2}$ is the same set as the one that we call $N_1$-SDD matrices, while the set $SGDD_n$ of all strictly generalized doubly diagonally dominant matrices in $\mathbf{C}^{n,n}$ is, in fact, our set $\mathscr{S}$-SDD. The set $SD_n$ is actually the set of all strictly diagonally dominant (SDD) matrices in $\mathbf{C}^{n,n}$.

Theorem 6 has been proven in [14] using various algebraic inequalities. We will show here the simplified proof for both statements in this theorem.

**Theorem 7** (The same as Theorem 6). *Let $A = [a_{i,j}] \in \mathbf{C}^{n,n}$ be an S-SDD matrix. Then for any nonempty proper subset $\alpha$ of N:*

- *such that $S \subseteq \alpha$ or $\overline{S} \subseteq \alpha$, $A/_\circ\alpha$ is an SDD matrix;*
- *$A/_\circ\alpha$ is also an $\mathscr{S}$-SDD matrix. More precisely, if A is an S-SDD matrix, then $A/_\circ\alpha$ is an $(S \setminus \alpha)$-SDD matrix.*

**Proof.** Let $A$ be an $S$-SDD matrix. Then, from Theorem 4, there exists a matrix $W \in \mathscr{W}$ (defined by (2)), such that $AW$ is an SDD matrix. As the diagonal-Schur complement of a strictly diagonally dominant matrix is strictly diagonally dominant, too, we conclude that $AW/_\circ\alpha$ is strictly diagonally dominant matrix. As in [6], it is easy to see that

$$(AW)/_\circ\alpha = (A/_\circ\alpha) \cdot W(\overline{\alpha}).$$

- Since $W(\overline{\alpha})$ is either the identity matrix, $I$ (if $S \subseteq \alpha$), or $\gamma \cdot I$ (if $\overline{S} \subseteq \alpha$), it will not affect the strict diagonal dominance. Therefore, $A/_\circ\alpha$ is a strictly diagonally dominant matrix.
- Since $W(\overline{\alpha}) \in \mathscr{W}$, i.e., the class $\mathscr{W}$ is closed under taking principal submatrices, from Theorem 4 we obtain that $A/_\circ\alpha$ is an $\mathscr{S}$-SDD matrix. To complete the proof it is enough to see that the matrix $W(\overline{\alpha})$ is of the form

$$W(\overline{\alpha}) = diag(w_{i_1}, w_{i_2}, \ldots, w_{i_l})$$

with

$$w_{i_j} = \gamma > 0 \text{ for } i_j \in S \setminus \alpha \text{ and } w_{i_j} = 1 \text{ otherwise}. \qquad \square$$

Obviously, using diagonal scaling, the proofs can be significantly shortened, but this technique allows us to get invariance theorems for some other subclasses of $H$-matrices, as we can see from the following theorem.

**Theorem 8.** *Let $A = [a_{i,j}] \in \mathbf{C}^{n,n}$ be a Dashnic–Zusmanovich matrix. Then for any nonempty proper subset $\alpha$ of N, $A/_\circ\alpha$ is also a Dashnic–Zusmanovich matrix.*

**Proof.** Let $A = [a_{i,j}] \in \mathbf{C}^{n,n}$ be a Dashnic–Zusmanovich matrix. Then, from Theorem 5, there exists a matrix $W \in \mathscr{F}$ (defined by (1)), such that $AW$ is an SDD matrix. As the diagonal-Schur complement of a strictly diagonally dominant matrix is strictly diagonally dominant, $AW/_\circ\alpha$ is strictly diagonally dominant, too. Since

$$(AW)/_\circ\alpha = (A/_\circ\alpha) \cdot W(\overline{\alpha})$$

with $W(\overline{\alpha}) \in \mathscr{F}$, Theorem 5 provides that $A/_\circ\alpha$ is a Dashnic–Zusmanovich matrix. $\square$

Moreover, if for the given matrix $A$ there exists a scaling matrix $W \in \mathscr{F}$ with $w_i = \gamma > 0$ where $\{i\} \subseteq \alpha$ or $N\setminus\{i\} = \alpha$, then $A/_\circ\alpha$ is a strictly diagonally dominant matrix. This can be derived from Theorem 8 with $S = \{i\}$.

## 4. Schur and diagonal-Schur complements of Nekrasov and S-Nekrasov matrices

As we have already mentioned before, S-Nekrasov matrices can be characterized by the form of scaling matrices which transform them to Nekrasov matrices. Here, we present this fact as a theorem, for its proof see [7].

**Theorem 9.** *A matrix A is an $\mathscr{S}$-Nekrasov matrix if and only if there exists a matrix $W \in \mathscr{W}$ such that AW is a Nekrasov matrix.*

To prove some properties of the Schur and diagonal-Schur complements, we need some additional notation and preliminaries.

**Definition 9.** The Schur complement of $A$ with respect to a proper subset of $N$, $\alpha$, is denoted by $A/\alpha$ and defined to be

$$A(\overline{\alpha}) - A(\overline{\alpha},\alpha)(A(\alpha))^{-1}A(\alpha,\overline{\alpha}).$$

**Definition 10.** For a given proper subset of the index set, $\alpha$, we say that a matrix class $\mathscr{C}$ is $\alpha$-SC-closed if for any $A \in \mathscr{C}$, $A/\alpha \in \mathscr{C}$.

**Definition 11.** A matrix class $\mathscr{C}$ is SC-closed if $\mathscr{C}$ is $\alpha$-SC-closed for all $\alpha$.

**Definition 12.** We say that a matrix class $\mathscr{C}$ is $\alpha$-diagonal-SC-closed if for any $A \in \mathscr{C}$, $A/_{\circ}\alpha \in \mathscr{C}$.

**Definition 13.** A matrix class $\mathscr{C}$ is diagonal-SC-closed if $\mathscr{C}$ is $\alpha$-diagonal-SC-closed for all $\alpha$.

**Theorem 10** (Sequential property of Schur complement [16]). *Let $A \in \mathbf{C}^{n,n}$ be principally nonsingular and suppose that $\alpha$ is a proper subset of N and that $\beta$ is a proper subset of $\overline{\alpha}$. Then,*

$$(A/\alpha)/\beta = A/(\alpha \cup \beta).$$

It is important to note that the same property does not hold for the diagonal-Schur complement.

In [10], it has been proven that Nekrasov property is hereditary for Gaussian elimination, which implies the following:

**Corollary.** *The Nekrasov class is {1}-SC-closed.*

Using the scaling characterization, from the above fact we obtain:

**Theorem 11.** *If A is S-Nekrasov matrix, then A/{1} is S\ {1}-Nekrasov matrix.*

**Proof.** Let $A$ be an S-Nekrasov matrix. Then, from Theorem 4, there exists a matrix $W \in \mathscr{W}$ (defined by (2)), such that $AW$ is an Nekrasov matrix. As the {1}-Schur complement of a Nekrasov matrix is an Nekrasov matrix, too, we conclude that $AW/\{1\}$ is a Nekrasov matrix. We have also

$$(AW)/\{1\} = (A/\{1\}) \cdot W(\{\overline{1}\}).$$

Since $W(\{\overline{1}\}) \in \mathscr{W}$ is of the form

$$W(\{\overline{1}\}) = diag(w_{i_1}, w_{i_2}, \dots, w_{i_l})$$

with

$$w_{i_j} = \gamma > 0 \quad \text{for } i_j \in S \setminus \{1\} \text{ and } w_{i_j} = 1 \text{ otherwise},$$

from Theorem 4 we obtain that $A/\{1\}$ is an $S \setminus \{1\}$-Nekrasov matrix. □

Or, in other words:

**Theorem 12.** *The $\mathscr{S}$-Nekrasov class is {1}-SC-closed. Moreover, from the sequential property of Schur complement, it is $\alpha$-SC-closed for all $\alpha = \{1,2,\dots,m\}$.*

The similar closure properties hold for diagonal-Schur complement:

**Theorem 13.** *The Nekrasov class is {1}-diagonal-SC-closed.*

**Theorem 14.** *If A is S-Nekrasov matrix, then $A/_{\circ}\{1\}$ is S\ {1}-Nekrasov matrix.*

**Theorem 15.** *The $\mathscr{S}$-Nekrasov class is {1}-diagonal-SC-closed.*

We are concluding this section with the following remark: As the sequential property doesn't hold for diagonal-Schur complement, we don't have immediately the $\alpha$-diagonal-SC-closure for $\alpha = \{1,2,\dots,m\}$ as before.

## References

[1] A. Berman, R.J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, SIAM, Philadelphia, 1994.
[2] D. Carlson, T. Markham, Schur complements of diagonally dominant matrices, Czech. Math. J. 29 (104) (1979) 246–251.
[3] Lj. Cvetković, Convergence theory for the relaxation methods for solving systems of equations (in Serbian), Ph.D. Thesis, University of Novi Sad, 1987.
[4] Lj. Cvetković, V. Kostić, R. Varga, A new Geršgorin-type eigenvalue inclusion area, ETNA 18 (2004) 73–80.
[5] Lj. Cvetković, *H*-matrix theory vs. eigenvalue localization, Numer. Algor. 42 (2006) 229–245.
[6] Lj. Cvetković, V. Kostić, M. Kovačević, T. Szulc, Further results on *H*-matrices and their Schur complements, Appl. Math. Comput. 198 (2) (2008) 506–510.
[7] Lj. Cvetković, V. Kostić, S. Rauški, A new subclass of *H*-matrices, SC:ALA Technical report 1 (2007), <http://scala.im.ns.ac.yu/tr>.
[8] L.S. Dashnic, M.S. Zusmanovich, O nekotoryh kriteriyah regulyarnosti matric i lokalizacii ih spectra, Zh. vychisl. matem. i matem. fiz. 5 (1970) 1092–1097.
[9] L.S. Dashnic, M.S. Zusmanovich, K voprosu o lokalizacii harakteristicheskih chisel matricy, Zh. vychisl. matem. i matem. fiz. 10 (6) (1970) 1321–1327.
[10] Kh.D. Ikramov, M.Yu. Ibragimov, The Nekrasov Property is Hereditary for Gaussian Elimination, ZAMM 77 (5) (1997) 394–396.
[11] W. Li, On Nekrasov matrices, Linear Alg. Appl. 281 (1998) 87–96.
[12] J. Liu, Y. Huang, F. Zhang, The Schur complements of generalized doubly diagonally dominant matrices, Linear Alg. Appl. 378 (2004) 231–244.
[13] J. Liu, Y. Huang, Some properties on Schur complements of *H*-matrices and diagonally dominant matrices, Linear Alg. Appl. 389 (2004) 365–380.
[14] J. Liu, J. Li, Z. Huang, X. Kong, Some properties of Schur complements and diagonal-Schur complements of diagonally dominant matrices, Linear Alg. Appl. 428 (2008) 1009–1030.
[15] R.S. Varga, Geršgorin and His Circles, Springer Series in Computational Mathematics, vol. 36, 2004.
[16] F. Zhang et al, The Schur Complement and Its Applications, Springer, New York, 2005.