Numerical solution of eigenvalue problems in acoustic field computation and car design

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Mathematics for key technologies

Outline

Optimality through mathematics

- Society is increasingly sensitive to inconveniences that come with modern technologies such as air and water pollution, noise by airplanes, cars, trains.
- There is an increasing demand for optimal solutions. Minimal energy consumption, minimal noise, pollution, waste.
- Optimal solutions need mathematical techniques, such as model based optimization/control.
- We need better mathematical models, faster and more accurate numerical methods, robust implementations on modern computer architectures.
- **The progress through better methods exceeds the progress through better hardware by large factors.**

Common attitude

- Our industry has built cars, trains airplanes for ages.
- Most engineers get away with the math from the first year.
- For the solution of differential equations, eigenvalue problems, optimization problems, etc., there are wonderful commercial packages? **They always deliver good solutions.**
- If the problems become more complex then we just buy a bigger computer.
- **We don't really need mathematics** except maybe as language for describing the models.
- **Optimization?** We just use genetic algorithms, they always find the optimal solution.

Anti-theses

No key technology development without modern mathematical technology!

We need:

- Very good mathematical models, that represent the technological process well.
- Deep understanding of the models and the dynamics of the processes.
- Accurate and efficient algorithms to simulate the models/processes.
- Accurate and efficient methods to control and optimize the processes and products.

An industrial example

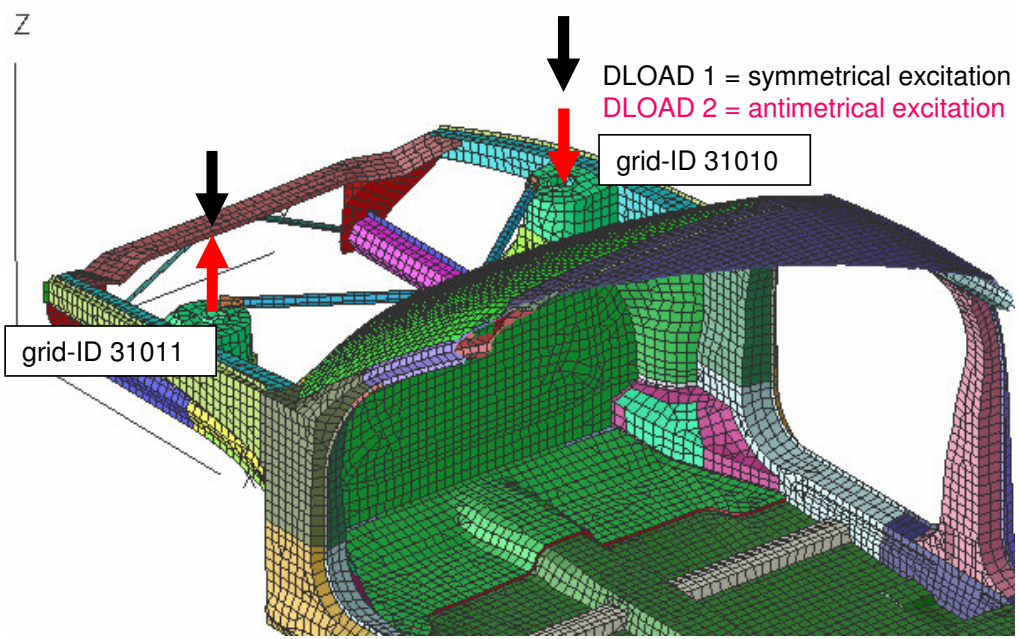
Acoustic field optimization: Industrial Project with SFE in Berlin.
$\rightarrow$ *film*

- Model for acoustic field within a car.
- SFE has its own parameterized FEM model which allows simple geometry and topology changes.
- Goal: Minimize noise in important regions inside the car using changes in geometry, topology, damping material, etc.
- Discretized model has up to size $10,000,000$ or bigger.
- Model order reduction to reduce size of model for optimization.

SFE AKUSMOD

FE Model: Excitation

Unit force = 1 N mm



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http://www.sfe-berlin.de

Optimization and discretization

- For SFE, discretization means modeling with discrete model.
- For costumers of SFE, **Optimization** means playing with the parameter until a certain acoustic field is achieved.
- A reduced order model is needed for small response times and optimization.
- One determines the **important modes** by solving nonlinear eigenvalue problems.
- Can this be automatized by prescribing a cost functional and really doing **optimization**?
- Is the fine discretization that is used in the 'forward PDE' really a good discretization if we then afterwards have to use model reduction to throw away all the unimportant staff.

The 3-D lossless wave equation (in air) is derived from:

1. The continuity equation (conservation of mass):

$$\frac{\partial \tilde{\rho}}{\partial t} + \nabla(\tilde{\rho} v) = 0.$$

2. The Euler equation (Newton's Second Law)

$$\tilde{\rho}(\frac{\partial v}{\partial t} + (v \cdot \nabla)v) = -\nabla \tilde{p}$$

$$\begin{aligned} v &= v(x; y; z; t) \text{ particle velocity}, \\ \tilde{\rho} &= \tilde{\rho}(x; y; z; t) \text{ particle density}, \\ \tilde{p} &= \tilde{p}(x; y; z; t) \text{ pressure}. \end{aligned}$$

Assumptions:

- There is no temperature change.
- The fluid is inviscid (no shear forces).
- No influence of external forces.
- We can make the expansions

$$\begin{aligned} \tilde{p} &= p_0 + p(x; y; z; t) \text{ with } p_0 \gg p \ (p_0 = 10^6 p), \\ \tilde{\rho} &= \rho_0 + \rho(x; y; z; t) \text{ with } \rho_0 \gg \rho. \end{aligned}$$

- Adiabatic fluid (no heat exchange during compression).
- Ideal gas $\rho = \frac{p}{c^2}$ where c is the speed of sound.
- $(v \cdot \nabla)v$ and $\rho \frac{\partial v}{\partial t}$ are small.

Simplified mathematical model

- Euler equations then give

$$\rho_0\left(\frac{\partial v}{\partial t}\right) = -\nabla p$$

or differentiating

$$\rho_0 \nabla\left(\frac{\partial v}{\partial t}\right) = -\Delta p.$$

- Continuity equation gives

$$\frac{\partial \rho}{\partial t} + \rho_0 \nabla v = 0.$$

- Then we have altogether

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} + \rho_0 \nabla \frac{\partial v}{\partial t} = \Delta p + \rho_0 \nabla \frac{\partial v}{\partial t} = 0.$$

Fluid structure interaction

The fluid-structure interaction is modeled via boundary conditions.

Let u be the displacement of the car body on the surface. Then $v = \frac{\partial u}{\partial t}$ and thus with the outer normal ν we get

$$\nu \rho_0 \frac{\partial^2 u}{\partial t^2} = -\nu \nabla p.$$

Multiply with a test function w, using integration by parts by integrating over control volumes V with surface elements S:

$$\int_V \frac{1}{c^2} w \frac{\partial^2}{\partial t^2} p \, dV + \int_V (\nabla w) \nabla p \, dV = -\rho_0 \int_S \nu w \frac{\partial^2 u}{\partial t^2} \, dS,$$

or equivalently

$$\int_V \frac{1}{\rho_0 c^2} w \frac{\partial^2}{\partial t^2} \, dV + \int_V \frac{1}{\rho_0} (\nabla w) \nabla p \, dV = - \int_S \nu w \frac{\partial^2 u}{\partial t^2} \, dS.$$

Damping/absorption is realized by the additional term:

$$\int_s w \frac{r}{\rho_0^2 c^2} \frac{\partial p}{\partial t} \, dS,$$

where r is a material dependent parameter $r = r(\alpha)$. Thus:

$$\int_V \frac{1}{\rho_0 c^2} w \frac{\partial^2}{\partial t^2} p \, dV + \int_V \frac{1}{\rho_0} (\nabla w) \nabla p dV + \int_s w \frac{r}{\rho_0^2 c^2} \frac{\partial p}{\partial t} \, dS = - \int_s \nu w \frac{\partial^2 u}{t^2} \, dS.$$

Finite element model: fluid

Discretization via FE shape functions in space yields the discretized second order system

$$M_f \ddot{p}_d + D_f \dot{p}_d + K_f p_d + D_{sf} \ddot{u}_d = 0.$$

Here $M_f = M_f^T$ and $K_f = K_f^T$ are positive definite and D_f is symmetric positive semidefinite, D_{sf} describes the coupling.

Finite element model: structure

The (discrete) finite element model for the vibration of the structure (linear materials) is:

$$M_s\ddot{u}_d + D_s\dot{u}_d + K_s u_d = f_e + f_p.$$

Here f_e is a (discrete) external load and f_p is the pressure load. M_s, D_s, K_s are real symm. pos. semidef. mass/damping/stiffness matrices of the structure. M_s is singular and diagonal.

Fluid structure coupling

The term f_p originates from the pressure load $F_p = \int_S p\nu \, dS$.
This yields $f_p = D_{sf}^T p_d$ and hence

$$M_s \ddot{u}_d + D_s \dot{u}_d + K_s u_d - D_{sf}^T p_d = f_e.$$

Here the stiffness matrix $K_s = K_1(\omega) + \imath K_2$ is complex symmetric and frequency dependent.

Full model: summary

$$\begin{bmatrix} M_s & 0 \\ D_{sf}^T & M_f \end{bmatrix} \begin{bmatrix} \ddot{u}_d \\ \ddot{p}_d \end{bmatrix} + \begin{bmatrix} D_s & 0 \\ 0 & D_f \end{bmatrix} \begin{bmatrix} \dot{u}_d \\ \dot{p}_d \end{bmatrix} + \begin{bmatrix} K_s(\omega) & D_{sf} \\ 0 & K_f \end{bmatrix} \begin{bmatrix} u_d \\ p_d \end{bmatrix} = \begin{bmatrix} f_s \\ 0 \end{bmatrix}.$$

- M_s, M_f, K_f are real symm. pos. semidef. mass/stiffness matrices of structure and air, M_s is singular and diagonal, M_s is a factor $1000 - 10000$ larger than M_f.
- $K_s(\omega) = K_s(\omega)^T = K_1(\omega) + \imath K_2$.
- D_s, D_f are real symmetric damping matrices.
- D_{sf} is real coupling matrix between structure and air.
- Parts depend on geometry, topology and material parameters.

Reformulation of evp

$$\left(\lambda^2 \begin{bmatrix} M_s & 0 \\ D_{sf}^T & M_f \end{bmatrix} + \lambda \begin{bmatrix} D_s & 0 \\ 0 & D_f \end{bmatrix} + \begin{bmatrix} K_s(\omega) & D_{sf} \\ 0 & K_f \end{bmatrix}\right) \begin{bmatrix} x_s \\ x_f \end{bmatrix} = 0,$$

or after scaling second block row with λ^{-1} and second block column with λ one has the complex symmetric quadratic evp

$$\left(\lambda^2 \begin{bmatrix} M_s & 0 \\ 0 & M_f \end{bmatrix} + \lambda \begin{bmatrix} D_s & D_{sf} \\ D_{sf}^T & D_f \end{bmatrix} + \begin{bmatrix} K_s(\omega) & 0 \\ 0 & K_f \end{bmatrix}\right) \begin{bmatrix} x_s \\ \lambda^{-1} x_f \end{bmatrix} = 0.$$

Numerical methods for nonlinear evps

Methods directly for nonlinear problems (incomplete list). For recent surveys see M./Voss 2005 or Dissertation Schreiber 2008.

- Second order Arnoldi method Bai 2006
- Rational Krylov method Ruhe 1998, 2000
- Residual iteration method Neumaier 1985
- Newton-Type methods Schreiber/Schwetlick 2006, 2008,
- Rayleigh quotient iterations Schreiber 2008, Freitag/Spence 2007, 2008
- Jacobi-Davidson methods Sleijpen/Van der Vorst et al 1996, Betcke/Voss 2004, Hochstenbach 2007
- Arnoldi type methods Voss 2003
- Eigenvalue continuation Beyn/Tümmler 2005,2008

Can we use these methods?

- None of these methods can be applied directly.
- Previous decoupled methods for structure/fluid subsystems do not work appropriately.
- One cannot guarantee that all desired ev's are obtained and a given relative residual?
- Evp is truely nonlinear.
- Problem is very ill-conditioned or even singular for some parameter sets.
- Mass matrix is block diagonal, but singular.
- The methods have to run as parallel methods on modern multi-core machines.

Nonlinear Newton evp. solver.

Truely nonlinear evp $P(\lambda)x = (\lambda^2 M + \lambda D + K(\lambda))x = 0$.
Apply Newton to function

$$f_w(x,\lambda) = \begin{bmatrix} P(\lambda)x \\ w^H x - 1 \end{bmatrix} = 0.$$

The Newton system for $\lambda_{k+1} = \lambda_k + \mu_k$ and $x_{k+1} = x_k + s_k$ is

$$\begin{bmatrix} P(\lambda_k) & \dot{P}(\lambda_k)x_k \\ w^H & 0 \end{bmatrix} \begin{bmatrix} s_k \\ \mu_k \end{bmatrix} = - \begin{bmatrix} P(\lambda_k)x_k \\ w^H x_k - 1 \end{bmatrix}$$

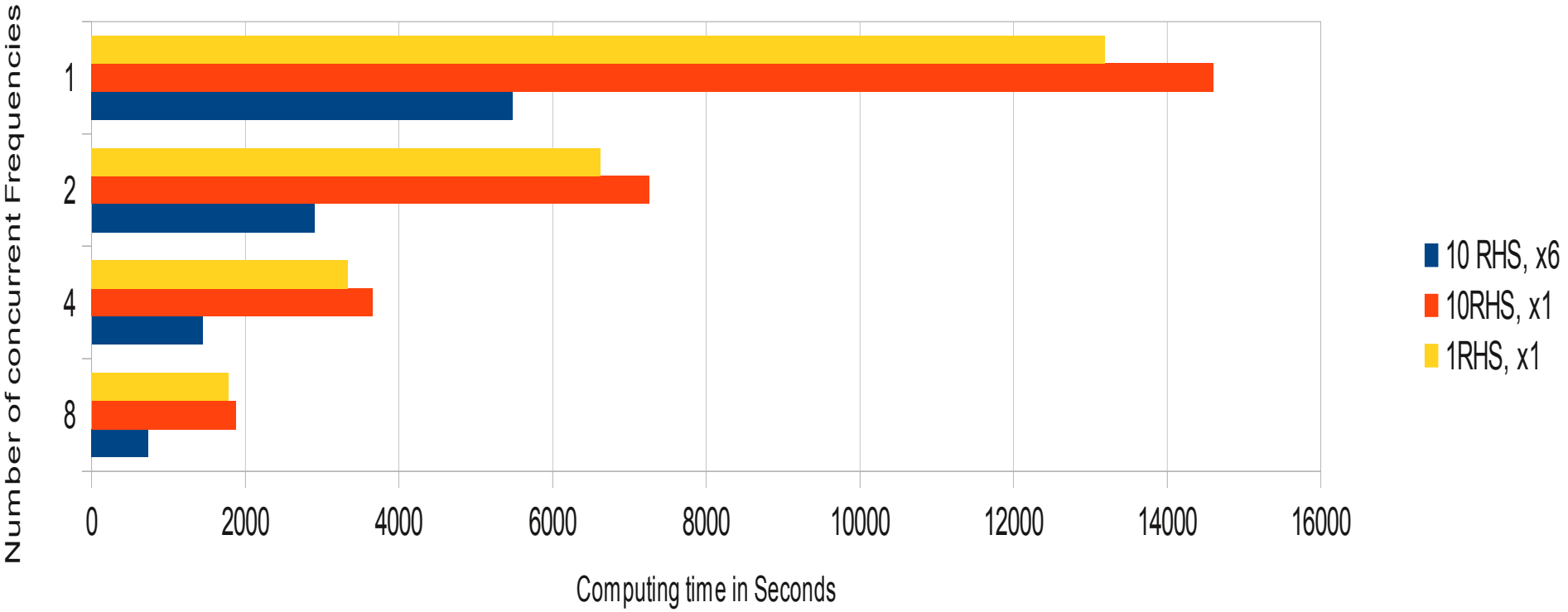
or

$$\begin{aligned} \lambda_{k+1} &= \lambda_k - \frac{1}{w^H P(\lambda_k)^{-1} \dot{P}(\lambda_k) x_k} \\ x_{k+1} &= (\lambda_k - \lambda_{k+1}) P(\lambda_k)^{-1} \dot{P}(\lambda_k) x_k. \end{aligned}$$

Difficulties

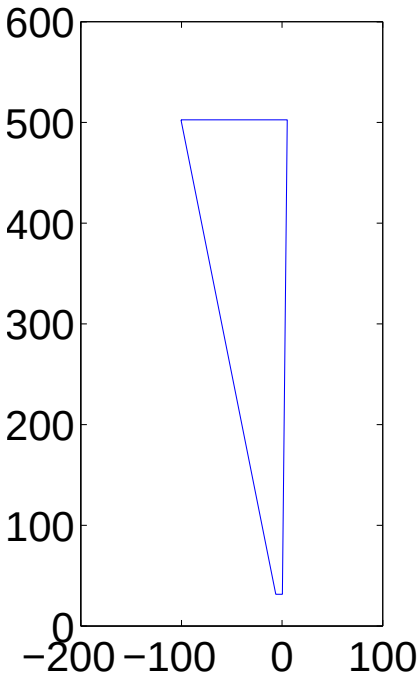
- We want many eigenvalues $50 - 100$.
- Need to use out-of-core sparse solvers for shift-and-invert.
- Need to get into convergence intervals for Newton, Jacobi-Davidson.
- We really need nonlinear model reduction (open problem).
- We need to use the fact that only a small part of the system is changed in every optimization step.
- We need to integrate ev computation, gradient computation, discretization.
- **Adaptive multilevel FEM for evs would be great.**

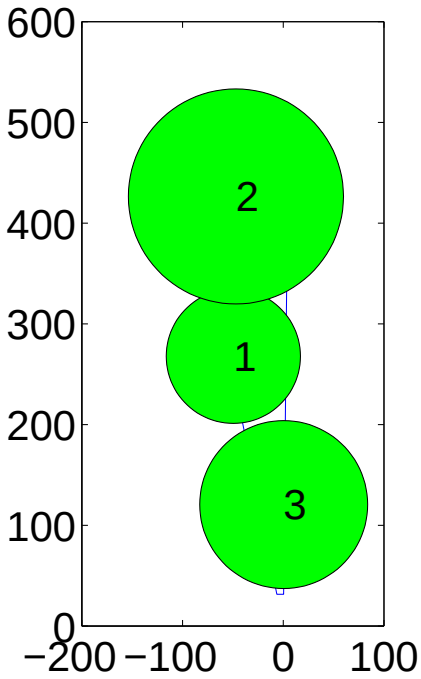
Frequency response



Computation times of the developed frequency response solver for 1,2,4, and 8 concurrent frequencies with different number of processors per frequency (x1 or x6) and different number of right hand sides (RHS).

Eigenvalue tracking





We use Krylov subspace methods with many different shifts: A typical trapezoidal region – within which all eigenvalues are sought – at beginning and after three shifts have been processed.

Intermediate conclusion

- Commercially available codes are not satisfactory.
- Discrete finite elements and quasi-uniform grids are a waste.
- For frequency analysis of high frequencies nothing really works.
- So far everything is partially heuristic, we cannot guarantee that we find all the desired eigenvalues.
- Eigenvalue methods need to use direct solvers for shift-and-invert.
- Homotopy and Newton like methods need to be developed and analyzed.
- Multi-way adaptive methods need to be developed, studied and made industrially available.

Outline

Adaptive FEM for ev. comp.

- Adaptive finite element (AFEM) methods for PDE boundary value problems are well studied. A priori and a posteriori error estimates are available for many problems.
- For PDE eigenvalue problems the problem is much harder.
- Most results and methods only for the self-adjoint elliptic case.
- a priori estimates Larsson 2001, Knyazev et al. 2006, 2007, 2008.
- a posteriori estimates Verfürht 1996, Giani/Graham 2008, Carstensen/Gedicke 2008, Garau/Morin/Zuppa 2008.
- Nonsymmetric problems: Heuveline/Rannacher 2001, Rannacher 2009.
- Very few applications in industrial codes Zschiedrich, Burger, Pomplun, Schmidt 2007/2008.

Getting EVP AFEM into industrial practice

- Need to understand and analyze the non-selfadjoint case.
- Need proof of functionality, prototype for industrial problems.
- Need to match analytic adaptation concepts with numerical linear algebra and optimization adaption concepts.
- Need to make this useful on modern computer architectures.

Model problem: Elliptic PDE evp

Model problem:

$$\begin{aligned} \Delta u &= \lambda u \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Classical FEM discretization (with mesh-width H) leads to generalized discrete evp

$$A_H \mathbf{u}_H = \lambda_H B_H \mathbf{u}_H$$

Solve $\rightarrow$ **Estimate** $\rightarrow$ **Mark** $\rightarrow$ **Refine**

- In classical AFEM it is assumed that the algebraic evp is solved exactly.
- But this requires the largest percentage of the computing time.
- The solution of the algebraic evp is only used to determine where the grid is refined. This is a complete waste of computational work.
- How we can incorporate the solution of the algebraic eigenvalue problem (AEVP) into the adaptation process?

Solve:

- compute eigenpair $(\tilde{\lambda}_H, \tilde{\mathbf{u}}_H)$ on the coarse mesh,
- use iterative solver, i.e. Krylov subspace method,
- do not solve the problem very accurately, stop after k steps or when tolerance *tol* is reached.

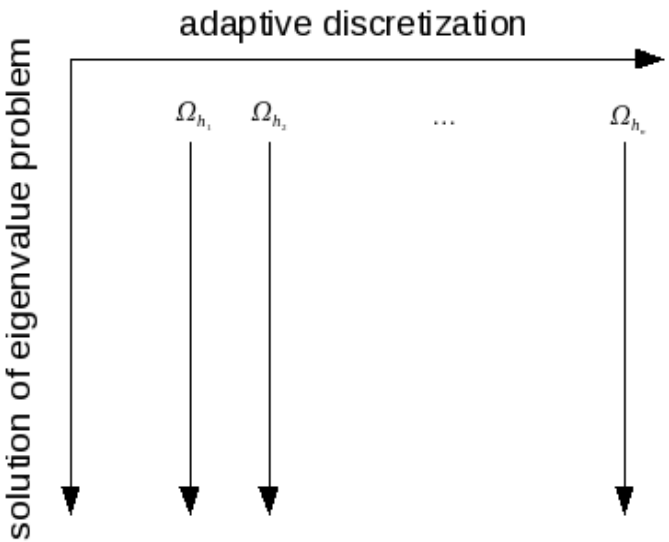
Estimate:

- prolongate $\tilde{\mathbf{u}}_H$ from the coarse mesh $\mathcal{T}_H$ to the uniformly refined mesh $\mathcal{T}_h$,
- compute residual vector $\hat{\mathbf{r}}_h$ and identify all its large coefficients and corresponding basis functions (nodes).

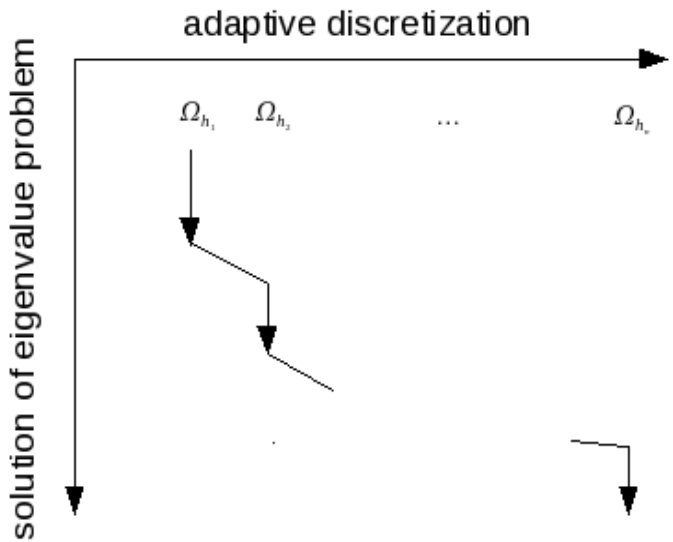
Mark and Refine: mark elements and refine the mesh.

Standard AFEM versus AFEMLA

Solve → Estimate → Mark → Refine



Standard AFEM



AFEMLA

small residual vector

$\xrightarrow{Q1?}$ good approximation of the discretized eigenpair $(\tilde{\lambda}_H, \tilde{\mathbf{u}}_H)$

$\xrightarrow{Q2?}$ good approximation of the PDE eigenpair (λ, u)

Q1: yes

- residual errors can be transformed to backward errors.
- if eigenvalues are well-conditioned.

Q2: yes, if

- saturation assumption holds, i.e., $\lambda_h - \lambda \leq \beta(\lambda_H - \lambda)$.

Computable bounds = backward error analysis + saturation assumption.

Backward error

Theorem

Let A, B be $n \times n$ matrices and let B be invertible. Let $\tilde{\lambda}$ be a computed eigenvalue for the matrix pair (A, B), let $\tilde{x}$ be an associated normalized eigenvector, i.e., $\|\tilde{x}\|_2 = 1$, and let $r = A\tilde{x} - \tilde{\lambda}B\tilde{x}$. Then $\tilde{\lambda}$ is an exact eigenvalue with associated eigenvector $\tilde{x}$ of a pair $(A + E, B)$, where $\|E\|_2 = \|r\|_2$.

Backward Error is of size of residual.

Theorem

Consider a pair (A, B) of real $n \times n$ matrices and assume that B is invertible. Let λ be a simple eigenvalue of the pair (A, B) with right eigenvector x and left eigenvector y, normalized so that $\|x\|_2 = \|y\|_2 = 1$. Let $\tilde{\lambda} = \lambda + \delta\lambda$ be the corresponding eigenvalue of the pair $(A + E, B)$ with eigenvector $\tilde{x} = x + \delta x$. Then

$$\tilde{\lambda} - \lambda = \frac{y^* E x}{y^* B x} + O(\|E\|_2^2),$$

and

$$|\tilde{\lambda} - \lambda| \leq \frac{\|E\|_2}{|y^* B x|} + O(\|E\|_2^2).$$

AFEMLA for smallest ev of Poisson evp

Require: An initial regular triangulation $\mathcal{T}_H^i$, a maximal number p of Arnoldi steps or a tolerance *tol* and a desired accuracy ϵ.

Ensure: Approximation $\hat{\lambda}_1$ to the smallest eigenvalue λ_1 together with the corresp. approx. eigenfunction $\hat{u}_1$.

1: **Solve:** Compute smallest eigenvalue $\tilde{\lambda}_H$ and associated eigenvector $\tilde{\mathbf{u}}_H$ for algebraic evp on the coarse mesh $\mathcal{T}_H^i$. Terminte Arnoldi after p steps or when a desired tolerance *tol* is reached.

2: Express $\tilde{\mathbf{u}}_H$ using mesh $\mathcal{T}_h^i$ obtained by uniformly refining $\mathcal{T}_H^i$. Use prolongation P from mesh $\mathcal{T}_H^i$ to mesh $\mathcal{T}_h^i$, compute $\hat{\mathbf{u}}_h = P\tilde{\mathbf{u}}_H$.

3: **Estimate:** Determine residual $\hat{\mathbf{r}}_h = A_h\hat{\mathbf{u}}_h - \hat{\lambda}_h B_h\hat{\mathbf{u}}_h$ for associated ev $\hat{\mathbf{u}}_h$, identify large coeff. in $\hat{\mathbf{r}}_h$ and ass. (nodes).

4: **if** $\|\hat{\mathbf{r}}_h\| < \epsilon$ **then**

5: **return** $(\hat{\lambda}_h, \hat{\mathbf{u}}_h)$

6: **else**

7: **Mark:** Mark all edges of identif. nodes, apply closure algorithm.

8: **Refine:** Refine coarse mesh $\mathcal{T}_H^i$ to get $\mathcal{T}_H^{i+1}$.

9: Start Algorithm 1 with $\mathcal{T}_H^{i+1}$.

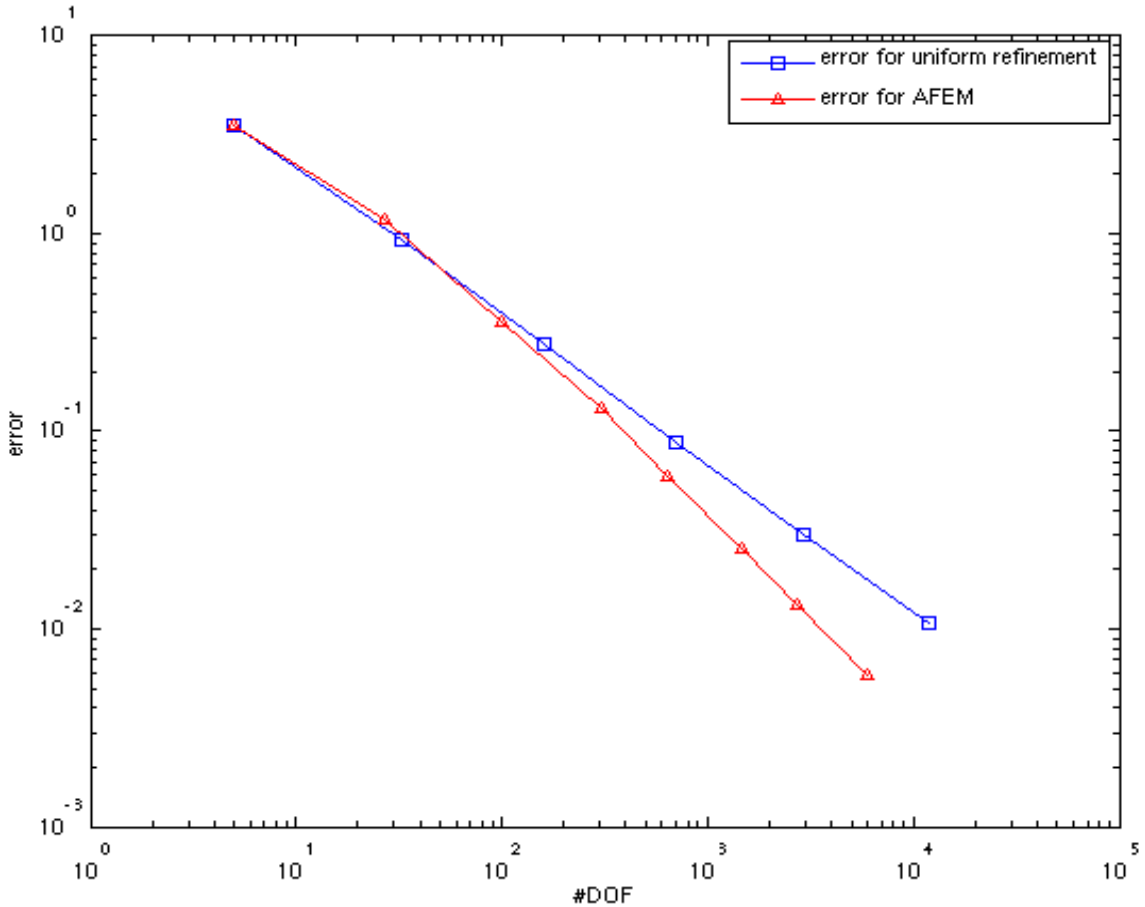
Computable bounds

- $\tilde{\lambda_H}$: computed ev on coarse grid.
- $\mathbf{r_H}$: Residual of eigenvector on coarse grid.
- $\hat{\mathbf{r}}_H, \hat{\mathbf{u}}_H$: Prolongated residual, eigenvector on fine grid.
- P Prolongation matrix, B_h fine mass matrix, β saturation constant.

Theorem

$$
\begin{aligned}
& |\tilde{\lambda}_H - \lambda| \\
\leq \quad & \frac{1}{1-\beta}\left(\|\mathbf{r}_H\|_2\|B_H^{-1}\|_2 + \frac{\|\mathbf{r}_H\|_2 + \|P^T\|_2\|\hat{\mathbf{r}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2} + \|\hat{\mathbf{r}}_h\|_2\|B_h^{-1}\|_2\right) \\
+ \quad & \|\mathbf{r}_H\|_2\|B_H^{-1}\|_2.
\end{aligned}
$$

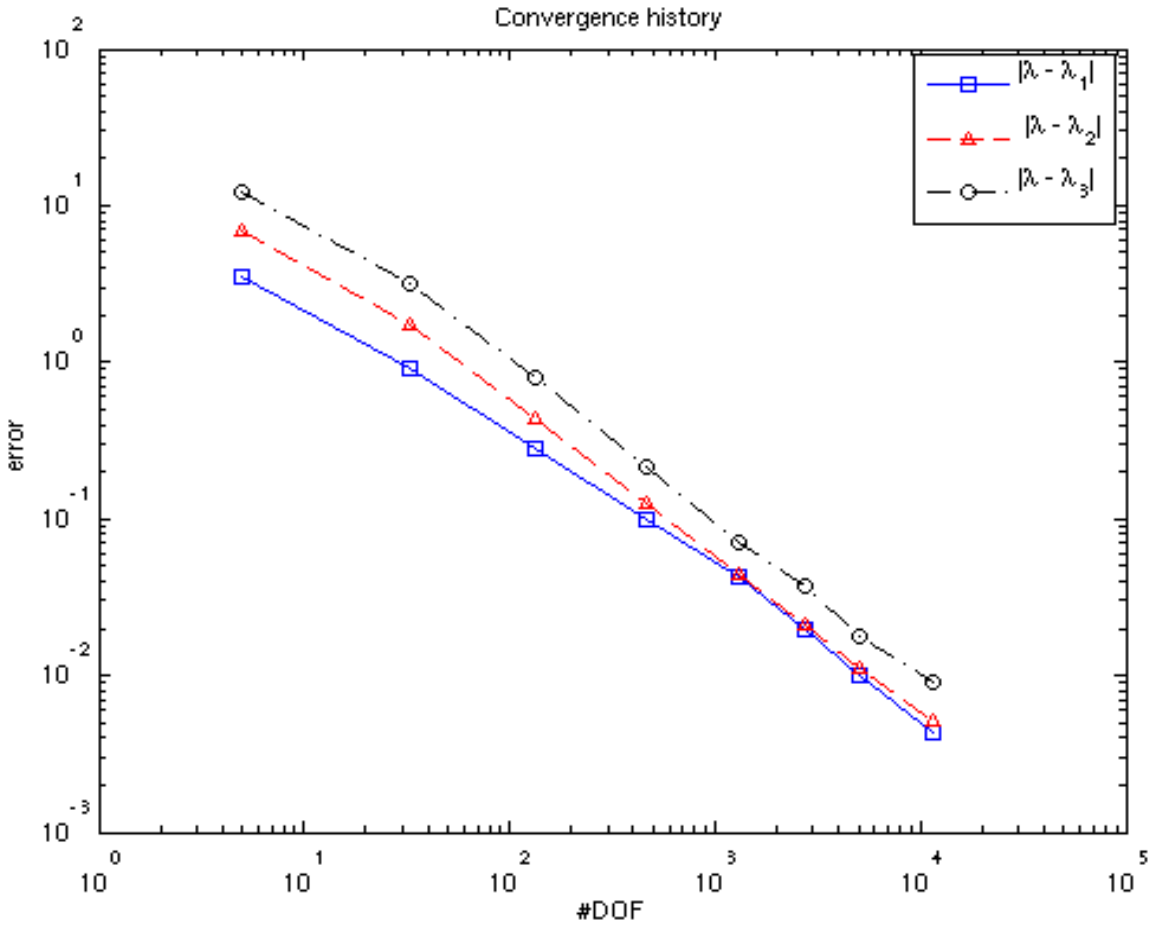
Convergence on L-shape domain.



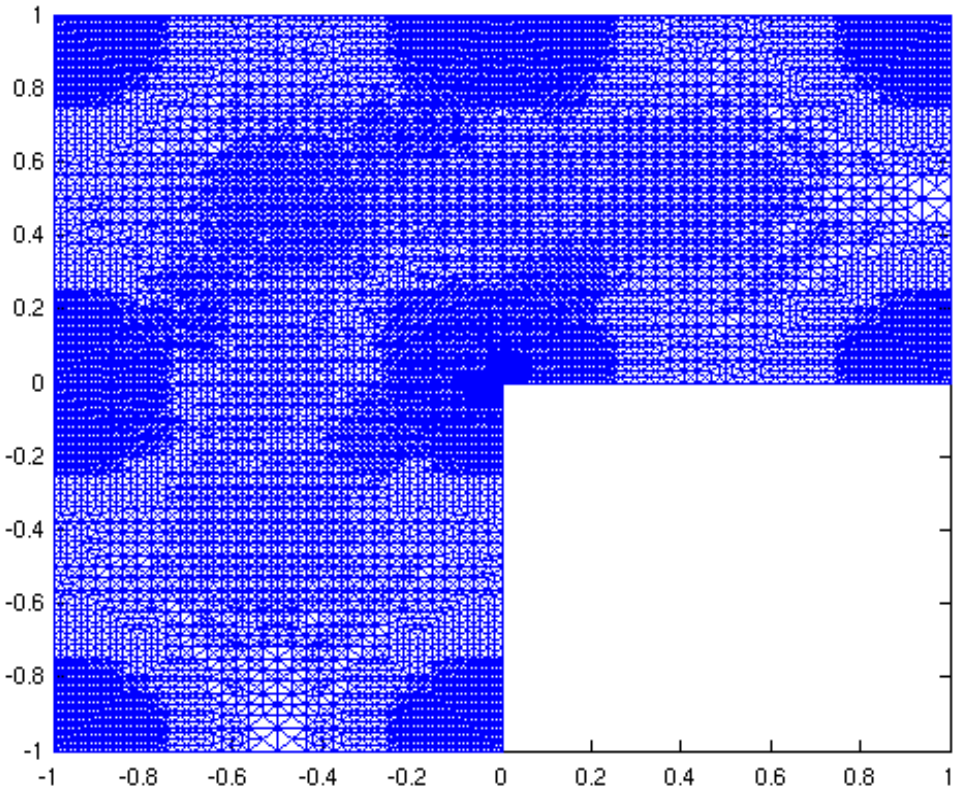
Approx. of smallest ev

ref. level	#DOF	$\tilde{\lambda}_1$	$\lvert\lambda_1 - \tilde{\lambda}_1\rvert$
1	5	13.1992	3.5595
2	27	10.8173	1.1775
3	99	9.9982	0.3584
4	306	9.7721	0.1323
5	641	9.6982	0.0585
6	1461	9.6652	0.0255
7	2745	9.6528	0.0131
8	5961	9.6455	0.0058

Conv. first 3 evs, L-shape domain.



Adaptive Mesh, first 3 evs



Outline

A simple nonsymmetric model problem

Carstensen/Gedicke/M./Miedlar 2009

Convection-diffusion eigenvalue problem:

$$-\Delta u + \beta \cdot \nabla u = \lambda u \text{ in } \Omega \qquad \text{and} \qquad u = 0 \text{ on } \partial\Omega$$

Discrete weak primal and dual problem:

$$\begin{aligned} a(u_\ell, v_\ell) + c(u_\ell, v_\ell) &= \lambda_\ell b(u_\ell, v_\ell) \quad \text{for all } v_\ell \in V_\ell, \\ a(w_\ell, u_\ell^\star) + c(w_\ell, u_\ell^\star) &= \overline{\lambda_\ell^\star} b(w_\ell, u_\ell^\star) \quad \text{for all } w_\ell \in V_\ell. \end{aligned}$$

Generalized algebraic eigenvalue problem:

$$(A_\ell + C_\ell)\mathbf{u}_\ell = \lambda_\ell B_\ell \mathbf{u}_\ell \qquad \text{and} \qquad \mathbf{u}_\ell^\star (A_\ell + C_\ell) = \lambda_\ell^\star \mathbf{u}_\ell^\star B_\ell$$

The eigenvalue with the smallest real part, which is proved to be simple and well separated Evans '00, is considered.

$$\mathcal{H}(t) = (1-t)\mathcal{L}_0 + t\mathcal{L}_1 \quad \text{for } t \in [0,1],$$

where $\mathcal{L}_0 u := -\Delta u$ and $\mathcal{L}_1 u := -\Delta u + \beta \cdot \nabla u$. Discrete homotopy for the model eigenvalue problem:

$$\mathcal{H}_\ell(t) = (A_\ell + C_\ell)(t) = (1-t)A_\ell + t(A_\ell + C_\ell) = A_\ell + tC_\ell.$$

Four errors

Homotopy, discretization, approximation and iteration error.

Homotopy error:

$$|\lambda(1) - \lambda(t)| \lesssim (1-t)\|\beta L^\infty(\Omega)\|\|u\|_A = \nu,$$

Discretization error:

$$\|\lambda(t) - \lambda_\ell(t)\| \lesssim \sum_{T\in\mathcal{T}_\ell} \left(\eta_\ell^2(T) + \eta_\ell^{\star 2}(T)\right).$$

Approximation error:

$$|\lambda_\ell(t) - \tilde{\lambda}_\ell(t)| + |\lambda_\ell^\star(t) - \tilde{\lambda}_\ell^\star(t)| \leq \mu_\ell.$$

Iteration error The iterative eigensolver can be stopped when the error is on the order of the others.

Lemma

Suppose that $|\lambda_\ell(t) - \tilde{\lambda}_\ell(t)| < 1$. *Then, for a fixed* $0 \leq t \leq 1$, *the perturbation of the a posteriori error estimator for the discretization error satisfies*

$$\|\eta(\lambda_\ell(t), u_\ell(t), u_\ell^\star(t)) - \eta(\tilde{\lambda}_\ell(t), \tilde{u}_\ell(t), \tilde{u}_\ell^\star(t))\|^2 \lesssim \mu^2(\tilde{\lambda}_\ell(t), \tilde{u}_\ell(t), \tilde{u}_\ell^\star(t))$$

Lemma

For the model problem, the difference between the exact eigenvalues $\lambda(t)$ of the homotopy $\mathcal{H}(t)$ and $\lambda(1)$ can be estimated via

$$\|\lambda(1) - \lambda(t)\| \lesssim \nu(t) := (1-t)\|\beta_\infty\| \left(|||u(t)||| + |||u^\star(t)|||\right)$$

for $0 \leq t \leq 1$. The constant in the inequality tends to $1/(2b(u(1), u^\star(1))$ as $t \to 1$.

Lemma

For the model problem, the difference between the iterative eigenvalue $\tilde{\lambda}_\ell(t)$ in the homotopy $\mathcal{H}_\ell(t)$ and the continuous eigenvalue $\lambda(1)$ of the original problem can be estimated a posteriori via

$$\begin{aligned} \|\lambda(1) - \tilde{\lambda}_\ell(t)\| &\lesssim \nu(\tilde{\lambda}_\ell(t), \tilde{u}_\ell(t), \tilde{u}^\star_\ell(t)) + \eta^2(\tilde{\lambda}_\ell(t), \tilde{u}_\ell(t), \tilde{u}^\star_\ell(t)) \\ &+ \mu^2(\tilde{\lambda}_\ell(t), \tilde{u}_\ell(t), \tilde{u}^\star_\ell(t)) \end{aligned}$$

in terms of

$$\begin{aligned} &\nu(\tilde{\lambda}_\ell(t), \tilde{u}_\ell(t), \tilde{u}^\star_\ell(t)) := (1-t)|\beta|_\infty \left(|||\tilde{u}_\ell(t)||| + |||\tilde{u}^\star_\ell(t)|||\right) \\ + \quad &(1-t)|\beta|_\infty \left(\eta(\tilde{\lambda}_\ell(t), \tilde{u}_\ell(t), \tilde{u}^\star_\ell(t)) + \mu(\tilde{\lambda}_\ell(t), \tilde{u}_\ell(t), \tilde{u}^\star_\ell(t))\right). \end{aligned}$$

Adaptive homotopy algorithms

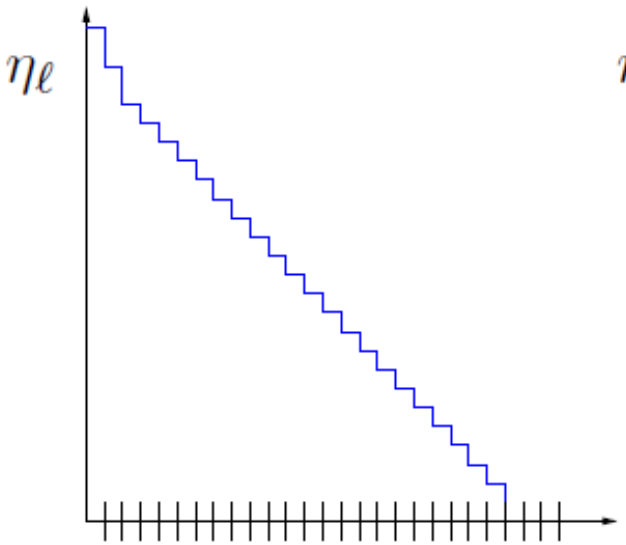
Solve $\rightarrow$ Estimate $\rightarrow$ Mark $\rightarrow$ Refine

Algorithm 1: Balances the homotopy, discretization, iteration and approximation errors but uses fixed stepsize in continuation method.

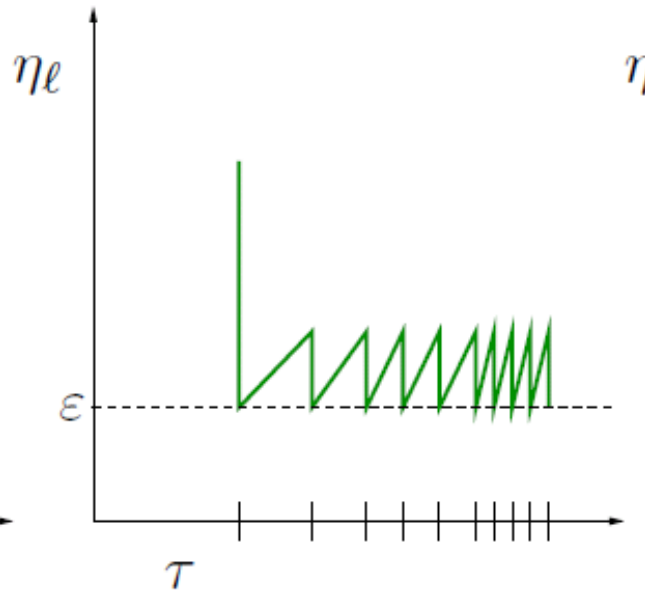
Algorithm 2: Adaptivity in homotopy and in the iteration is achieved by simple stepsize control, no homotopy error is considered.

Algorithm 3: Adaptivity in the homotopy error, the discretization error, the iteration error including step size control.

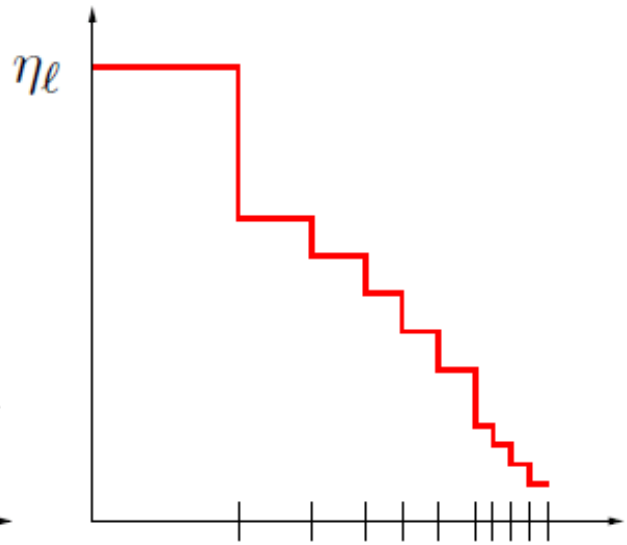
Error dynamics



Algorithm 1



Algorithm 2



Algorithm 3

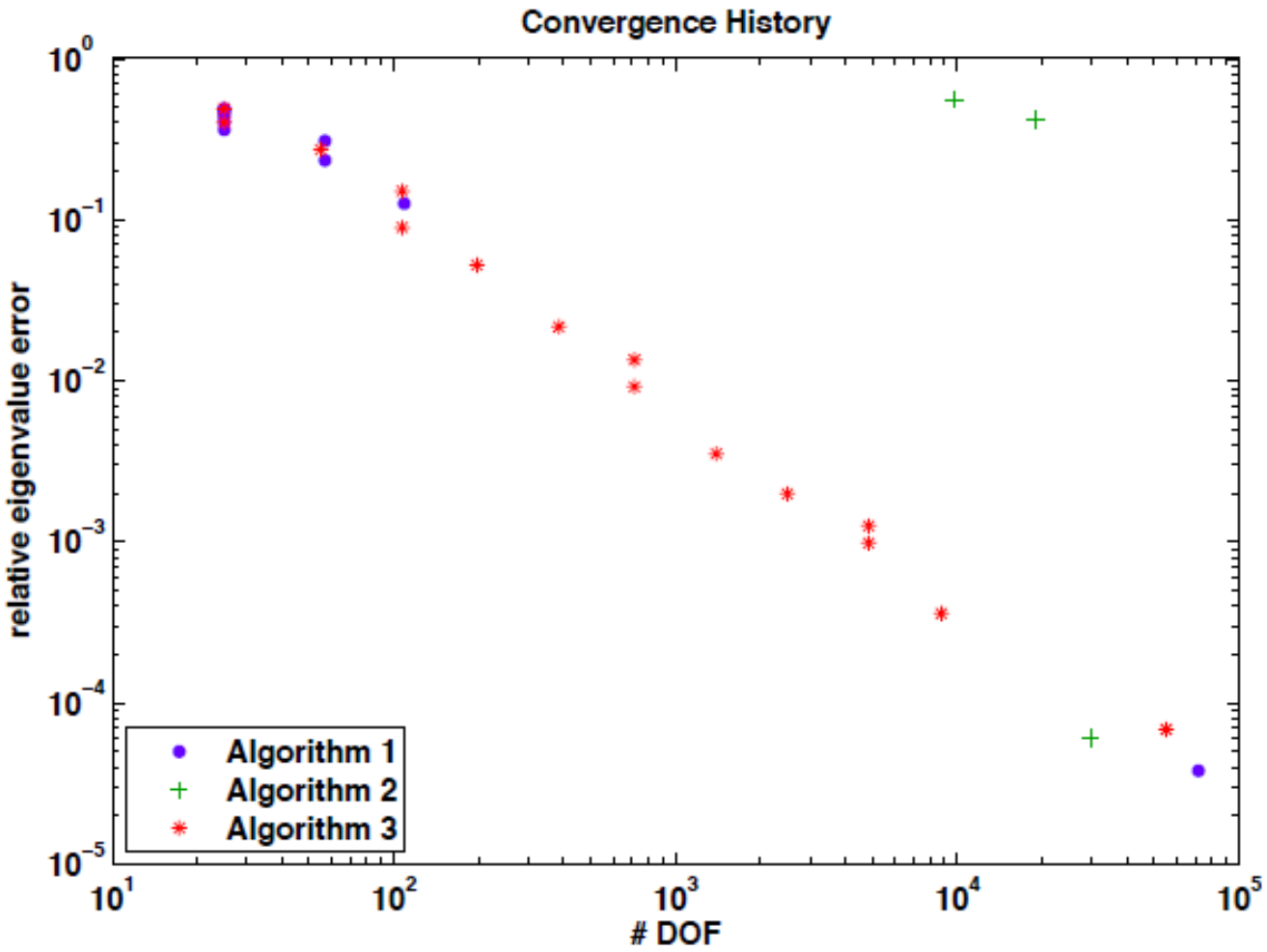


Figure: Conv. history of Algorithm 1, 2 and 3 with respect to #DOF.

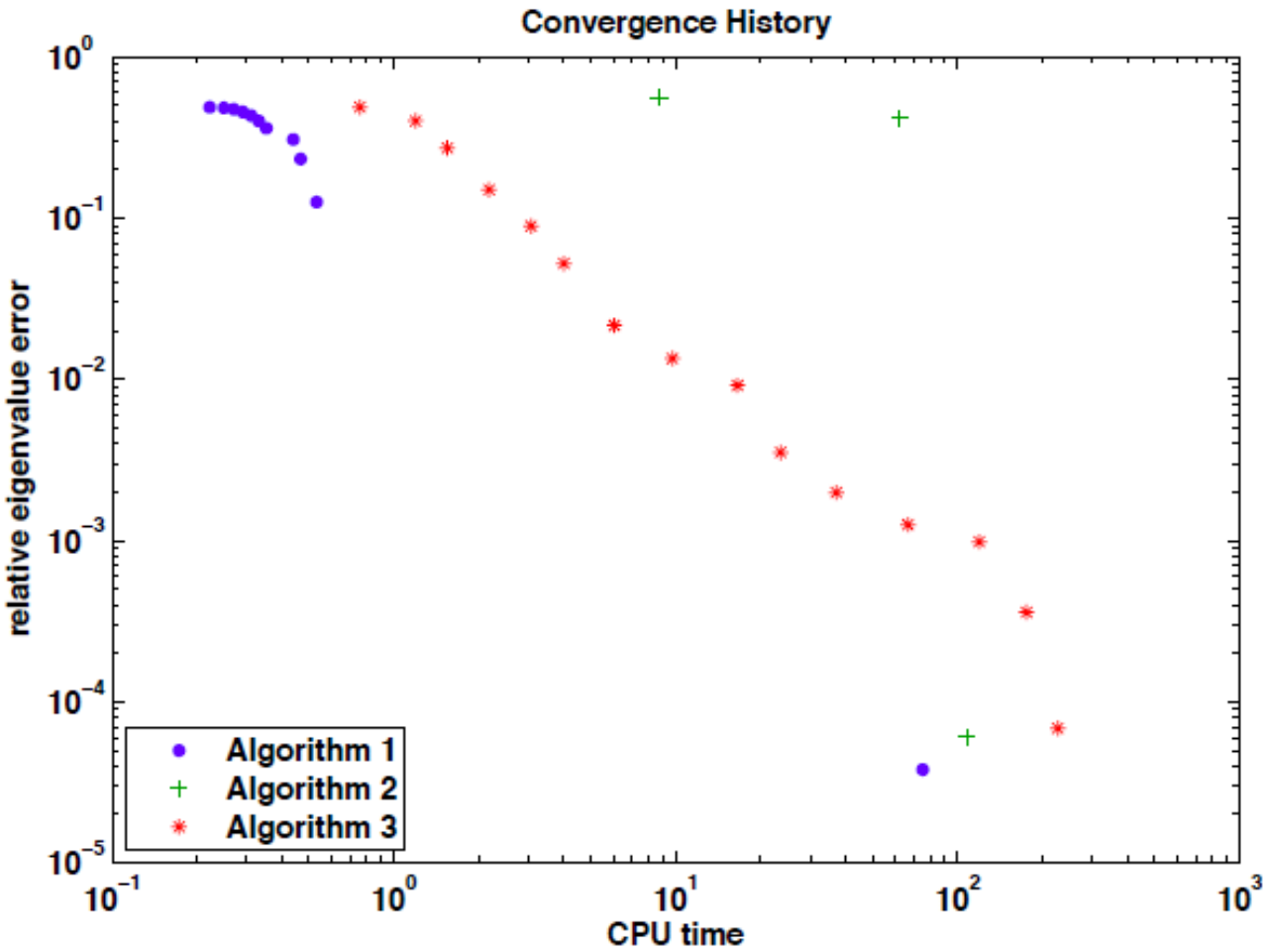


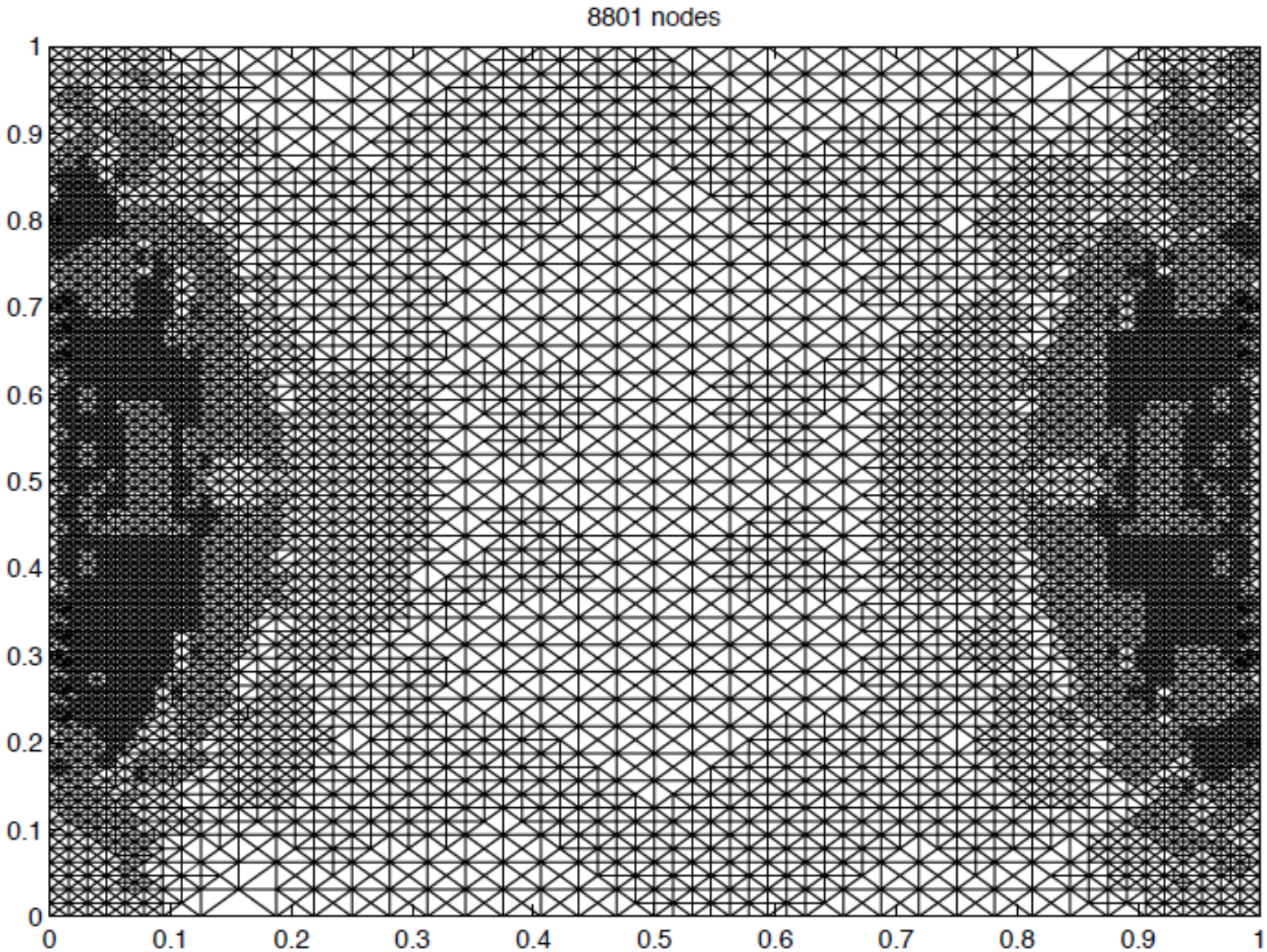
Figure: Conv. history of Algorithm 1, 2 and 3 with respect to CPU time.

t	$\eta_\ell(t)$	$\nu_\ell(t)$	$\mu_\ell(t)$	est. error
0.0000	23.0271	95.6366	0.2701265	118.93382
0.5000	32.6896	51.7112	0.0843690	84.48512
0.7500	11.6020	28.5244	0.4515713	40.57800
0.8750	6.7380	15.4099	0.4711298	22.61912
0.9375	7.8500	7.9782	0.0272551	15.85547
0.9688	3.2088	4.0697	0.2891100	7.56762
0.9844	1.2060	2.0673	0.4278706	3.70119
0.9922	0.4560	1.0380	0.0004539	1.49451
0.9961	0.4602	0.5202	0.0029006	0.98322
0.9980	0.1864	0.2608	0.0012530	0.44843
0.9990	0.0707	0.1305	0.0204610	0.22162
0.9995	0.0282	0.0653	0.0003639	0.09386
0.9998	0.0282	0.0326	0.0001766	0.06105
0.9999	0.0106	0.0163	0.0001521	0.02703
1.0000	0.0007	0.0000	0.0000243	0.00073

Errors and DOFs

t	$\tilde{\lambda}_\ell(t)$	$\frac{\lvert\lambda_\ell(1)-\tilde{\lambda}_\ell(t)\rvert}{\lvert\lambda_\ell(1)\rvert}$	#DOF	CPU time
0.0000	22.86578	0.48891	25	0.76
0.5000	26.73866	0.40234	25	1.20
0.7500	32.54928	0.27247	55	1.55
0.8750	38.00079	0.15062	107	2.18
0.9375	40.73818	0.08943	107	3.07
0.9688	42.39339	0.05243	197	4.01
0.9844	43.77023	0.02166	385	6.06
0.9922	44.13547	0.01349	715	9.74
0.9961	44.32847	0.00918	715	16.59
0.9980	44.58151	0.00352	1398	23.57
0.9990	44.65025	0.00199	2494	37.14
0.9995	44.68298	0.00126	4848	66.70
0.9998	44.69522	0.00098	4848	119.47
0.9999	44.72311	0.00036	8785	175.75
1.0000	44.73615	0.00007	55235	226.87

Final mesh



What about the SFE problem ?

- Today there is no easy way to get adaptivity into the code.
- There is an urgent need to combine discrete FEM modelling and adaptivity.
- We need methods for nonlinear PDE eigenvalue problems within optimization loop.
- The mathematical theory and algorithms are still far from the needs in reality.
- Commercially available codes are not satisfactory.
- Homotopy and Newton like methods need to be developed and analyze.
- Multi-way adaptive methods need to be developed, studied and made industrially available.

References

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- C. Carstensen, J. Gedicke, V. M., and A. Międlar, *An adaptive homotopy approach for non-selfadjoint eigenvalue problems* PREPRINT 718, DFG Research Center MATHEON. In revision in NUMERISCHE MATHEMATIK.
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Thank you very much
for your attention.