Constraint Satisfaction with Weakly Oligomorphic Template

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Abstract

Constraint satisfaction problems form a very interesting and much studied class of decision problems. Feder and Vardi realized their relation to general coloring problems of relational structure. This enabled the use of algebraic, combinatorial, and model theoretic methods for studying the complexity of such decision problems.

In this paper we are interested in constraint satisfaction problems with countable homomorphism homogeneous template and, more generally, with weakly oligomorphic templates.

A first result is a Fraïssé-type theorem for homomorphism homogeneous relational structures.

Further we show the existence and uniqueness of homogeneous, homomorphism-homogeneous cores in weakly oligomorphic homomorphism homogeneous structures. A consequence of this result is that every constraint satisfaction problem with weakly oligomorphic template is equivalent to a problem with finite or ω-categorical template.

Another result is the characterization of positive existential theories of weakly oligomorphic structures as the positive existential parts of ω-categorical theories — akin to the Engeler-Ryll-Nardzewski-characterization of the theories of oligomorphic structures.

1 Introduction

Constraint satisfaction problems (CSPs, for short) cover a wide range of decision problems that arise in artificial intelligence, operations research, and combinatorics. Standard examples include boolean k-SAT, or the decision problem whether a graph is k-colorable.

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While it is difficult, to give a general precise mathematical definition of what is a constraint satisfaction problem, many of the considered CSPs fit into the following framework (introduced by Feder and Vardi in [13]):

> Let $\mathbf{T}$ be a relational structure (over a given relational signature Σ). Then the constraint satisfaction problem for the template $\mathbf{T}$ is the following computational problem:
>
> CSP($\mathbf{T}$)
>
> **Instance:** A finite relational structure $\mathbf{A}$ over the signature Σ.
>
> **Problem:** Is there a homomorphism from $\mathbf{A}$ to $\mathbf{T}$?

If $\mathbf{T}$ is finite over a finite signature, then CSP($\mathbf{T}$) is called a finite constraint satisfaction problem. Clearly, finite CSPs are in NP. It is conjectured that every finite constraint satisfaction problem is either in P or it is NP-complete. This dychotomy-conjecture (originally stated by Feder and Vardi in [13]) is very intensively studied in theoretical computer science and in universal algebra. It is impossible to give a reasonable bibliographic overview of this topic but a few important step-stones are [21, 15, 9, 10, 8, 2, 1, 4].

On the other hand, there are a number of constraint satisfaction problems arising in literature that can not be modeled using a finite template. Very natural examples were uncovered in [5, 7, 6]. They include, e.g., graph acyclicity, Allen's interval algebra, and tree-description languages arising in computational linguistics. It turns out that many of these problems can be formalized as constraint satisfaction problems with a countable, ω-categorical template, or even with homogeneous, ω-categorical template. For this reason, it seems to be quite natural to study CSPs with ω-categorical templates (of all countable templates). Another benefit of such an approach is that ω-categorical structures have a well developed and strong structural theory rooting deeply in model theory and group theory (cf. [11, 16]).

In this paper we widen the class of allowed templates to the class of weakly oligomorphic relational structure. The notion of weak oligomorphy was introduced in [18] for studying the unary parts of local clones. It soon turned out that this notion is intimately related to the notion of homomorphism homogeneity (in the sense of [12]) and quantifier elimination of positive existential formulæ, much similar, like ω-categoricity is related to homogeneity and to quantifier-elimination (cf. [17],[19]). At the same time, weak oligomorphy is a much weaker notion than ω-categoricity. E.g., any reduct of a homomorphism homogeneous structure over a finite signature will be weakly oligomorphic, and there exist homomorphism-homogeneous graphs that have a trivial automorphism group (cf. [12, Cor. 2.2]). Therefore allowing weakly oligomorphic templates gives allot more freedom for the formalization of concrete constraint satisfaction problems.

One of the main results of this paper is to show that on the other hand every constraint satisfaction problem that can be formalized using a weakly oligomorphic template, in fact can also be formalized using an ω-categorical template. This strengthens the feeling that the (at the first sight arbitrary) restriction to ω-categorical templates is indeed very natural.

In order to be able to prove the above mentioned reduction result, we develop to a certain point the basic model theory of weakly oligomorphic relational structures.

In Section 3, we introduce the notion of weak oligomorphy and we show that from an algebraic point of view, this is the weakest reasonable relaxation of the notion of ω-categoricity.

In Section 4 we give a characterization of the ages of homomorphism homogeneous structures in the vein of Fraïssé's Theorem by showing that a class of finite structures is the age of a homomorphism homogeneous structure if and only if it is a hom-amalgamation class.

In Section 5 we study, when there exist homomorphisms between weakly oligomorphic structures. In particular, we show that the existence of a homomorphism between countable weakly oligomorphic structures depends only of a relation between their ages.

In Section 6 we show that every countable weakly oligomorphic homomorphism homogeneous structure has a unique (up to isomorphism) hom-equivalent substructure that is oligomorphic, homogeneous, and a core. A consequence is that every CSP with weakly oligomorphic, homomorphism homogeneous template is equivalent to a CSP with oligomorphic, homogeneous template.

In Section 7 we show that every countable weakly oligomorphic structure is homomorphism-equivalent to a finite or ω-categorical structure. From this result it follows at once that every CSP with weakly oligomorphic template is equivalent to a CSP with finite or with ω-categorical template.

In Section 8, as a model theoretic result, the positive existential theories of countable weakly oligomorphic structures are characterized as the positive existential parts of ω-categorical theories. This result has the flavor of the Engeler-Ryll-Nardzewski-Svenonius Theorem that characterizes the theories of countable oligomorphic structures as precisely the ω-categorical theories.

2 Preliminaries

The object of study in this paper are relational structures. As a basis for our notions and notations we use the model theoretic book [16] (which is a standard in this field). A relational signature is a model-theoretic signature without constant- and function symbols. A model over a relational signature will be called a *relational structure*. Note, that throughout this paper we make no other assumptions about the signatures. In particular, if not stated otherwise, we allow signatures of any cardinality. Relational structures will be denoted by bold capital letters $\mathbf{A}$, $\mathbf{B}$, etc., while their carriers will be denoted by usual capital letters A, B, etc., respectively.

As usual, a homomorphism between two relational structures is a function between the carriers that preserves all relations. We will use the notation $\mathbf{A} \to \mathbf{B}$ as a way to say that there exists a homomorphism from $\mathbf{A}$ to $\mathbf{B}$. If $\mathbf{A} \to \mathbf{B}$ and $\mathbf{B} \to \mathbf{A}$, then we call $\mathbf{A}$ and $\mathbf{B}$ *hom-equivalent*. It is easy to see, that hom-equivalent relational structures define equivalent constraint satisfaction problems.

If $f : \mathbf{A} \to \mathbf{B}$, then we call $\mathbf{A}$ the *domain* of f (denoted by $\operatorname{dom}(f)$, and $\mathbf{B}$, the *codomain* (denoted by $\operatorname{cdom}(f)$. Moreover, the structure induced by $f(A)$ is called the *image* of f (denoted by $\operatorname{img}(f)$).

Epimorphisms are surjective homomorphisms and *monomorphisms* are injective homomorphisms. Isomorphisms are bijective homomorphisms whose inverse is a homomorphism, too. *Embeddings* are monomorphisms that not only preserve relations but also reflect them. That is, a monomorphism is an embedding if and only if it is an isomorphism to its image.

As a final note, in this paper under a countable set we understand a finite or countably infinite set.

3 Weakly oligomorphic structures

In [11], Peter Cameron introduced the notion of oligomorphic permutation groups. Recall that a permutation group is called *oligomorphic* if it has just finitely many orbits in its action on n-tuples for any n. A structure $\mathbf{A}$ is called oligomorphic if its automorphism group is oligomorphic.

Before coming to the definition of weakly oligomorphic structures, we have to recall some model theoretic notions: Let Σ be a relational signature, and let $L(\Sigma)$ be the language of first order logics with respect to Σ. Let $\mathbf{A}$ be a Σ-structure. For a formula $\varphi(\bar{x})$ (where $\bar{x} = (x_1, \dots, x_n)$), we define $\varphi^{\mathbf{A}} \subseteq A^n$ as the set of all n-tuples $\bar{a}$ over A such that $\mathbf{A} \models \varphi(\bar{a})$. More generally, for a set Φ of formulæ from L with free variables from $\{x_1, \dots, x_n\}$, we define $\Phi^{\mathbf{A}}$ as the intersection of all relations $\varphi^{\mathbf{A}}$ where φ ranges through Φ. We call Φ a *type*, and $\Phi^{\mathbf{A}}$ the relation defined by Φ in $\mathbf{A}$.

If $\Phi^{\mathbf{A}} \neq \emptyset$, then we say that $\mathbf{A}$ realizes Φ. We call Φ positive existential, or positive primitive, if it consists just of positive existential, or positive primitive formulæ, respectively.

For a relation $\varrho \subseteq A^n$ by $\operatorname{Tp}_{\mathbf{A}}(\varrho)$ we denote the set of all formulæ $\varphi(\bar{x})$ such that $\varrho \subseteq \varphi^{\mathbf{A}}$. This is the type defined by ϱ with respect to $\mathbf{A}$. Analogously, the positive existential type $\operatorname{pTp}_{\mathbf{A}}(\varrho)$ and the positive primitive type $\operatorname{ppTp}_{\mathbf{A}}(\varrho)$ are defined. With $\operatorname{Tp}^{(0)}_{\mathbf{A}}(\varrho)$, $\operatorname{pTp}^{(0)}_{\mathbf{A}}(\varrho)$, and $\operatorname{ppTp}^{(0)}_{\mathbf{A}}(\varrho)$ the respective quantifier free parts of $\operatorname{Tp}_{\mathbf{A}}(\varrho)$, $\operatorname{pTp}_{\mathbf{A}}(\varrho)$, and $\operatorname{ppTp}_{\mathbf{A}}(\varrho)$ are denoted.

Let us come now to the definition of the structures under consideration in this paper.

Definition ([18, 19])**.** A relational structure $\mathbf{A}$ is called *weakly oligomorphic* if for every arity there are just finitely many relations that can be defined by positive existential types.

One can argue that it would be more appropriate to define a structure $\mathbf{A}$ to be weakly oligomorphic if its endomorphism monoid is *oligomorphic* (i.e. there are just finitely many invariant relations of $\operatorname{End}(\mathbf{A})$ of any arity). However, there is no need to worry, since, at least for countable structures, these two definitions are equivalent:

Proposition 3.1 ([19, Thm. 6.3.4],cf. [18, Prop. 2.2.5.1])**.** *A countable structure $\mathbf{A}$ is weakly oligomorphic if and only if* $\operatorname{End}(\mathbf{A})$ *is oligomorphic.*

Clearly, if a structure is oligomorphic, then it is also weakly oligomorphic.

In principle, it is possible to introduce an even weaker notion of weak oligomorphy. Recall that *an n-ary polymorphism* of a structure $\mathbf{A}$ is a homomorphism from $\mathbf{A}^n$ to $\mathbf{A}$. The collection of all polymorphisms of $\mathbf{A}$ forms a clone (cf. [20, 22]) and is denoted by $\operatorname{Pol}(\mathbf{A})$. We could call $\operatorname{Pol}(\mathbf{A})$ oligomorphic if it has of every arity just finitely many invariant relations. However, the following Proposition will show that (at least in the countable case) this does not lead to a strictly weaker notion.

Proposition 3.2. *Let* $\mathbf{A}$ *be a countable relational structure. Then the following are equivalent:*

(a) $\mathrm{Pol}(\mathbf{A})$ *has of every arity just finitely many invariant relations,*

(b) for every arity there are just finitely many relations that can be defined by positive primitive types,

(c) $\mathbf{A}$ *is weakly oligomorphic.*

Proof. ((a)⇒(b)) Every relation definable by a set of positive primitive formulæ is invariant under $\mathrm{Pol}(\mathbf{A})$. Hence, the set of definable relations of a given arity is a subset of the set of invariant relations of this arity.

((b)⇒(c) Every positive existential formula is equivalent to a finite disjunction of positive primitive formulæ. Thus, the set of relations of arity n definable by positive existential formulæ is obtained from the set of relations definable by positive primitive formulæ by closing the latter one with respect to finite unions. If there are just finitely many relations definable by sets of positive primitive formulæ, then the closure with respect to finite unions will not produce an infinite set of relations.

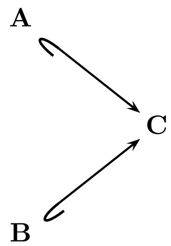
((c)⇒(a)) From Proposition 3.1, it follows that $\mathrm{End}(\mathbf{A})$ is oligomorphic. However, $\mathrm{End}(\mathbf{A}) \subset \mathrm{Pol}(\mathbf{A})$. Hence the set of invariant relations of $\mathrm{Pol}(\mathbf{A})$ is contained in the set of invariant relations of $\mathrm{End}(\mathbf{A})$. This finishes the proof. □

4 Homomorphism-homogeneity and amalgamation

Recall that *the age of a structure* is the class of finite structures embeddable into it.

Definition. Let $\mathcal{C}$ be a class of finite relational structures over the same signature. We say that $\mathcal{C}$ has the

Joint embedding property (JEP) if $\mathbf{A}, \mathbf{B} \in \mathcal{C}$, then there exists a $\mathbf{C} \in \mathcal{C}$ such that both $\mathbf{A}$ and $\mathbf{B}$ are embeddable in $\mathbf{C}$:



Hereditary property (HP) if $\mathbf{A} \in \mathcal{C}$, and $\mathbf{B} < \mathbf{A}$, then $\mathbf{B}$ is isomorphic to some $\mathbf{C} \in \mathcal{C}$,

Amalgamation property (AP) If $\mathbf{A}, \mathbf{B_1}, \mathbf{B_2} \in \mathcal{C}$, and $f_1 : \mathbf{A} \to \mathbf{B_1}$ and $f_2 : \mathbf{A} \to \mathbf{B_2}$ are embeddings, then there are $\mathbf{C} \in \mathcal{C}$, and embeddings

$g_1 : \mathbf{B_1} \to \mathbf{C}$ and $g_2 : \mathbf{B_2} \to \mathcal{C}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{A} & \overset{f_1}{\hookrightarrow} & \mathbf{B_1} \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1} \\ \mathbf{B_2} & \overset{g_2}{\hookrightarrow} & \mathbf{C} \end{array}$$

i.e. $g_1 \circ f_1 = g_2 \circ f_2$.

Homo-amalgamation property (HAP) If $\mathbf{A}, \mathbf{B_1}, \mathbf{B_2} \in \mathcal{C}$, $f_1 : \mathbf{A} \to \mathbf{B_1}$ is a homomorphism, and $f_2 : \mathbf{A} \to \mathbf{B_2}$ is an embedding, then there are $\mathbf{C} \in \mathcal{C}$, an embedding $g_1 : \mathbf{B_1} \to \mathbf{C}$, and a homomorphism $g_2 : \mathbf{B_2} \to \mathcal{C}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{f_1} & \mathbf{B_1} \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1} \\ \mathbf{B_2} & \xrightarrow{g_2} & \mathbf{C} \end{array}$$

i.e. $g_1 \circ f_1 = g_2 \circ f_2$.

A basic theorem by Roland Fraïssé states that a class of finite structures of the same type is the age of a countable structure if and only if

1. it has only countably many isomorphism types,
2. it is isomorphism-closed,
3. it has properties HP and JEP.

Another central result of Fraïssé is the characterization of the ages of homogeneous structures. We quote the formulation due to Cameron (cf. [11, (2.12-13)]):

Theorem 4.1 (Fraïssé ([14]))**.** *A class $\mathcal{C}$ of finite relational structures is the age of some countable homogeneous relational structure if and only if*

(i) it is closed under isomorphism,

(ii) it has only countably many non-isomorphic members,

(iii) it has properties HP and AP.

Moreover, any two countable homogeneous relational structures with the same age are isomorphic. □

A class of finite structures over the same signature, that is isomorphism-closed and that has properties HP, and AP, is called *Fraïssé-class.*

In [12], Peter Cameron and Jaroslav Nešetřil introduced the notion of homomorphism-homogeneous structure. *A local homomorphism* of a structure $\mathbf{A}$ is a homomorphism from a finite substructure of $\mathbf{A}$ to $\mathbf{A}$.

Definition (Cameron, Nešetřil)**.** A structure $\mathbf{A}$ is called *homomorphism-homogeneous* if every local homomorphism of $\mathbf{A}$ can be extended to an endomorphism of $\mathbf{A}$.

As a straight forward adaptation of the notion of weakly homogeneous structure, it will be useful to introduce the notion of a weakly homomorphism-homogeneous structure:

Definition. A structure $\mathbf{A}$ is called *weakly homomorphism-homogeneous* if whenever $\mathbf{B} < \mathbf{C}$ are finite substructures of $\mathbf{A}$, then every homomorphism $f : \mathbf{B} \to \mathbf{A}$ extends to $\mathbf{C}$.

Clearly, a countable structure is weakly homomorphism-homogeneous if and only if it is homomorphism-homogeneous.

The concept of weak homomorphism homogeneity is closely related to other model theoretic notions, via the notion of weak oligomorphy.

Definition. A structure $\mathbf{A}$ is called *endolocal* if for any relation $\varrho \subseteq A^n$ we have that ϱ is definable by a positive existential type if and only if for all $\bar{a} \in \varrho$, $\bar{b} \in A^n$ holds that if $\operatorname{pTp}^{(0)}_{\mathbf{A}}(\bar{a}) \subseteq \operatorname{pTp}^{(0)}_{\mathbf{A}}(\bar{b})$, then $\bar{b} \in \varrho$.

Another, perhaps more transparent definition of endolocality is the following:

Lemma 4.2. *A structure $\mathbf{A}$ is endolocal if and only if the set of relations definable by positive existential types in $\mathbf{A}$ coincides with the set of relations that are invariant under local homomorphisms of $\mathbf{A}$.* □

The following characterization of weak homomorphism homogeneity for weakly oligomorphic structures will be of good help in later proofs. It first appeared in [18].

Proposition 4.3 ([19, Main Thm.])**.** *Let $\mathbf{A}$ be a weakly oligomorphic structure. Then the following are equivalent:*

(a) $\mathbf{A}$ *is homomorphism homogeneous,*

(b) $\mathbf{A}$ *is endolocal,*

(c) every positive existential formula in the language of $\mathbf{A}$ is equivalent in $\mathbf{A}$ to a quantifier free positive formula. □

We continue with an analogue of Fraïssé's Theorem for homomorphism-homogeneous structures:

Theorem 4.4. *(a) The age of any homomorphism-homogeneous structure has property HAP.*

(b) If a class $\mathcal{C}$ of finite relational structures is isomorphism-closed, has only a countable number of isomorphism classes, and has properties HP, JEP, and HAP, then there is a countable homomorphism-homogeneous structure $\mathbf{H}$ whose age is equal to $\mathcal{C}$.

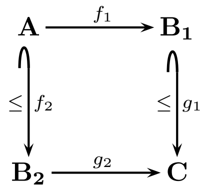
Proof. **About (a)** Let $\mathbf{H}$ be a homomorphism-homogeneous relational structure and let $\mathcal{C}$ be its age. Let $\mathbf{A}, \mathbf{B_1}, \mathbf{B_2} \in \mathcal{C}$, and let $f_1 : \mathbf{A} \to \mathbf{B_1}$ be a homomorphism and $f_2 : \mathbf{A} \to \mathbf{B_2}$ an embedding. Without loss of generality, we can assume that $\mathbf{A} \leq \mathbf{B_2}$, f_2 is the identical embedding, and $\mathbf{B_1}, \mathbf{B_2} \leq \mathbf{H}$.

Since f_1 is also a homomorphism from $\mathbf{A}$ to $\mathbf{H}$, and $\mathbf{H}$ is homomorphism-homogeneous, it follows that f_1 can be extended to a $g \in \operatorname{End}(\mathbf{H})$.

Further, we define

- $g_2 := g{\restriction}_{\mathbf{B_2}}$,
- $\mathbf{C} := \mathrm{img}(g_2) \cup \mathbf{B_1}$, and
- g_1 to be the identical embedding from $\mathbf{B_1}$ to $\mathbf{C}$.

Then we obtain the following commuting diagram



Indeed, for $a \in A$ we have

$$g_1(f_1(a)) = f_1(a) = g(a) = g_2(a) = g_2(f_2(a)),$$

so this diagram commutes, and hence the age of $\mathbf{H}$ has property HAP.

about (b) Our goal is to effectively construct a countable homomorphism-homogeneous structure $\mathbf{H}$ whose age is equal to $\mathcal{C}$.

Since $\mathcal{C}$ has a countable number of isomorphism classes, we can choose a representative from each class thus obtaining a countable set of structures. Denote this set by $\mathcal{R}$ and well-order $\mathcal{R}$ like ω.

We aim to construct a chain $\mathbf{H}_i$, $i \in \mathbb{N}$, of structures from $\mathcal{C}$ such that the following holds:

(I) If $\mathbf{A}, \mathbf{B} \in \mathcal{C}$, where $\mathbf{A} < \mathbf{B}$, then for each homomorphism $f : \mathbf{A} \to \mathbf{H}_i$, for some $i \in \mathbb{N}$, there are $j > i$ and a homomorphism $g : \mathbf{B} \to \mathbf{H}_j$ which extends f.

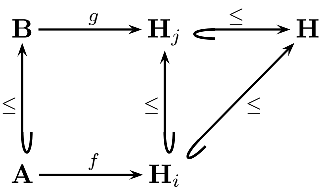
(II) For every $\mathbf{A} \in \mathcal{C}$ there exists an $i \in \mathbb{N}$ such that $\mathbf{A}$ is embeddable in $\mathbf{H}_i$.

We claim that the union $\mathbf{H} = \cup_{i\in\mathbb{N}} \mathbf{H}_i$ of such a chain of structures is going to be a countable homomorphism-homogeneous structure with age $\mathcal{C}$.

First of all, note that if for each $i \in \mathbb{N}$, the age of $\mathbf{H}_i$ is included in $\mathcal{C}$, then the age of $\mathbf{H} = \cup_{i\in\mathbb{N}} \mathbf{H}_i$ is also included in $\mathcal{C}$.

On the other hand, take some $\mathbf{A} \in \mathcal{C}$. Then from (II) it follows that $\mathbf{A}$ is embeddable into some $\mathbf{H}_i$, and, therefore, also into $\mathbf{H}$, showing that it is in the age of $\mathbf{H}$.

It is left to show, that $\mathbf{H}$ is weakly homomorphism-homogeneous. Let $\mathbf{A} < \mathbf{B}$ be a two finite substructures of $\mathbf{H}$, and let $f : \mathbf{A} \to \mathbf{H}$ be a local homomorphism. Then by (II) there exists some $i \in \mathbb{N}$ such that $f(\mathbf{A}) < \mathbf{H}_i$. Thus, by (I) there is a $j > i$ and a homomorphism $g : \mathbf{B} \to \mathbf{H}_j$ that extends f, i.e. the following diagram commutes:



This implies that **H** is weakly homomorphism-homogeneous.

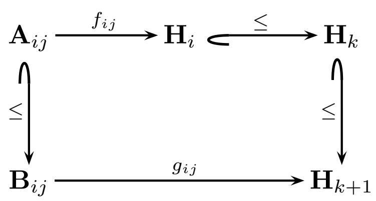
To conclude, the structure **H**, which is the union of the chain of structures fulfilling conditions (I) and (II), is weakly homomorphism-homogeneous (and thus, since it is countable, also homomorphism-homogeneous), and it has $\mathcal{C}$ as its age.

It remains to construct the chain. In addition to the above defined set $\mathcal{R}$, we define set $\mathcal{P}$ of pairs of structures $(\mathbf{A}, \mathbf{B})$ such that $\mathbf{A}, \mathbf{B} \in \mathcal{C}$, and $\mathbf{A} \leq \mathbf{B}$. We choose $\mathcal{P}$ so that it contains representatives from each isomorphism class of such pairs. Hence, $\mathcal{P}$ is a countable set. Now choose a bijection $\pi : 2\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for every $i \in 2\mathbb{N}, j \in \mathbb{N}$ holds $\pi(i, j) \geq i$. Then the construction goes by induction as follows:

Take the first structure from $\mathcal{R}$ and denote it by $\mathbf{H}_0$.

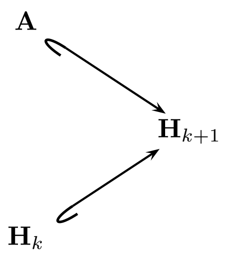
Step 1 Suppose that we have constructed $\mathbf{H}_k$, for k even. We proceed as follows: For given $\mathbf{H}_k$ we list all the triples $(\mathbf{A}_{kl}, \mathbf{B}_{kl}, f_{kl})$, where $(\mathbf{A}_{kl}, \mathbf{B}_{kl}) \in \mathcal{P}$ and $f_{kl} : \mathbf{A}_{kl} \to \mathbf{H}_k$ is a homomorphism (there are just countably many such triples).

In the next step, we find i and j such that $\pi(i, j) = k$. Since $i \leq \pi(i, j) = k$, it follows that the triple $(\mathbf{A}_{ij}, \mathbf{B}_{ij}, f_{ij})$ was already determined as a member of the list for $\mathbf{H}_i$ in one of the previous iterations of Step 1. We apply now to this triple HAP:



with $g_{ij}\restriction_{\mathbf{A}_{ij}} = f_{ij}$, and obtain the structure $\mathbf{H}_{k+1}$. (Note that $\mathbf{H}_{k+1}$ can be always chosen in such a way that $\mathbf{H}_k \leq \mathbf{H}_{k+1}$ by taking for $\mathbf{H}_{k+1}$ the appropriate structure from the isomorphism class, i.e. by changing the representative of the class.)

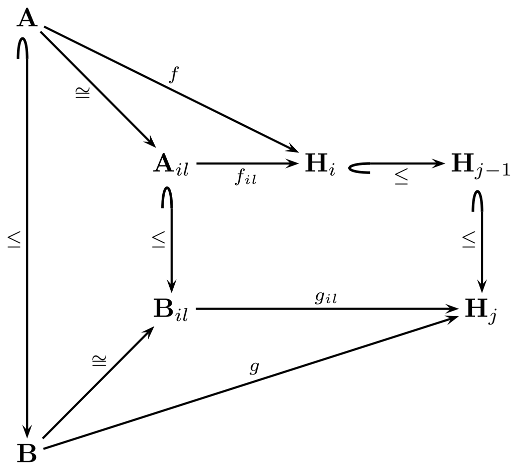
Step 2 Suppose that we have constructed $\mathbf{H}_k$, for k odd. Take the next structure $\mathbf{A} \in \mathcal{R}$. By JEP, there exists $\mathbf{H}_{k+1}$ such that both $\mathbf{H}_k$ and $\mathbf{A}$ are embeddable into it:



Together, Step 1 and Step 2 form the induction step.

This completes the description of the construction. It is left to prove that the constructed chain fulfils conditions (I) and (II):

About (I) We identify a given triple $(\mathbf{A}, \mathbf{B}, f)$ in the list for $\mathbf{H}_i$ in the construction. Let it be $(\mathbf{A}_{il}, \mathbf{B}_{il}, f_{il})$. Compute $j := \pi(i, l)$. The requested g and $\mathbf{H}_j$ are those that appear on the diagram for the construction of $\mathbf{H}_{j+1}$:



About (II) Follows immediately from Step 2 in the construction.

□

A class of finite relational structures over the same signature, that is closed under isomorphism and that has properties HP, JEP, and HAP will be called *hom-amalgamation class.*

Note that, in contrast to Fraïssé's construction, the construction of a homomorphism-homogeneous structure from an age does not guaranty the uniqueness of the result up to isomorphism. Indeed, all countably infinit linear orders are homomorphism-homogeneous and have as the age the class of all finite linear orders. Let us therefore have a look, how the different homomorphism-homogeneous structures with the same age are interrelated.

Definition. Let $\mathbf{H}$ and $\mathbf{H}'$ be two relational structures. We write $\mathbf{H} \preceq_h \mathbf{H}'$ if

- $\operatorname{Age}(\mathbf{H}) \supseteq \operatorname{Age}(\mathbf{H}')$, and
- for all finite $\mathbf{A} \leq \mathbf{B} \leq \mathbf{H}$ we have that every homomorphism from $\mathbf{A}$ to $\mathbf{H}'$ extends to a homomorphism from $\mathbf{B}$ to $\mathbf{H}'$.

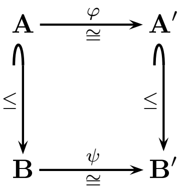
Proposition 4.5. *Let $\mathbf{H}$ and $\mathbf{H}'$ be two relational structures.*

(a) If $\mathbf{H} \preceq_h \mathbf{H}'$, and $\mathbf{H}$ is weakly homomorphism-homogeneous, then $\mathbf{H}'$ is weakly homomorphism-homogeneous, too.

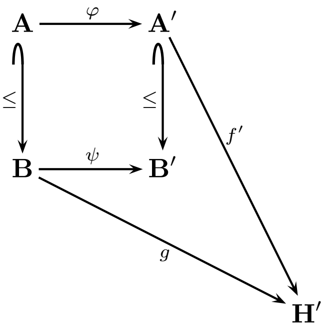
(b) If $\mathbf{H}'$ is weakly homomorphism-homogeneous, and $\operatorname{Age}(\mathbf{H}) = \operatorname{Age}(\mathbf{H}')$, then $\mathbf{H} \preceq_h \mathbf{H}'$.

Proof. **About (a)** Let $\mathbf{A}', \mathbf{B}'$ be finite substructures of $\mathbf{H}'$, such that $\mathbf{A}' \leq \mathbf{B}'$, and let $f' : \mathbf{A}' \to \mathbf{H}'$ be a local homomorphism: Since $\operatorname{Age}(\mathbf{H}) \supseteq \operatorname{Age}(\mathbf{H}')$, and $\mathbf{A}', \mathbf{B}' \in \operatorname{Age}(\mathbf{H}')$, it follows that there is a $\mathbf{B} \leq \mathbf{H}$, such that $\mathbf{B} \cong \mathbf{B}'$.

Further, since $\mathbf{A}' \leq \mathbf{B}'$, it follows that there is an $\mathbf{A} \leq \mathbf{B}$, such that $\mathbf{A} \cong \mathbf{A}'$, and that the following diagram commutes:



where ψ is any isomorphism, φ is obtained by restricting ψ. Since $\mathbf{H} \preceq_h \mathbf{H}'$, the homomorphism $f' \circ \varphi : \mathbf{A} \to \mathbf{H}'$ has an extension g to $\mathbf{B}$, as depicted in the following commuting diagram:

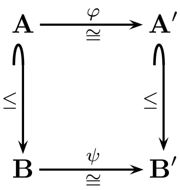


(1)

Now we define $g' := g \circ \psi^{-1}$. Using diagram (1), it can easily be checked that this is an extension of f' to $\mathbf{B}'$. This shows that $\mathbf{H}'$ is weakly homomorphism-homogeneous.

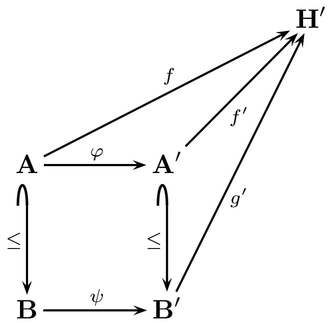
About (b) It suffices to show that for given finite structure $\mathbf{A}$ with $\mathbf{A} \leq \mathbf{B} \leq \mathbf{H}$, and a homomorphism $f : \mathbf{A} \to \mathbf{H}'$, we can always extend f to a homomorphism from $\mathbf{B}$ to $\mathbf{H}'$.

Note that $\mathbf{A}, \mathbf{B} \in \mathrm{Age}(\mathbf{H}) = \mathrm{Age}(\mathbf{H}')$. Then there exists a $\mathbf{B}' \leq \mathbf{H}'$, such that $\mathbf{B} \cong \mathbf{B}'$. Let $\hat{\mathbf{A}} \leq \hat{\mathbf{B}}$ such that $\hat{\mathbf{A}} \cong \mathbf{A}$, and let $\varphi : \mathbf{A} \to \mathbf{A}'$, $\psi : \mathbf{B} \to \mathbf{B}'$ be isomorphisms that make the following diagram commutative:



Let $f' := f \circ \varphi^{-1}$, Then f' is a homomorphism from $\mathbf{A}'$ to $\mathbf{H}'$. Since $\mathbf{H}'$ is weakly homomorphism-homogeneous, there exists a homomorphism

$g' : \mathbf{B}' \to \mathbf{H}'$ that extends f':



We define $g := g' \circ \psi$. It is easily checked that g extends f to a homomorphism from $\mathbf{B}$ to $\mathbf{H}'$. Thus $\mathbf{H} \preceq_h \mathbf{H}'$.

□

The second part of the previous Proposition gives the interrelation of any two homomorphism-homogeneous countable structures with the same age:

Corollary 4.6. *Let* $\mathbf{A}$*, and* $\mathbf{B}$ *be two weakly homomorphism-homogeneous structures with the same age. Then* $\mathbf{A} \preceq_h \mathbf{B}$ *and* $\mathbf{B} \preceq_h \mathbf{A}$*. In particular, any two countable homomorphism-homogeneous relational structures with the same age are homomorphism-equivalent.* □

Remark. A binary relation similar to $\preceq_h$ appeared in [12] in the context of a Fraïssé-type theorem for monomorphism homogeneous structures.

5 Homomorphisms between weakly oligomorphic structures

In this section we collect some auxiliary results about homomorphisms between weakly oligomorphic structures.

Definition. Let $\mathcal{A}$, $\mathcal{B}$ be two classes of relational structures over a common signature. We say, that $\mathcal{A}$ *projects* onto $\mathcal{B}$ (written $\mathcal{A} \to \mathcal{B}$) if for every $\mathbf{A} \in \mathcal{A}$ there exists a $\mathbf{B} \in \mathcal{B}$, such that $\mathbf{A} \to \mathbf{B}$.

Clearly, for two relational structures $\mathbf{A}$ and $\mathbf{B}$, the condition $\operatorname{Age}(\mathbf{A}) \to \operatorname{Age}(\mathbf{B})$ is necessary for $\mathbf{A} \to \mathbf{B}$. In the following we will prove that if $\mathbf{A}$, is countable, and $\mathbf{B}$ is weakly oligomorphic, then this condition is also sufficient. Before coming to the prove of this claim, let us prove a more specific result about weakly homomorphism homogeneous relational structures. The proof uses Kőnig's tree-lemma — a typical technique from the theory of ω-categorical structures.

Proposition 5.1. *Let* $\mathbf{A}$*,* $\mathbf{B}$ *be a relational structures over the same signature such that* $\operatorname{Age}(\mathbf{A}) \to \operatorname{Age}(\mathbf{B})$*, and suppose that* $\mathbf{A}$ *is countable, and that* $\mathbf{B}$ *is weakly oligomorphic and weakly homomorphism homogeneous. Then* $\mathbf{A} \to \mathbf{B}$*.*

Proof. If $\mathbf{A}$ is finite, then nothing needs to be proved. So we assume that $\mathbf{A}$ is countably infinite. In that case we can write the carrier A as $A = \{a_0, a_1, a_2, \dots\}$. Define $A_n := \{a_0, \dots, a_{n-1}\}$, and let $\mathbf{A}_n$ be the substructure of $\mathbf{A}$ that is induced by A_n.

Let us define an equivalence relation on the homomorphisms from $\mathbf{A}_n$ to $\mathbf{B}$: Let $h_1, h_2 : \mathbf{A}_n \to \mathbf{B}$. Then the two homomorphisms induce tuples $\bar{h}_1 = (h_1(a_0), \dots, h_1(a_{n-1}))$ and $\bar{h}_2 = (h_2(a_0), \dots, h_2(a_{n-1}))$. We call h_1 and h_2 equivalent (written $h_1 \cong h_2$) if there exists a local isomorphism that maps $\bar{h}_1$ to $\bar{h}_2$.

We claim that there are just finitely many equivalence classes of homomorphisms from $\mathbf{A}_n$ to $\mathbf{B}$. Since $\mathbf{B}$ is weakly oligomorphic, and weakly homomorphism homogeneous, by Proposition 4.3 it follows that $\mathbf{B}$ is endolocal. In other words, the n-ary relations that are definable by positive-existential formulæ coincide with the relations that are closed with respect to local homomorphisms. Note that if there is a local homomorphism that maps a tuple $\bar{a}$ to a tuple $\bar{b}$, and if there is a local homomorphism that maps $\bar{b}$ to $\bar{a}$, then $\bar{a} \cong \bar{b}$. In other words, two tuples $\bar{a}$ and $\bar{b}$ are equivalent if and only if the respective closure of $\{\bar{a}\}$ and $\{\bar{b}\}$ under local homomorphisms coincide. Hence there are just finitely many equivalence classes of tuples.

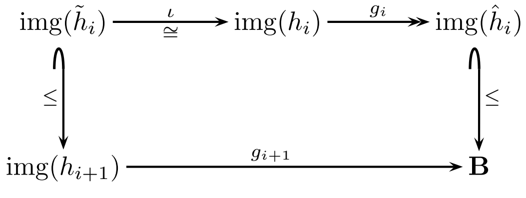
Next we claim, that if $h_1, h_2 : \mathbf{A}_{n+1} \to \mathbf{B}$ with $h_1 \cong h_2$, then $h_1{\restriction_{A_n}} \cong h_2{\restriction_{A_n}}$. Indeed, the same local isomorphism that maps $\bar{h}_1$ to $\bar{h}_2$ will also map $\overline{h_1{\restriction_{A_n}}}$ to $\overline{h_2{\restriction_{A_n}}}$.

Next, we define a tree, whose nodes on level n are all equivalence classes of homomorphisms from $\mathbf{A}_n$ to $\mathbf{B}$. For the equivalence class $[h]_\cong$ that is generated by $h : \mathbf{A}_{n+1} \to \mathbf{B}$, the unique lower neighbor is $[h{\restriction_{A_n}}]_\cong$.

By the above proved claims, this tree is well-defined and finitely branching. Moreover, the tree has nodes on every level, since $\operatorname{Age}(\mathbf{A}) \to \operatorname{Age}(\mathbf{B})$. Hence, by Kőnig's tree-lemma, the tree has an infinite branch $([h_i]_\cong)_{i \in \mathbb{N}}$.

It remains, to construct the homomorphism from $\mathbf{A}$ to $\mathbf{B}$. We proceed by induction. Our goal is to construct a tower $(\hat{h}_i)_{i\in\mathbb{N}}$ of homomorphisms $\hat{h}_i$ from $\mathbf{A}_i$ to $\mathbf{B}$. Such that for every i, there exists a local homomorphism g_i such that $g \circ h_i = \hat{h}_i$.

Define $\hat{h}_0 := h_0$. Suppose that $\hat{h}_i$ is already defined. Then $h_{i+1}{\restriction_{A_i}} \cong h_i$. Let us denote $h_{i+1}{\restriction_{A_i}}$ by $\tilde{h}_i$. Then there exists a local isomorphism ι such that $\iota \circ \tilde{h}_i = h_i$. without loss of generality, $\operatorname{dom}(\iota) = \operatorname{img}(\tilde{h}_i)$, $\operatorname{dom}(g_i) = \operatorname{img}(h_i)$, and $\operatorname{img}(g_i) = \operatorname{img}(\hat{h}_i)$. Then $g_i \circ \iota : \operatorname{img}(\tilde{h}_i) \to \operatorname{img}(\hat{h}_i)$ is a local homomorphism, and $\operatorname{img}(h_{i+1}) \supseteq \operatorname{img}(\tilde{h}_i)$. Since $\mathbf{B}$ is weakly homomorphism homogeneous, $g_i \circ \iota$ extends to $\operatorname{img}(h_{i+1})$ (cf. the following diagram).



Now we define $\hat{h}_{i+1}(x) := g_{i+1}(h_{i+1}(x))$. Then for any $x \le i$, we calculate

$$g_{i+1}(h_{i+1}(x)) = g_{i+1}(\tilde{h}_i(x)) = g_i(\iota(\tilde{h}_i(x))) = g_i(h_i(x)) = \hat{h}_i(x).$$

Thus, $\hat{h}_{i+1}$ is indeed an extension of $\hat{h}_i$, and $g_{i+1} \circ h_{i+1} = \hat{h}_{i+1}$.

It remains to note that the union over all $\hat{h}_i$ is a homomorphism from $\mathbf{A}$ to $\mathbf{B}$. □

Proposition 5.2. *Let* $\mathbf{A}$, $\mathbf{B}$ *be a relational structures over the same signature such that* $\mathrm{Age}(\mathbf{A}) \to \mathrm{Age}(\mathbf{B})$, *and suppose that* $\mathbf{A}$ *is countable, and that* $\mathbf{B}$ *is weakly oligomorphic. Then* $\mathbf{A} \to \mathbf{B}$.

Proof. First we expand the signature by all positive existential formulæ. $\hat{B}$ shall be the structure obtained from $\mathbf{B}$ by expansion by all positive existential definitions.

We also expand $\mathbf{A}$ to a new structure $\hat{\mathbf{A}}$ over the new signature. However, we interprete each additional relational symbol as the empty relation in $\hat{\mathbf{A}}$.

With this setting it is clear, that $\mathrm{Age}(\hat{\mathbf{A}}) \to \mathrm{Age}(\hat{\mathbf{B}})$. Moreover, in $\hat{\mathbf{B}}$, every positive existential formula is equivalent to a quantifier-free positive formula. Hence, observing that $\hat{\mathbf{B}}$ is weakly oligomorphic, and using Proposition 4.3, we conclude that $\hat{\mathbf{B}}$ is weakly homomorphism homogeneous.

By Proposition 5.1, it follows that there is a homomorphism from $h : \hat{\mathbf{A}} \to \hat{\mathbf{B}}$. Clearly, h is also a homomorphism from $\mathbf{A}$ to $\mathbf{B}$. □

6 Cores of homomorphism homogeneous structures

A finite relational structure is called a *core* if every of its endomorphisms is an automorphism. There are many ways to generalize the definition of a core to infinite structures. Several possibilities were explored in [3]. For us, the following definition seems most reasonable:

Definition. A relational structure $\mathbf{C}$ is called a core if every endomorphism of $\mathbf{C}$ is an embedding.

We say that $\mathbf{C}$ is a core of $\mathbf{A}$, if $\mathbf{C}$ is a core, $\mathbf{C} \leq \mathbf{A}$, and there is an endomorphism f of $\mathbf{A}$ such that $\mathrm{img}(f) = \mathbf{C}$.

Cores of relational structures play an important role in the theory of CSPs. If a template $\mathbf{T}$ has a core $\mathbf{C}$, then that means that $\mathbf{T}$ and $\mathbf{C}$ define the same CSP. Moreover, $\mathbf{C}$ defines the CSP in "the most efficient way".

For finite relational structures, the core always exist and is unique, up to isomorphism. For infinite structures a core may exist or may not exist. Moreover, if it exists, it may not be unique up to isomorphism.

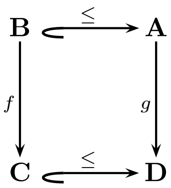
In this section we will employ the machinery, that was developed in the previous section in order to study cores of homomorphism-homogeneous structures. The crucial definition in this section is that of a hom-irreducible element in some class of structures:

Definition. Let $\mathcal{C}$ be a class of relational structures over the same signature and let $\mathbf{A} \in \mathcal{C}$. We say that $\mathbf{A}$ is *hom-irreducible in* $\mathcal{C}$ if for every $\mathbf{B} \in \mathcal{C}$ and every homomorphism $f : \mathbf{A} \to \mathbf{B}$ holds that f is an embedding.

For a relational structure $\mathbf{A}$, by $\mathcal{C}_{\mathbf{A}}$ we denote the class of all finite structures of the same type like $\mathbf{A}$ that are hom-irreducible in the age of $\mathbf{A}$.

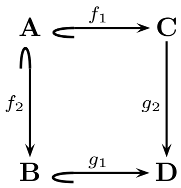
Lemma 6.1. *Let $\mathcal{C}$ be a hom-amalgamation class, let $\mathcal{D}$ be the class of all structures from $\mathcal{C}$ that are hom-irreducible in $\mathcal{C}$. If $\mathcal{C} \to \mathcal{D}$, then $\mathcal{D}$ is a Fraïssé class, i.e. it has HP, and AP.*

Proof. (HP) Let $\mathbf{A} \in \mathcal{D}$, and let $\mathbf{B}$ be a substructure of $\mathbf{A}$ (in particular, $\mathbf{B} \in \mathcal{C}$). Let $\mathbf{C} \in \mathcal{C}$, and let $f : \mathbf{B} \to \mathbf{C}$ be any homomorphism. Then by HAP we have that there exist a $\mathbf{D} \in \mathcal{C}$, and a homomorphism $g : \mathbf{A} \to \mathbf{D}$ such that the following diagram commutes:



Since $\mathbf{A}$ is hom-irreducible in $\mathcal{C}$, it follows that g is an embedding, and that f, being a restriction of g to $\mathbf{B}$, is an embedding, too. We conclude now that $\mathbf{B}$ is hom-irreducible in $\mathcal{C}$, and, hence, $\mathcal{D}$ has HP.

(AP) Let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{D} \subseteq \mathcal{C}$ and let $f_1 : \mathbf{A} \to \mathbf{C}$ and $f_2 : \mathbf{A} \to \mathbf{B}$ be embeddings. Then, by HAP of $\mathcal{C}$, there exist a $\mathbf{D} \in \mathcal{C}$, an embedding $g_1 : \mathbf{B} \to \mathbf{D}$ and a homomorphism $g_2 : \mathbf{C} \to \mathbf{D}$:



Since $\mathbf{C} \in \mathcal{D}$, it follows that g_2 is an embedding, too. Using that $\mathcal{C} \to \mathcal{D}$, we obtain that there exist a structure $\hat{\mathbf{D}} \in \mathcal{D}$ with a homomorphism $h : \mathbf{D} \to \hat{\mathbf{D}}$. Then $\hat{\mathbf{D}}$ will be the amalgam, with $h \circ g_1$ and $h \circ g_2$ as embeddings.

□

Lemma 6.2. *Let $\mathbf{A}$ be a weakly oligomorphic, weakly homomorphism homogeneous relational structure. Then* $\mathrm{Age}(\mathbf{A}) \to \mathcal{C}_{\mathbf{A}}$.

Proof. Consider A^n and note that all tuples from A^n can be quasiordered by $\bar{a} \leq_n \bar{b}$ if and only if there is a local homomorphism of $\mathbf{A}$ that maps $\bar{a}$ to $\bar{b}$.

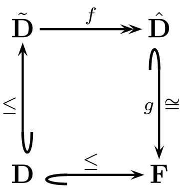
Since $\mathbf{A}$ is weakly oligomorphic, by Proposition 4.3, $\mathbf{A}$ is endolocal. Hence the set of n-ary relations on $\mathbf{A}$, definable by positive existential types, coincides with the filters of $(A^n, \leq_n)$. Hence the equivalence relation $\leq_n \cap \geq_n$ has finitely many equivalence classes. In particular, every properly ascending chain is finite and every tuple $\bar{a}$ lies below a maximal tuple $\bar{a}^\sharp$.

Let $\mathbf{B}$ be a finite substructure of $\mathbf{A}$ with $B = \{b_1, \dots, b_n\}$. Define $\bar{b} := (b_1, \dots, b_n)$ and let $\bar{b}^\sharp$ be a maximal tuple above $\bar{b}$, say $\bar{b}^\sharp = (b_1^\sharp, \dots, b_n^\sharp)$. Let, further, $\mathbf{D}$ be a substructure of $\mathbf{A}$ induced by $\{b_1^\sharp, \dots, b_n^\sharp\}$. We define $f : \mathbf{B} \to \mathbf{D} : b_i \mapsto b_i^\sharp$, for $i = 1, \dots, n$. Then f is an epimorphism and $\mathbf{D}$ is hom-irreducible in $\mathrm{Age}(\mathbf{A})$. □

Proposition 6.3. *Let $\mathbf{A}$ be a countable homomorphism-homogeneous relational structure, such that* $\mathrm{Age}(\mathbf{A}) \to \mathcal{C}_{\mathbf{A}}$ *then $\mathbf{A}$ has a core $\mathbf{C}$ with age $\mathcal{C}_{\mathbf{A}}$.*

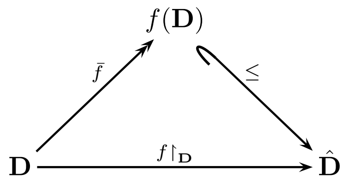
Before coming to the proof of Proposition 6.3, we need to prove a technical lemma:

Lemma 6.4. *Let* $\mathbf{A}$ *be a weakly homomorphism-homogeneous relational structure, and let* $\mathbf{D} \leq \mathbf{A}$ *be hom-irreducible in* $\operatorname{Age}(\mathbf{A})$*. Further let* $\tilde{\mathbf{D}}$ *be a finite superstructure of* $\mathbf{D}$ *in* $\mathbf{A}$*. Finally, let* $\hat{\mathbf{D}} \leq \mathbf{A}$ *be hom-irreducible in* $\operatorname{Age}(\mathbf{A})$*, and let* $f : \tilde{\mathbf{D}} \twoheadrightarrow \hat{\mathbf{D}}$*. Then there exists a finite substructure* $\mathbf{F} \leq \mathbf{A}$*, and an isomorphism* $g : \hat{\mathbf{D}} \to \mathbf{F}$ *such that* $\mathbf{D} \leq \mathbf{F}$ *and the following diagram commutes:*



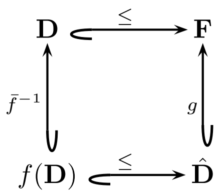
(2)

Proof. Consider the mapping $\bar{f}$ given by the following diagram:



and note that $\bar{f}$ is an isomorphism because $\mathbf{D}$ is hom-irreducible in $\operatorname{Age}(\mathbf{A})$.

Since $\mathbf{A}$ is weakly homomorphism-homogeneous, it follows that $\bar{f}^{-1}$ extends to a homomorphism $g : \hat{\mathbf{D}} \to \mathbf{F}$. $\mathbf{F}$ can be chosen in such a way that g is surjective. Note that since $\hat{\mathbf{D}}$ is hom-irreducible in $\operatorname{Age}(\mathbf{A})$, it follows that g is an isomorphism.



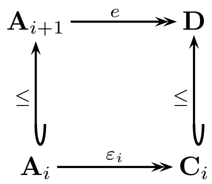
Let $d \in D$. Then $g(f(d)) = g(\bar{f}(d)) = d$, since $\bar{f}(d) \in f(D)$ and $g{\restriction}_{f(\mathbf{D})} = \bar{f}^{-1}$. Hence, the diagram (2) commutes. □

Proof of Proposition 6.3. Take all finite substructures of $\mathbf{A}$, and denote them by $\mathbf{E}_0, \mathbf{E}_1, \mathbf{E}_2, \dots$

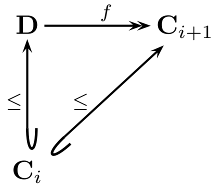
We will show that a core exists by constructing an endomorphism whose image has an age contained in $\mathcal{C}_{\mathbf{A}}$. This endomorphism will be obtained as the union of a tower of local homomorphisms $\varepsilon : A_i \twoheadrightarrow \mathbf{C}_i$, where $\mathbf{C}_i \in \mathcal{C}_{\mathbf{A}}$:

Induction basis. Define $\mathbf{A}_0 := \mathbf{E}_0$. Then, by assumptions, there exist a $\mathbf{C}_0 \in \mathcal{C}_{\mathbf{A}}$ and an epimorphism $\varepsilon_0 : \mathbf{A}_0 \twoheadrightarrow \mathbf{C}_0$.

Induction step. Suppose that we have constructed $\varepsilon_i : \mathbf{A}_i \twoheadrightarrow \mathbf{C}_i$. Define $\mathbf{A}_{i+1} := \mathbf{A}_i \cup \mathbf{E}_{i+1}$. Since $\mathbf{A}$ is weakly homomorphism-homogeneous, there exist a $\mathbf{D} \geq \mathbf{C}_i$ and an epimorphism $e : \mathbf{A}_{i+1} \twoheadrightarrow \mathbf{D}$ such that the following diagram commutes:



Further, there exists an epimorphism from $\mathbf{D}$ to a substructure of $\mathbf{A}$ that is hom-irreducible in Age($\mathbf{A}$). By Lemma 6.4, there exist a structure $\mathbf{C}_{i+1} \geq \mathbf{C}_i$ that is hom-irreducible in Age($\mathbf{A}$) and there exists an epimorphism $f : \mathbf{D} \twoheadrightarrow \mathbf{C}_{i+1}$ such that the following diagram commutes:



Define $\varepsilon_{i+1} := f \circ e$ and observe that ε_{i+1} is an extension of ε_i. Note, further, that $\mathbf{A} = \cup_{i\in\mathbb{N}}\mathbf{A}_i$.

Let $C := \cup_{i\in\mathbb{N}}C_i$ and let $\mathbf{C} \leq \mathbf{A}$ be a structure induced by $\mathcal{C}$. Further, let $\varepsilon := \cup_{i\in\mathbb{N}}\varepsilon_i$. Then $\varepsilon : \mathbf{A} \twoheadrightarrow \mathbf{C}$.

Instead of directly showing that $\mathbf{C}$ is a core, we prove the stronger claim, that every homomorphism $f : \mathbf{C} \to \mathbf{A}$ is an embedding.

Suppose to the contrary that there exists $f : \mathbf{C} \to \mathbf{A}$ that is not an embedding. Then either f is not injective or f is injective, but not strong.

Case 1. If f is not injective, then there are $c, d \in \mathbf{C}$ such that $f(c) = f(d)$.

On the other hand, there exists an $i \in \mathbb{N}$ such that $\{c, d\} \subseteq C_i$. Since $f\restriction_{\mathbf{C}_i}$ is a homomorphism and $\mathbf{C}_i$ is hom-irreducible in Age($\mathbf{A}$), it follows that $f\restriction_{\mathbf{C}_i}$ is an embedding, and we arrive at a contradiction.

Case 2. If f is injective, but not strong (i.e. if f is a monomorphism, but not an embedding), then there exist a basic n-ary relation ϱ of $\mathbf{A}$ and a tuple $\bar{a} = (a_1, a_2, \ldots, a_n) \in A^n \setminus \varrho$ such that $(f(a_1), f(a_2), \ldots, f(a_n)) \in \varrho$. However, then there is an $i \in \mathbb{N}$ such that $\{a_1, a_2, \ldots, a_n\} \subseteq C_i$, and $f\restriction_{\mathbf{C}_i}$ is an embedding, which is a contradiction.

Summing up, we conclude f must be an embedding.

Let us finally show that $\text{Age}(\mathbf{C}) = \mathcal{C}_{\mathbf{A}}$: By construction, $\text{Age}(\mathbf{C}) \subseteq \mathcal{C}_{\mathbf{A}}$. On the other hand, as a core of $\mathbf{A}$, every finite substructure of $\mathbf{A}$ that is hom-irreducible in Age($\mathbf{A}$) embedds into $\mathbf{C}$. Hence $\mathcal{C}_{\mathbf{A}} \subseteq \text{Age}(\mathbf{C})$. □

Corollary 6.5. *Every countable weakly oligomorphic homomorphism-homogeneous structure* $\mathbf{A}$ *has a core* $\mathbf{C}$ *of age* $\mathcal{C}_{\mathbf{A}}$. □

Before coming to the main result of this section, we need to prove a few auxiliary results:

Lemma 6.6. *Let* $\mathbf{A}$ *be a relational structure, then the following are true:*

(a) *If* $\mathbf{A}$ *is weakly oligomorphic, then for every* $n \in \mathbb{N}$*, the class* $\mathrm{Age}(\mathbf{A})$ *contains up to isomorphism only finitely many structures of cardinality* n*.*

(b) *If for every* $n \in \mathbb{N}$*, the class* $\mathrm{Age}(\mathbf{A})$ *contains up to isomorphism only finitely many structures of cardinality* n*, and* $\mathbf{A}$ *is homomorphism homogeneous, then* $\mathbf{A}$ *is weakly oligomorphic.*

Proof. **About (a)** Suppose, there are infinitely many isomorphism classes of substructures of cardinality k. Let $(\mathbf{B}_i)_{i\in\mathbb{N}}$ be a sequence of distinct representatives of isomorphism classes of substructures of $\mathbf{A}$ of cardinality k. Let us enumerate the elements of B_i like $B_i = \{b_{i,1}, \dots, b_{i,k}\}$. Consider the tuples $\bar{b}_i = (b_{i,1}, \dots, b_{i,k})$, and the relations $\varrho_i := \{\bar{c} \mid \mathrm{pTp}_{\mathbf{A}}(\bar{b}_j) \subseteq \bar{c}\}$.

Then for any two distinct i, j from $\mathbb{N}$ we have that $\mathrm{pTp}_{\mathbf{A}}(\bar{b}_i) \neq \mathrm{pTp}_{\mathbf{A}}(\bar{b}_j)$, and hence $\varrho_i \neq \varrho_j$. This way we have infinitely many distinct k-ary relations on $\mathbf{A}$ that can be defined by sets of positive existential formulæ over $\mathbf{A}$ — contradiction.

About (b) Equip the n-tuples over A with the following quasiorder: $\bar{a} \leq \bar{b}$ if there is a local homomorphism that maps $\bar{a}$ to $\bar{b}$. Since $\mathbf{A}$ is homomorphism homogeneous, this is the case if and only if $\bar{b}$ is in the invariant relation of $\mathrm{End}(\mathbf{A})$ generated by $\bar{a}$.

We will show, that $\mathrm{End}(\mathbf{A})$ is oligomorphic. Suppose, it is not. Then there is a k and a sequence of tuples $(\bar{b}_i)_{i\in\mathbb{N}}$ of k-tuples over A, such that for all distinct natural numbers i and j we have that $\bar{b}_i$ and $\bar{b}_j$ generate different invariant relations of $\mathrm{End}(\mathbf{A})$. If this is so, then by the infinite pigeon hole principle, there exists an $n \leq k$ and a sequence $(\bar{c}_i)_{i\in\mathbb{N}}$ of irreflexive n-tuples over A such that any two tuples generate different invariant relations of $\mathrm{End}(\mathbf{A})$. Let M be the number of isomorphism classes of substructures of cardinality n in $\mathbf{A}$. Then the number of invariant n-ary relations of generated by n-ary irreflexive tuples is bounded from above by $M \cdot n!$ — contradiction. Hence $\mathrm{End}(\mathbf{A})$ is oligomorphic.

Since all relations definable by sets of positive existential formulæ over $\mathbf{A}$ are invariant under $\mathrm{End}(\mathbf{A})$, it follows that $\mathbf{A}$ is weakly oligomorphic. □

An immediate consequence of the previous lemma is:

Corollary 6.7. *If* $\mathbf{A}$ *is a homomorphism homogeneous relational structure over a finite signature, then* $\mathbf{A}$ *is weakly oligomorphic.* □

This, together with the characterization of the ages of countable homomorphism homogeneous structures, gives a rich source of weakly oligomorphic structures, since any reduct of a countable weakly oligomorphic structure will again be weakly oligomorphic.

Lemma 6.8. *Let* $\mathbf{C}$ *be a homogeneous core. Then* $\mathbf{C}$ *is weakly oligomorphic if and only if it is oligomorphic.*

Proof. Obviously, if a structure is oligomorphic, then it is weakly oligomorphic, too.

Suppose now that $\mathbf{C}$ is weakly oligomorphic, and let $\mathbb{M} := \mathrm{End}(\mathbf{C})$ and $\mathbb{G} := \mathrm{Aut}(\mathbf{C})$. Take $\bar{a} = (a_1, \dots, a_n) \in C^n$ and $\bar{b} = (b_1 \dots, b_n) \in \bar{a}^{\mathbb{M}}$. Then there exists an $f \in \mathbb{M}$, such that $f(\bar{a}) = \bar{b}$. Since **C** is a core, we conclude that f is an embedding, and, therefore, $a_i \mapsto b_i$, for $i = 1, \dots, n$ is a local isomorphism. Since **C** is homogeneous, it follows that there is a $g \in G$ such that $g(\bar{a}) = \bar{b}$, so $\bar{b} \in \bar{a}^{\mathbb{G}}$ implying that **C** is oligomorphic. □

The following result links homomorphism homogeneous structures to homogeneous structures.

Theorem 6.9. *Let* **A** *be a weakly oligomorphic countable homomorphism homogeneous structure. Then* **A** *contains a substructure* **F** *that is isomorphic to the Fraïssé-limit of* $\mathcal{C}_{\mathbf{A}}$*. Moreover,* **F** *and* **A** *are hom-equivalent, and* **F** *is oligomorphic.*

Proof. Let **F** be the Fraïssé-limit of $\mathcal{C}_{\mathbf{A}}$.

Since **A** is weakly oligomorphic and $\mathcal{C}_{\mathbf{A}} \to \mathrm{Age}(\mathbf{A})$, we conclude from Proposition 5.2, that $\mathbf{F} \to \mathbf{A}$. Since $\mathrm{Age}(\mathbf{F}) = \mathcal{C}_{\mathbf{A}}$, every homomorphism from **F** to **A** is an embedding. So we can assume without loss of generality that $\mathbf{F} \leq \mathbf{A}$.

Every local homomorphism of **F** is an embedding. Hence **F** is homomorphism homogeneous. Since $\mathcal{C}_{\mathbf{A}} \subseteq \mathrm{Age}(\mathbf{A})$, and since **A** is weakly oligomorphic, from Lemma 6.6, it follows that **F** is weakly oligomorphic. Since **F** is a homogeneous core, from Lemma 6.8, it follows that **F** is oligomorphic.

It remains to show that $\mathbf{A} \to \mathbf{F}$. By Corolary 6.5, **A** has a core **C** such that $\mathrm{Age}(\mathbf{C}) = \mathcal{C}_{\mathbf{A}}$. Hence, by Proposition 5.2, it follows that $\mathbf{C} \to \mathbf{F}$. □

The following corollary is of independent interest in the theory of homomorphism homogeneous structures

Corollary 6.10. *Every countable, weakly oligomorphic, homomorphism homogeneous structure* **A** *contains, up to isomorphism, a unique hom-equivalent homomorphism homogeneous core* **F***. Moreover,* **F** *is oligomorphic and homogeneous.*

Proof. Theorem 6.9 guaranties the existence of **F**. Indeed, **F** is homomorphism homogeneous, because every local homomorphism of **F** is an embedding.

Suppose that $\mathbf{F}'$ is another such core. Then $\mathcal{C}_{\mathbf{A}} \subseteq \mathrm{Age}(\mathbf{F}')$. On the other hand, $\mathrm{Age}(\mathbf{A}) \to \mathcal{C}_{\mathbf{A}}$, hence $\mathrm{Age}(\mathbf{F}') \to \mathcal{C}_{\mathbf{A}}$. Hence any substructure of **F** that is not hom-irreducible in $\mathrm{Age}(\mathbf{A})$, homomorphically maps to a hom-irreducible element. This defines a local homomorphism that is not an embedding. By the homomorphism-homogeneity of $\mathbf{F}'$, this extends to an endomorphism, that is not an embedding — contradiction. Thus $\mathrm{Age}(\mathbf{F}') = \mathcal{C}_{\mathbf{A}}$, and every local homomorphism is an embedding. From this follows that $\mathbf{F}'$ is weakly homogeneous, and hence homogeneous. From Fraïssé's theorem it follows that $\mathbf{F} \cong \mathbf{F}'$. □

Remark. The previous Theorem shows, that whenever a CSP can be formalized using a weakly oligomorphic homomorphism homogeneous template, then it can also be formalized by an oligomorphic homogeneous core.

7 Weak oligomorphy and ω-categoricity

In this section we will create a link from weakly oligomorphic structures to ω-categorical structures. Let us start by recalling some classical notions and results from model theory, and by proving some additional auxiliary results:

Definition. A first order theory is called ω-categorical if it has up to isomorphism exactly one countably infinite model. A countably infinite structure $\mathbf{A}$ is called ω-categorical if $\mathrm{Th}(\mathbf{A})$ is ω-categorical.

The following classical result links ω-categoricity with oligomorphy (cf. [11, (2.10)]):

Theorem 7.1 (Engeler, Ryll-Nardzewski, Svenonius)**.** *Let* $\mathbf{A}$ *be a countably infinite structure. Then* $\mathbf{A}$ *is* ω*-categorical if and only if it is oligomorphic.* □

In the previous section we linked weakly oligomorphic homomorphism homogeneous structures with oligomorphic homogeneous structures. The following Theorem makes a similar link between weakly oligomorphic structures and ω-categorical structures.

Theorem 7.2. *Let* $\mathbf{A}$ *be a countable weakly oligomorphic relational structure. Then* $\mathbf{A}$*, is hom-equivalent to a finite or* ω*-categorical structure* $\mathbf{F}$*. Moreover,* $\mathbf{F}$ *embedds into* $\mathbf{A}$*.*

Proof. Let $\hat{\mathbf{A}}$ be the structure that is obtained by expanding $\mathbf{A}$ by all positive existential defininable relations over $\mathbf{A}$.

In $\hat{\mathbf{A}}$ every positive existential formula is equivalent to a positive quantifier-free formula. Hence, by Proposition 4.3, $\hat{\mathbf{A}}$ is homomorphism homogeneous. Clearly, $\hat{\mathbf{A}}$ is weakly oligomorphic, too. Hence, by Theorem 6.9, $\hat{\mathbf{A}}$ has a substructure $\hat{\mathbf{F}}$ that is oligomorphic, homogeneous, and homomorphism equivalent to $\hat{\mathbf{A}}$.

Let $\mathbf{F}$ be the reduct of $\hat{\mathbf{F}}$ to the signature of $\mathbf{A}$. Then still $\mathbf{F}$ is oligomorphic, and since $\hat{\mathbf{A}}$ and $\mathbf{A}$ have the same endomorphisms, $\mathbf{F}$ still is homomorphism-equivalent to $\mathbf{A}$.

If $\mathbf{F}$ is countably infinite, then, by Theorem 7.1, it is ω-categorical. □

8 Positive existential theories of weakly oligomorphic structures

The Engeler, Ryll-Nardzewski, Svenonius Theorem (cf. Theorem 7.1) can be understood as a characterization of the first order theories of countable oligomorphic structures. Using Theorem 7.2, we can give a similar characterization of the positive existential theories of weakly oligomorphic structures.

Theorem 8.1. *Let* T *be a set of positive existential propositions. Then the following are equivalent:*

1. T *is the positive existential theory of a countable weakly oligomorphic structure.*
2. T *is the positive existential part of an* ω*-categorical theory.*
3. T *is the positive existential theory of a countable oligomorphic structure.*

Proof. From Theorem 7.1, it follows that statements 2 and 3 are equivalent. Obviously, from 3 follows 1, so, to complete the proof, it is left to show that from 1 follows 3:

Let T be the positive existential theory of a countable weakly oligomorphic structure $\mathbf{A}$. Then, by Theorem 7.2, $\mathbf{A}$ is hom-equivalent to a finite or ω-categorical structure $\mathbf{F}$. If $\mathbf{F}$ is finite, then it is hom-equivalent to an ω-categorical structure $\hat{\mathbf{F}}$ (take ω disjoint copies of $\mathbf{F}$; this structure surely is oligomorphic and hence ω-categorical; moreover, $\mathbf{F}$ is a retract of $\hat{\mathbf{F}}$).

Clearly, two homomorphism-equivalent structures have the same positive existential theories. □

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