

ON THE THRESHOLD FOR THE MAKER-BREAKER H -GAME

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ABSTRACT. We study the Maker-Breaker H -game played on the edge set of the random graph $G_{n,p}$. In this game two players, Maker and Breaker, alternately claim unclaimed edges of $G_{n,p}$, until all edges are claimed. Maker wins if he claims all edges of a copy of a fixed graph H ; Breaker wins otherwise.

In this paper we show that, with the exception of trees and triangles, the threshold for an H -game is given by the threshold of the corresponding Ramsey property of $G_{n,p}$ with respect to the graph H .

Keywords. Positional games; random graphs; Maker-Breaker

1. INTRODUCTION

Combinatorial games are games like Tic-Tac-Toe or Chess in which each player has perfect information and players move sequentially. Outcomes of such games can thus, at least in principle, be predicted by enumerating all possible ways in which the game may evolve. But, of course, such complete enumerations usually exceed available computing powers, which keeps these games interesting to study.

In this paper we take a look at a special class of combinatorial games, the so-called Maker-Breaker positional games. Given a finite set X and a family \mathcal{E} of subsets of X , two players, Maker and Breaker, alternate in claiming unclaimed elements of X until all the elements are claimed. Unless explicitly stated otherwise, Maker starts the game. Maker wins if he claims all elements of a set from \mathcal{E} , and Breaker wins otherwise. The set X is referred to as the *board*, and the elements of \mathcal{E} as the *winning sets*. More on positional game theory can be found in the recent monograph [6].

Given a (large) graph G and a (small) graph H , the H -game on G is played on the board $E(G)$ and the winning sets are the edge sets of all copies of H appearing in G as subgraphs. Maker and Breaker alternately claim unclaimed edges of the graph G until all the edges are claimed. Maker wins if he claims all the edges of a copy of H in which case we say that the graph G is Maker's win, and otherwise Breaker wins, in which case we say that the graph G is Breaker's win.

When H is fixed, G is a complete graph on n vertices and n is large enough, Ramsey's theorem together with the strategy stealing argument readily implies that Maker wins the game. To give Breaker more power and make the game more interesting, two standard approaches [6] are to study *biased games*, or *games on a random board*.

In the biased version of the H -game on K_n , we introduce an integer b and let Breaker claim b edges (instead of one) per move. It is not hard to see that for b

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large enough Breaker will be able to win the game. The bias-monotonicity in Maker-Breaker games actually implies that there exists a threshold bias $b_0 = b_0(n, H)$ such that the biased H -game is Maker's win if and only if $b < b_0$.

For a graph G on at least three vertices, we let $d_2(G) := (|E(G)| - 1) / (|V(G)| - 2)$ and denote by $m_2(G)$ the so-called *2-density*, defined as

$$m_2(G) = \max_{J \subseteq G, |V(J)| \geq 3} d_2(J).$$

If $m_2(G) = d_2(G)$ we say that G is *2-balanced*, and if in addition $m_2(G) > d_2(J)$ for every subgraph $J \subset G$ with $|V(J)| \geq 3$, we say that G is *strictly 2-balanced*.

For the H -game played on K_n , Bednarska and Łuczak [2] found the order of the threshold bias, $b_0 = \Theta(n^{1/m_2(H)})$. Determining the leading constant inside the $\Theta(\cdot)$ remains an open problem that appears to be challenging.

We now turn our attention to the games played on random boards. In particular, we will look at unbiased games – where each player claims one edge per move. Positional games played on edges of random graphs were first introduced and studied in [18]. Here we look at the H -game played on the binomial random graph $G_{n,p}$. More precisely, we aim at determining a threshold function $p_0 = p_0(n, H)$ such that

$$\lim_{n \rightarrow \infty} \Pr[G_{n,p} \text{ is Maker's win in the } H\text{-game}] = \begin{cases} 1, & p \gg p_0(n, H), \\ 0, & p \ll p_0(n, H). \end{cases}$$

Since the graph property of “being Maker's win” is monotone, the existence of the threshold function follows from [4]. For the case that H is a clique such thresholds were recently obtained by Müller and Stojaković [10].

There is an easy intuitive argument for the location of a threshold: if the random graph $G_{n,p}$ is so sparse that w.h.p. (with high probability, i.e. with probability tending to 1 as n tends to infinity) it only contains few scattered copies of H , then this should be a Breaker's win. If on the other hand the graph contains many copies of H that heavily overlap, then this should make Maker's task easier. As it turns out, the same intuition can also be applied to the threshold for the Ramsey property of $G_{n,p}$, thus it is not surprising that the two are related. We formalize this as follows.

For graphs G and H we denote by $G \rightarrow (H)_r^e$ the property that every edge-coloring of G with r colors contains a copy of H with all edges having the same color. The Ramsey property of random graphs $G_{n,p}$ is well understood, as the following theorem shows, cf. also [12] for a short proof.

Theorem 1 (Rödl, Ruciński [13, 14, 15]). *Let $r \geq 2$ be an integer and H be a graph that is not a forest of stars or, in case $r = 2$, paths of length 3. Then there exist constants $c, C > 0$ such that*

$$\lim_{n \rightarrow \infty} \Pr[G_{n,p} \rightarrow (H)_r^e] = \begin{cases} 1, & \text{if } p \geq Cn^{-1/m_2(H)}, \\ 0, & \text{if } p \leq cn^{-1/m_2(H)}. \end{cases}$$

Note that $p = \Theta(n^{-1/m_2(H)})$ is the density which corresponds to the argument for the threshold, as given above. In this paper, we show that this intuition also provides the correct answer in the case of H -games, for most graphs H .

Theorem 2. *Let H be a graph for which there exists $H' \subseteq H$ such that $d_2(H') = m_2(H)$, H' is strictly 2-balanced and it is not a tree or a triangle. Then there exist constants $c, C > 0$ such that*

$$\lim_{n \rightarrow \infty} \Pr[G_{n,p} \text{ is Maker's win in the } H\text{-game}] = \begin{cases} 1, & p \geq Cn^{-1/m_2(H)}, \\ 0, & p \leq cn^{-1/m_2(H)}. \end{cases}$$

Note that Theorem 1 together with a strategy stealing argument readily implies the 1-statement of Theorem 2. However, we will give a stronger resilience-type statement in Section 3 (Theorem 16), along with a proof utilizing a recent tool given in [1] and [16] – *hypergraph containers*. This statement will later come in handy for the proof of Theorem 4. Also, it is worth noting that in our proof Maker relies on the Erdős-Selfridge criterion, which offers an *explicit* winning strategy. Of all the known results guaranteeing Maker’s win in the H -game (played on various base graphs), this is the first case of an explicit winning strategy. The proof of the 0-statement is based on ideas from [12].

Next, we take a look at those graphs H that are not covered by Theorem 2.

For $H = K_3$ we have $m_2(K_3) = 2$. Moreover, the threshold is at a different location, as was shown by Stojaković and Szabó [18].

Theorem 3 ([18]). *Consider the K_3 -game (i.e. the triangle game) played on the edge set of $G_{n,p}$. Then*

$$\lim_{n \rightarrow \infty} \Pr[G_{n,p} \text{ is Maker's win in the } K_3\text{-game}] = \begin{cases} 1, & p \gg n^{-5/9}, \\ 0, & p \ll n^{-5/9}. \end{cases}$$

The reason turns out to be that K_5 minus an edge is Maker’s win (which can be easily checked by hand) – and this graph appears in $G_{n,p}$ w.h.p. whenever $p \gg n^{-5/9}$. Moreover, Müller and Stojaković [10] showed that in the random graph process, w.h.p. the appearance of such a subgraph coincides with the moment when G becomes Maker’s win.

As a consequence, for graphs H that contain a triangle, various things can happen. If the 2-density is above 2, then such a graph is covered by the above theorem. If $m_2(H) = 2$ and H contains a subgraph with 2-density exactly two that does not contain a triangle, then this case is also covered by Theorem 2. Otherwise, the threshold can be placed almost arbitrarily between $n^{-5/9}$ and $n^{-1/2}$ while the 2-density of H remains at 2, as our next theorem confirms.

Theorem 4. *Let H be a graph which satisfies the conditions of Theorem 2, and denote by H_P a graph obtained by adding a path of length 3 between a vertex of a K_3 and an arbitrary vertex of H (see Figure 1). Then for $t = \min\{\frac{5}{9}, 1/m_2(H)\}$ we have*

$$\lim_{n \rightarrow \infty} \Pr[G_{n,p} \text{ is Maker's win in the } H_P\text{-game}] = \begin{cases} 1, & p \gg n^{-t}, \\ 0, & p \ll n^{-t}. \end{cases}$$

In particular, Theorem 4 shows that there exists an infinite class of graphs for which the threshold is not determined by the 2-densest subgraph.

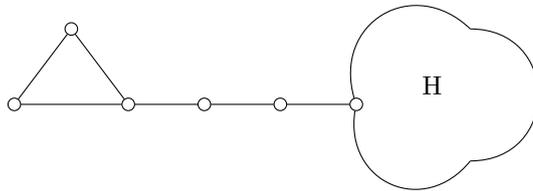


FIGURE 1. Graph H_P

For H a tree we have $m_2(H) = 1$, and the following was shown in [17].

Theorem 5 (Lemma 36 in [17]). *Let T be a tree.*

- (i) *There exists a tree T' such that Maker can win the T -game played on the edge set of T' .*
- (ii) *If \widehat{T} is a tree of minimal size such that Maker can win the T -game played on the edge set of \widehat{T} , then the threshold probability for Maker's win on $G_{n,p}$ is $p_0(n, T) = n^{-\frac{e(\widehat{T})+1}{e(\widehat{T})}}$.*

When H is a forest, the threshold probability is equal to the threshold probability for the game in which Maker's goal is to claim the tree of largest size in the forest H .

Finally, we mention the following paradigm in positional game theory, commonly referred to as the *probabilistic intuition* and attributed to Paul Erdős. As it turns out, for many standard games \mathcal{F} on graphs the inverse of the threshold bias b_0 in the biased game played on the complete graph K_n is “closely related” to the threshold for the *appearance* of a member of \mathcal{F} in $G_{n,p}$.

For certain games, such as the connectivity game and the Hamilton cycle game (where Maker's goal is to claim a spanning tree and a Hamilton cycle, respectively), the probabilistic intuition indeed gives the right answer as the leading term of both parameters is equal to $\frac{\log n}{n}$ (see, e.g., [6]). On the other hand, the probabilistic intuition does not hold for the H -game, in case H is a non-trivial fixed graph. It turns out that in the H -game, at least for graphs satisfying the conditions of Theorem 2, the inverse of the threshold bias instead corresponds to the threshold for Maker's win in $G_{n,p}$.

It is tempting to suggest that this is partly due to the “global” and “local” nature of these games. In the case of connectivity or Hamilton cycle game, the goal is to build a structure which grows with the size of the board graph. As a consequence, the desired structures overlap so heavily in $G_{n,p}$ as soon as we are past the threshold for the appearance, that playing randomly implicitly uses this property. On the other hand, in the case of the H -game the goal remains fixed regardless of the board graph. But now there is a gap between the threshold for the appearance and the point when the copies start to overlap, thus the discrepancy between the threshold for the appearance and the inverse of the threshold bias.

It is worth mentioning that there is even more disagreement for H being a triangle, as here all three parameters (inverse of the threshold bias in the game on K_n , the threshold for appearance of H in $G_{n,p}$, and the threshold for Maker's win on $G_{n,p}$) are of different order – they are, respectively, $n^{-\frac{1}{2}}$, n^{-1} and $n^{-\frac{5}{9}}$. A similar fact holds for trees, where again all three parameters are of different order.

Our paper is structured as follows. In the next section we collect some preliminaries. Then, in Sections 3-5 we prove Theorem 2, while in Section 6 we prove Theorem 4.

2. PRELIMINARIES

In this section we collect some known properties about positional games, graph decompositions and random graphs. We follow the standard notation. In particular, for a graph G and a subset $A \subseteq V(G)$, we denote with $N_G(A)$ the neighborhood of A in $V(G) \setminus A$,

$$N_G(A) := \{v \in V(G) \setminus A \mid \exists a \in A \text{ such that } \{v, a\} \in E(G)\}.$$

If the graph G is clear from the context, we omit it in the subscript. Furthermore, for a graph G we use v_G and e_G to denote the number of vertices and edges of G , respectively. We denote with $\delta(G)$ the smallest degree in G and with $\chi(G)$ the chromatic number of G .

2.1. Positional games. For a Maker-Breaker game with board X and winning sets \mathcal{E} , the hypergraph (X, \mathcal{E}) is referred to as the *hypergraph of the game*. The following is a classical result in the theory of positional games.

Theorem 6 (Erdős-Selfridge criterion [5]). *Let (X, \mathcal{E}) be a hypergraph. Then, if Breaker has the first move in the game,*

$$\sum_{A \in \mathcal{E}} 2^{-|A|} < 1 \quad (1)$$

is a sufficient condition for Breaker's win in the game (X, \mathcal{E}) .

The following theorem appears in [7], stated in a more general form and using different terminology.

Theorem 7 (Theorem 1.8 in [7]). *Let $T_1 = (V, E_1)$ and $T_2 = (V, E_2)$ be two edge disjoint trees on the same vertex set V . If two players alternately claim unclaimed edges from $E_1 \cup E_2$, the second player can enforce that (V, E_f) is a tree, where E_f are the edges claimed by the first player.*

As a simple corollary of Theorem 7, we obtain that the second player can enforce the first player to build a forest, in the game played on the union of two disjoint forests.

Lemma 8. *Let $F_1 = (V, E_1)$ and $F_2 = (V, E_2)$ be two edge disjoint forests on the same vertex set V . If two players alternately claim unclaimed edges from $E_1 \cup E_2$, the second player can enforce that the edges of the first player span a forest.*

Proof. It is easy to see that by adding an extra vertex v to V , there exist disjoint edge sets $E'_1 \supseteq E_1$ and $E'_2 \supseteq E_2$ such that $T_1 = (V \cup \{v\}, E'_1)$ and $T_2 = (V \cup \{v\}, E'_2)$ are trees. By Theorem 7, the second player can enforce that $(V \cup \{v\}, E_f)$ is a tree (and, hence, contains no cycle) in the game played on the edge set $E'_1 \cup E'_2$, where E_f is the set of the edges claimed by the first player. Now, the second player can use this winning strategy to play on $E_1 \cup E_2$, by simply ignoring his moves that do not belong to $E_1 \cup E_2$ (and playing arbitrary moves instead). This way, the first player will not be able to claim a cycle, which means that at the end of the game his graph will be a forest. \square

We make use of Lemma 8 in the proof of Theorem 19, which is one of the main ingredients in the proof of the 0-statement of Theorem 2.

2.2. Graph decompositions. We make use of the following statements in the proof of Theorem 19.

Theorem 9 (Nash-Williams' arboricity theorem [11]). *Any graph G can be decomposed into $\lceil ar(G) \rceil$ edge-disjoint forests, where*

$$ar(G) = \max_{G' \subseteq G} \frac{e_{G'}}{v_{G'} - 1}.$$

The next lemma follows immediately from Hall's theorem. For convenience of the reader we add its short proof.

Lemma 10. *The edges of any graph G can be oriented such that the maximal outdegree is at most $\lceil m(G) \rceil$, where*

$$m(G) = \max_{G' \subseteq G} \frac{e_{G'}}{v_{G'}}.$$

Proof. Let $k := \lceil m(G) \rceil$. We construct a bipartite graph \hat{G} as follows. One vertex class consists of all edges of G (class P_e) and the other of k copies of each vertex of G (class P_v). Furthermore, we add an edge between edge e and a vertex v if and only if v is an endpoint of e in G . It follows immediately from the definition of $m(G)$ and the construction that \hat{G} satisfies Hall's condition with respect to the class P_e . Thus, \hat{G} contains a matching M that covers the set P_e . Orient an edge $e = \{v, u\}$ of G towards u if $\{e, v\}$ belongs to M (for some copy of v in P_v). Since each vertex appears only k times in P_v , we deduce from the construction that the out-degree of each vertex is bounded by k . Since M covers P_e , this process describes the orientation of every edge. \square

2.3. Hypergraph containers. For the proof of the 1-statement of Theorem 2, we need the following consequence of the container theorems of Balogh, Morris, and Samotij [1] and Saxton and Thomason [16]. The result that we use is from [16]. A similar statement is obtained in [1] for all 2-balanced graphs H .

Definition 11. For a given set S , let $\mathcal{T}_{k,s}(S)$ be the family of k -tuples of subsets of S defined as follows,

$$\mathcal{T}_{k,s}(S) := \left\{ (S_1, \dots, S_k) \mid S_i \subseteq S \text{ for } 1 \leq i \leq k \text{ and } \left| \bigcup_{i=1}^k S_i \right| \leq s \right\}.$$

Theorem 12 (Theorem 2.3 in [16]). *Let H be a graph with $e_H \geq 2$ and let $\varepsilon > 0$ be a constant. Then there exist constants $n_0, k \in \mathbb{N}$ such that the following is true. For every $n \geq n_0$ there exists $t = t(n)$, pairwise distinct tuples $T_1, \dots, T_t \in \mathcal{T}_{k, kn^{2-1/m_2(H)}}(E(K_n))$ and sets $C_1, \dots, C_t \subseteq E(K_n)$, such that*

- (a) *each C_i contains at most $(1 - \frac{1}{\chi(H)-1} + \varepsilon) \binom{n}{2}$ edges, and*
- (b) *for every H -free graph G on n vertices there exists $1 \leq i \leq t$ such that $T_i \subseteq E(G) \subseteq C_i$. (Here $T_i \subseteq E(G)$ means that all sets contained in T_i are subsets of $E(G)$.)*

2.4. Random graphs.

Theorem 13 (Markov's Inequality). *Let X be a non-negative random variable. Then for all $t > 0$ we have $\Pr[X \geq t] \leq \frac{\mathbb{E}[X]}{t}$.*

Theorem 14 (Chernoff's Inequality). *Let X_1, \dots, X_n be independent Bernoulli distributed random variables with $\Pr[X_i = 1] = p$ and $\Pr[X_i = 0] = 1 - p$. Then for $X = \sum_{i=1}^n X_i$ we have*

$$\Pr[X \leq (1 - \delta)\mathbb{E}[X]] \leq e^{-\mathbb{E}[X]\delta^2/2}, \quad \text{for every } 0 < \delta \leq 1.$$

The following is a standard result from random graph theory. We include its simple proof for convenience of the reader.

Lemma 15. *Let α, c, L be positive constants and assume $p \leq cn^{-1/\alpha}$. Then w.h.p. every subgraph G' of $G_{n,p}$ on at most L vertices has density $m(G') \leq \alpha$.*

Proof. Observe that there exist only constantly many different graphs on L vertices. Let H be one such graph, and choose $\hat{H} \subseteq H$ such that $m(H) = e_{\hat{H}}/v_{\hat{H}}$. Then the expected number of \hat{H} -copies in $G_{n,p}$ is at most $n^{v_{\hat{H}}} p^{e_{\hat{H}}}$. Observe that for $p = cn^{-1/\alpha}$ we have $n^{v_{\hat{H}}} p^{e_{\hat{H}}} = o(1)$ whenever $m(H) = e_{\hat{H}}/v_{\hat{H}} > \alpha$. It thus follows from Markov's inequality that for $p \leq cn^{-1/\alpha}$ w.h.p. there is no \hat{H} -copy, and hence no H -copy in $G_{n,p}$. Therefore, it follows from the union bound that w.h.p. every subgraph G' of $G_{n,p}$ of size $v_{G'} \leq L$ satisfies $m(G') \leq \alpha$. \square

3. PROOF OF THE 1-STATEMENT OF THEOREM 2

We prove that the following, slightly stronger, *resilience*-type version of the 1-statement of Theorem 2 also holds. Such a property will turn out to be useful in the proof of Theorem 4.

Theorem 16. *Let H be a graph and $\varepsilon > 0$ a constant. Then there exist a positive constant C such that $G \sim G_{n,p}$ with probability $1 - e^{-\Omega(n^2p)}$ satisfies the following, provided that $p \geq Cn^{-1/m_2(H)}$: there exists a winning strategy for Maker in the H -game played on $E(G) \setminus R$, for any $R \subseteq E(G)$ with $e_R \leq \left(\frac{1}{\chi(H)-1} - \varepsilon\right) \binom{n}{2}p$.*

Note that Theorem 16 is optimal both with respect to p and the number of edges of R , in the following sense. Observe that for $p \ll n^{-1/m_2(H)}$ we have that the expected number of copies of $H' \subseteq H$, a strictly 2-balanced subgraph of H with $m_2(H') = m_2(H)$, in $G_{n,p}$ is asymptotically smaller than the number of edges. Thus, to make the graph H' -free (and consequently H -free, therefore Breaker's win), it suffices to remove an edge from each copy of H' yielding in total $o(n^2p)$ removed edges. On the other hand, for $p \geq n^{-1/m_2(H)}$, a random graph $G \sim G_{n,p}$ contains w.h.p. a subgraph $R \subseteq G$ with $e_R \leq \left(\frac{1}{\chi(H)-1} + \varepsilon\right) \binom{n}{2}p$ such that $G \setminus R$ is $(\chi(H) - 1)$ -partite (consider an equitable partition of the vertices into $\chi(H) - 1$ parts and remove all the edges lying inside parts). In particular, any subgraph of $G \setminus R$ has a chromatic number at most $\chi(H) - 1$, thus it does not contain H which makes it a trivial Breaker's win.

In the proof we make use of the hypergraph containers, a new tool that seems to have potential for applications in positional games. A simplified version of the containers approach was first utilized in positional game theory under a different name in [9] (referred to as *small hypergraphs with large cover*), where the following observation has been put to good use – if there are two hypergraphs $\mathcal{H}_1 = (X, \mathcal{E}_1)$ and $\mathcal{H}_2 = (X, \mathcal{E}_2)$ such that every cover (set of vertices that intersects every hyper-edge) of \mathcal{H}_1 is also a cover of \mathcal{H}_2 , then a Breaker's win in the game played on \mathcal{H}_1 implies a Breaker's win on \mathcal{H}_2 .

Proof of Theorem 16. Our proof is based on ideas of the proof from [12] of the 1-statement of Theorem 1. Note, however, that here we need to be much more careful: for the proof of Theorem 1 one has to show that *every* coloring contains a monochromatic copy of H in *some* color. Here we have to argue that we can find a *strategy* for Maker that ensures that he gets a monochromatic copy in *his* color. We achieve this by using the Erdős-Selfridge criterion. For simpler notation, throughout the proof we identify the graph with its edge set.

Let k and $t(\cdot)$ be as given by Theorem 12 when applied on the graph H with $\varepsilon/4$, let $\mathcal{T} := \mathcal{T}_{k, kn^{2-1/m_2(H)}}(E(K_n))$ and $G \sim G_{n,p}$ for $p \geq Cn^{-1/m_2(H)}$, where C is yet to be determined.

Consider some subset $R \subseteq G$ with $e_R \leq \left(\frac{1}{\chi(H)-1} - \varepsilon\right) \binom{n}{2}p$. Observe that if Maker loses in the H -game on $G \setminus R$, then by Theorem 12 there exists $1 \leq w \leq t(n)$ such that $T_w \subseteq E_M \subseteq C_w$, where E_M is the set of Maker's edges.

Let us consider an auxiliary game played on the hypergraph $\mathcal{H} = (G \setminus R, \mathcal{E})$, with the vertex set being the edge set of $G \setminus R$ and the edge set

$$\mathcal{E} = \{(K_n \setminus C_i) \cap (G \setminus R) : T_i \subseteq G \setminus R\}.$$

In this game Breaker wins if he claims at least one edge from each set $(K_n \setminus C_i) \cap (G \setminus R)$. Note that, by the previous observation, in case of Breaker's win the edge set of Breaker cannot be H -free. We can thus conclude that Maker has a winning strategy in the H -game if he has a winning strategy (as Breaker) in the auxiliary

game. In light of the Erdős-Selfridge criterion (Theorem 6) it remains to check that the hypergraph $(G \setminus R, \mathcal{E})$ satisfies condition (1).

First, we show that all hyperedges typically have size at least $\frac{\varepsilon}{2} \binom{n}{2} p$. It follows from Theorem 12 applied with $\varepsilon/4$ (as ε) that $|K_n \setminus C_i| \geq \left(\frac{1}{\chi(H)-1} - \varepsilon/4\right) \binom{n}{2}$, and thus from Chernoff's inequality we have

$$\Pr \left[|(K_n \setminus C_i) \cap G| < \left(\frac{1}{\chi(H)-1} - \varepsilon/2\right) \binom{n}{2} p \right] < e^{-\delta^2 \binom{n}{2} p/2}, \quad (2)$$

where $\delta = \frac{\varepsilon}{4} \cdot \frac{1}{1/(\chi(H)-1) - \varepsilon/4}$, for every fixed $1 \leq i \leq t(n)$. Let \mathcal{B} be the event that there exists a hyperedge which has less than $\left(\frac{1}{\chi(H)-1} - \varepsilon/2\right) \binom{n}{2} p$ vertices "before" the removal of R , i.e.

$$\mathcal{B} := \exists T_i \subseteq G : |(K_n \setminus C_i) \cap G| < \left(\frac{1}{\chi(H)-1} - \varepsilon/2\right) \binom{n}{2} p.$$

Then

$$\Pr[\mathcal{B}] \leq \sum_{i=1}^{t(n)} \Pr \left[T_i \subseteq G \wedge |(K_n \setminus C_i) \cap G| < \left(\frac{1}{\chi(H)-1} - \varepsilon/2\right) \binom{n}{2} p \right].$$

As $T_i \subseteq C_i$, the two events are independent, thus we obtain

$$\begin{aligned} \Pr[\mathcal{B}] &\leq \sum_{i=1}^{t(n)} \Pr[T_i \subseteq G] \cdot \Pr \left[|(K_n \setminus C_i) \cap G| < \left(\frac{1}{\chi(H)-1} - \varepsilon/2\right) \binom{n}{2} p \right] \\ &\stackrel{(2)}{\leq} e^{-\delta^2 \binom{n}{2} p/2} \cdot \sum_{i=1}^{t(n)} p^{|T_i^+|} \leq e^{-\delta^2 \binom{n}{2} p/2} \cdot \sum_{T \in \mathcal{T}} p^{|T^+|}, \end{aligned}$$

where T^+ is the union of all sets of the k -tuple T . Note that the last inequality follows from the fact that all k -tuples T_i given by Theorem 12 are distinct. The sum can now be upper-bounded by first deciding on $s := |T^+|$, which is by the definition of \mathcal{T} at most $kn^{2-1/m_2(H)} \leq kn^2 p/C$, then choosing s edges (at most $\binom{n^2}{s}$ choices) and finally deciding to which sets of the k -tuple each chosen edge belongs (at most $(2^k)^s$ choices). Hence, we have

$$\sum_{T \in \mathcal{T}} p^{|T^+|} \leq \sum_{s=1}^{kn^2 p/C} \binom{n^2}{s} 2^{ks} p^s \leq \sum_{s=1}^{kn^2 p/C} \left(\frac{en^2 2^k p}{s} \right)^s.$$

Let $\gamma := \min\{\delta^2/4, \varepsilon/8\}$, and note that we can choose $C := C(k, \gamma)$ such that

$$\sum_{T \in \mathcal{T}} p^{|T^+|} \leq n^2 \left(\frac{Ce2^k n^2 p}{kn^2 p} \right)^{kn^2 p/C} \leq 2^{\gamma \binom{n}{2} p}. \quad (3)$$

Finally, from the choice of γ we conclude $\Pr[\mathcal{B}] = e^{-\Omega(n^2 p)}$. It now easily follows that

$$\Pr \left[\exists A \in \mathcal{E} : |A| < \frac{\varepsilon}{2} \binom{n}{2} p \right] \leq \Pr[\mathcal{B}] = e^{-\Omega(n^2 p)},$$

regardless of the choice of R (recall that $e_R \leq \left(\frac{1}{\chi(H)-1} - \varepsilon\right) \binom{n}{2} p$).

Furthermore, the expected number of hyperedges is upper-bounded by

$$\mathbb{E}[|\mathcal{E}|] \leq \sum_{i=1}^{t(n)} \Pr[T_i \subseteq G] = \sum_{i=1}^{t(n)} p^{|T_i^+|} \stackrel{(3)}{\leq} 2^{\varepsilon \binom{n}{2} p/8},$$

thus by Markov's inequality we get

$$\Pr[|\mathcal{E}| \geq 2^{\varepsilon \binom{n}{2} p/4}] \leq 2^{-\varepsilon \binom{n}{2} p/8}.$$

Therefore, with probability $1 - e^{-\Omega(n^2 p)}$, G is such that

$$\sum_{A \in \mathcal{E}} 2^{-|A|} \leq 2^{-\varepsilon \binom{n}{2} p/2 + \varepsilon \binom{n}{2} p/4} < 1.$$

By the Erdős-Selfridge criterion, Breaker has a winning strategy in the auxiliary game, hence by the previous discussion Maker has a winning strategy in the H -game played on $G \setminus R$. \square

As noticed by Andrzej Ruciński (personal communication), a weaker version of Theorem 16 (in particular, the upper-bound on the number of edges of R is of the form $\delta n^2 p$ for some $\delta > 0$) follows directly from the strategy stealing argument, high probability of the event $G_{n,p} \rightarrow (H)_r^c$ and Lemma 2.52 from [8]. We also note that a more direct proof of such a weaker version could be obtained by using the approach of derandomized Maker's strategy from [2].

4. CRITERIA FOR BREAKER'S WIN IN AN H -GAME

In this section we collect some graph properties that suffice for characterizing the graph as a Breaker's win in an H -game. These will be used later in the proof of the 0-statement of Theorem 2.

Proposition 17. *Let G and H be graphs such that*

$$\left\lceil \frac{ar(G)}{2} \right\rceil < ar(H),$$

then Breaker can win the H -game played on the edge set of G , even if Maker starts.

Proof. Let $k := \left\lceil \frac{ar(G)}{2} \right\rceil$, and let F_0, \dots, F_{2k-1} be an edge-disjoint decomposition of G into forests which exists by Theorem 9. Assume Breaker uses the strategy from Lemma 8 for every pair of forests F_{2i} and F_{2i+1} , $0 \leq i < k$. Then Lemma 8 implies that Maker's edges can be partitioned into k forests. Any subset S of the vertex set can thus contain at most $k(|S| - 1)$ Maker's edges. That is, the arboricity value for Maker's edges is at most k and, as $ar(H) > k$ by assumption, Maker's graph cannot contain H . \square

Proposition 18. *Let G and H be graphs such that*

$$\left\lceil \frac{m(G)}{2} \right\rceil < m(H),$$

then Breaker can win the H -game played on the edge set of G , even if Maker starts.

Proof. Let us fix any orientation of the edges of G such that each vertex has out-degree at most $\lceil m(G) \rceil$. Such an orientation exists by Lemma 10. Now Breaker can play according to the following simple pairing strategy. For each vertex, he will arbitrarily pair up all outgoing edges (except possibly one, if their number is odd). Now throughout the game, whenever Maker claims one edge of a pair, Breaker responds by claiming the other one. Otherwise, he plays arbitrarily.

It is clear that this way Breaker will claim at least half of the paired outgoing edges of each vertex. In other words, the out-degree of each vertex, with respect to Maker's edges, is at most $\left\lceil \frac{\lceil m(G) \rceil}{2} \right\rceil = \left\lceil \frac{m(G)}{2} \right\rceil$. Therefore, by the condition of the proposition, the density of each subgraph of Maker's graph is less than $m(H)$, and thus it cannot contain H as a subgraph. \square

With these two basic criteria at hand we can now prove the main theorem of this section.

Theorem 19. *Let G and H be graphs such that $m(G) \leq m_2(H)$ and H is strictly 2-balanced and contains at least 4 vertices. Then Breaker has a winning strategy for the H -game on the edge set of G .*

Proof. Since H is strictly 2-balanced graph on at least 4 vertices, it is easy to check that H has to contain a cycle and thus $m_2(H) > 1$. Let $m_2(H) = k + x$, for some $k \in \mathbb{N}$ and $0 \leq x < 1$. We first handle the case when $0 \leq x < 1/2$.

Since H is a strictly 2-balanced, by removing a vertex of the smallest degree in H we obtain

$$m_2(H) = \frac{e_H - 1}{v_H - 2} > \frac{e_H - \delta(H) - 1}{v_H - 3},$$

which easily (estimating $\delta(H)$ from the last inequality) implies $m_2(H) < \delta(H)$. For the sake of contradiction, let G be the smallest graph such that Maker has a winning strategy. We first deduce that then $\delta(G) \geq 2(\delta(H) - 1) + 1$. Assuming otherwise, let v be a vertex of degree at most $2(\delta(H) - 1)$. Then Breaker has the following winning strategy: whenever Maker claims an edge incident to v , Breaker does the same (if possible). If on the other hand Maker claims an edge from $G - \{v\}$, then Breaker follows his winning strategy for $G - \{v\}$ (which exists by choice of G). Then, clearly, Maker cannot build a copy of H in $G - \{v\}$. Further, the degree of v in Maker's graph is at most $\delta(H) - 1$, thus it cannot be part of an H -copy either. Hence $\delta(G) \geq 2(\delta(H) - 1) + 1$. We can now lower-bound $m(G)$ as

$$m(G) \geq \frac{\sum_{v \in G} \deg(v)}{2n} \geq \delta(H) - 1/2.$$

It follows from $m_2(H) < \delta(H)$ that $\delta(H) \geq k + 1$ and thus $m(G) \geq k + 1/2$, which is a contradiction to $m(G) \leq m_2(H) < k + 1/2$.

From now on we can thus assume that $x \geq 1/2$. Next, we consider the case that $k \geq 3$. Observe that for every graph H on at least 3 vertices we have $\frac{3}{4}v_H^2 - v_H > \binom{v_H}{2} \geq e_H$, and thus

$$\frac{e_H}{v_H} + 3/2 > \frac{e_H - 1}{v_H - 2}. \quad (4)$$

Therefore $m(H) > m_2(H) - 3/2 \geq k - 1$, and so we have

$$\lceil m(G)/2 \rceil \leq \lceil (k + 1)/2 \rceil \stackrel{(k \geq 3)}{\leq} k - 1 < m(H).$$

Breaker's win now follows from Proposition 18.

If H is not very dense, then a better estimate than the one in (4) can be made. In particular, $e_H < v_H^2/4$ implies that $\frac{e_H}{v_H} + 1/2 > \frac{e_H - 1}{v_H - 2}$. Since we also assumed that $x \geq 1/2$, this implies $m(H) > m_2(H) - 1/2 \geq k$. Similarly as before we have

$$\lceil m(G)/2 \rceil \leq \lceil (k + 1)/2 \rceil \stackrel{(k \geq 1)}{\leq} k < m(H),$$

and Breaker's win again follows from Proposition 18.

To summarize, so far we have shown that Breaker has a winning strategy for the H -game on graph G if one of the following holds,

- (a) $0 \leq x < 1/2$,
- (b) $k \geq 3$, or
- (c) $e_H < v_H^2/4$.

Let us consider a graph H which does not satisfy any of the above properties. Then $e_H \geq \lceil v_H^2/4 \rceil$ and thus

$$m_2(H) = \frac{e_H - 1}{v_H - 2} \geq \frac{\lceil v_H^2/4 \rceil - 1}{v_H - 2} \geq 2$$

for $v_H \geq 5$, and since H does not satisfy (a) and (b) we have $2.5 \leq m_2(H) < 3$. Furthermore, it is easy to check that $ar(G) \leq m(G) + 1/2$, and thus $ar(G) \leq m(G) + 1/2 \leq m_2(H) + 1/2 < 4$. On the other hand, from $m_2(H) \geq 2.5$ we have $e_H \geq \frac{5}{2}v_H - 4$, and thus $e_H > 2v_H - 2$ for $v_H \geq 5$, which implies $ar(H) > 2$. It follows now from $\lceil ar(G)/2 \rceil \leq 2 < ar(H)$ and Proposition 17 that Breaker has a winning strategy in this case.

Finally, checking all graphs on 4 vertices we see that the only strictly 2-balanced graphs are K_4 and C_4 . The case $H = K_4$ is covered by Lemma 2.1 in [10]. For $H = C_4$ we have $ar(H) = 4/3$ and $ar(G) \leq m(G) + 1/2 \leq 2$, thus Proposition 17 implies that Breaker has a winning strategy also in this case. \square

5. PROOF OF THE 0-STATEMENT OF THEOREM 2

We need to show that with high probability Breaker has a strategy such that, when played on the random graph $G_{n,p}$ with $p = cn^{-1/m_2(H)}$, for $0 < c = c(H) < 1$ small enough, Maker's edges do not span an H -copy. Observe that we may assume, without loss of generality, that H is strictly 2-balanced. If not, replace H by a minimal subgraph H' with the same 2-density. Clearly, if Breaker has a strategy for winning the H' -game on $G_{n,p}$, then the same strategy prevents Maker from obtaining an H -copy.

Let us first give an intuition behind the Breaker's strategy. Observe that the expected number of copies of H on any given edge is bounded by

$$v_H^2 \cdot n^{v_H-2} \cdot p^{e_H-1} = v_H^2 \cdot c^{e_H-1}.$$

That is, for $0 < c < 1$ small enough we expect that the copies of H are scattered 'loosely' and that we even have many edges that are not contained in any copy of H . Clearly, whether such edges are claimed by Maker or Breaker is irrelevant for the outcome of the game. Assume now we find a copy of H that contains two edges which are not contained in any other copy of H . Then Breaker can easily ensure that this H -copy will never be claimed by Maker: fix two such edges arbitrarily and as soon as Maker claims the first of these edges, claim the other edge. Clearly, in this way this specific H -copy will never be a Maker's copy. We formalize these ideas as follows.

Definition 20. We call an edge *free* if it does not belong to any copy of H , *open* if it is contained in exactly one copy of H and *closed* otherwise. Furthermore, we call a copy of H *unproblematic* if it contains at least two open edges. Otherwise we call the copy *problematic*.

Preprocessing. Before starting the game, Breaker preprocesses the graph $G \sim G_{n,p}$ to obtain a subgraph \hat{G} (with some special properties that we exhibit below) and a sequence of pairwise disjoint sets of edges S_1, \dots, S_k of cardinality two each:

```

 $k := 0; G_k := G;$ 
while there exists an unproblematic  $H$ -copy  $\hat{H}$  in  $G_k$ 
   $k \leftarrow k + 1;$ 
  let  $S_k \leftarrow \{ \text{two open edges (chosen arbitrarily) of } \hat{H} \};$ 
   $G_k \leftarrow G_{k-1} - \{ \text{all open edges of } \hat{H} \};$ 
while there exists a free edge  $e \in G_k$ 
   $k \leftarrow k + 1;$ 
   $G_k \leftarrow G_{k-1} - e;$ 
 $\hat{G} \leftarrow G_k$ 

```

Note that, within this algorithm, open, free and closed edges are always defined with respect to the current graph G_k .

Strategy. Assuming that Breaker has a winning strategy for the H -game when played on \hat{G} , the winning strategy for the whole graph G is defined as follows:

if Maker claims an edge from \hat{G}
 claim an edge from \hat{G} according to the winning strategy for \hat{G} ;
else if Maker claims an edge from a set S_j for some $1 \leq j \leq k$
 claim the other edge from the set S_j ;
else
 take an arbitrary edge.

We first show that this strategy extends a winning strategy for \hat{G} to a winning strategy for the whole graph.

Claim 21. *Assuming that Breaker has a winning strategy for the H -game on \hat{G} , Breaker claims at least one edge from every copy of H in G .*

Proof. First, consider an H -copy \hat{H} which is contained in \hat{G} . Since Breaker is playing according to the winning strategy on \hat{G} , it follows that this copy has to contain at least one edge which belongs to Breaker. Secondly, consider an H -copy \hat{H} which is contained in G_i but not in G_{i+1} , for some $1 \leq i < k$. It follows from the construction of S_i that $S_i \subset \hat{H}$, and since Breaker claims at least one edge from S_i , he also claims at least one edge from \hat{H} . \square

It remains to show that there exists a winning strategy for \hat{G} . In order to state the argument concisely, we introduce some notation.

Definition 22. An H -core of G is a maximal subgraph $G' \subseteq G$ (with respect to subgraph inclusion) that has the following two properties: every edge of G' is contained in at least one copy of H and every copy of H in G' is problematic.

Recall that, by construction, \hat{G} is an H -core. The following claim shows that it is the unique H -core.

Claim 23. *There exists a unique H -core.*

Proof. Let us assume that there exist two different H -cores, say G' and G'' . Then $G' \not\subseteq G''$ and $G'' \not\subseteq G'$, so $G_s = G' \cup G''$ is a proper superset of G' and G'' . Therefore, to reach a contradiction to the maximality of G' and G'' it suffices to show that G_s is an H -core.

First, it is easy to see that every edge of G_s is contained in at least one copy of H . Further, observe that every H -copy which is problematic in G' or G'' remains problematic in G_s as well. Thus, if an H -copy in G_s is unproblematic then it cannot be contained in G' nor in G'' . Consider such an H -copy \hat{H} and an arbitrary edge $e \in \hat{H}$. Then e is contained in at least one of G' and G'' and thus, by the definition of G' and G'' , e is also contained in a copy of H different from \hat{H} . Therefore e is closed in G_s , and since this holds for every edge $e \in \hat{H}$ we conclude that \hat{H} is problematic implying that G_s is an H -core. \square

We say that a subgraph G' of the H -core of G is H -closed if every copy of H from the H -core is either contained in G' or edge-disjoint with G' . It is easy to see that the edges of the H -core can be partitioned into minimal H -closed subgraphs (where minimal is with respect to subgraph inclusion).

Claim 24. *Every two distinct minimal H -closed subgraphs are edge disjoint.*

Proof. Let us assume that there exist two distinct minimal H -closed subgraphs of the H -core, say G' and G'' , such that $E(G') \cap E(G'') \neq \emptyset$. Let $G_0 = G' \cap G''$ and note that G_0 is a proper subset of both G' and G'' .

First, by the minimality of G' we have that G_0 is not H -closed. Since G' is H -closed this implies that there exists an H -copy in G' , let us denote it with H' , such that $H' \not\subseteq G_0$ and $H' \not\subseteq G' \setminus G_0$. However, such an H -copy then contains an edge from G'' but is not contained in G'' , thus violating the assumption that G'' is H -closed. \square

Using the previous claim, we conclude that Breaker can play an independent H -game on each minimal H -closed subgraph.

The core of our argument is the following lemma which states that w.h.p. every minimal H -closed subgraph in the H -core of $G_{n,p}$ has constant size.

Lemma 25. *Let H be a strictly 2-balanced graph which is not a tree or a triangle. Then there exist constants $c > 0$ and $L > 0$ such that w.h.p. every minimal H -closed subgraph of the H -core of $G_{n,p}$ has size at most L , provided that $p \leq cn^{-1/m_2(H)}$.*

Before we prove Lemma 25, we first show how it implies the 0-statement of Theorem 2.

Proof of the 0-statement of Theorem 2. Let $G \sim G_{n,p}$, and let Breaker play as described. Recall that, by Claim 21, it suffices to show that there exists a winning strategy for the H -core \hat{G} of G . Furthermore, by Claim 24 we only have to find a winning strategy for all minimal H -closed subgraphs of the H -core.

From Lemma 25 we know that w.h.p. the graph G is such that all minimal H -closed subgraphs have size at most $L = L(H)$. From Lemma 15 we know that w.h.p. the graph G is such that this implies that all minimal H -closed subgraphs have density at most $m_2(H)$. Theorem 19 implies that there exists a winning strategy for Breaker for all minimal H -closed subgraphs – and thus also for the H -core \hat{G} . \square

It remains to prove Lemma 25. We do this in the remainder of this section. Actually, our proof of Lemma 25 follows the proof of Lemma 3.1 from [12]. The main difference is that in [12] a problematic copy of H was defined as a copy of H in which *all* edges are contained in two copies of H , while the definition in this paper allows the existence of one (but only one) edge that may be open. As we shall see, this difference in definition is responsible for the fact that the proof goes through for triangles in [12], but does not here. Of course, this is no coincidence: for the Random Ramsey result that was considered in [12], the threshold for triangles is of order $n^{-1/m_2(K_3)} = n^{-1/2}$ (Theorem 1), while for the Maker-Breaker game considered in this paper the threshold for triangles is $n^{-5/9}$ (Theorem 3). In the following we repeat the main arguments from [12], for the convenience of the reader.

We define a process that generates H -closed structures iteratively starting from a single copy of H . We fix an (arbitrary) total ordering ω of the edges of $G_{n,p}$, and let G' be a minimal H -closed subgraph of the H -core of $G_{n,p}$. Then G' can be generated by starting with an H -copy in G' and repeatedly attaching H -copies to the graph constructed so far, as described in the following procedure.

Let H_0 be an H -copy in G' ,
 $k \leftarrow 0$; $G'_0 \leftarrow H_0$;
while $G'_k \neq G'$ **do**
 if G'_k contains a copy of H that is unproblematic in G'_k **then**
 let $\ell \leq k$ be the smallest index such that H_ℓ is
 a copy of H that is unproblematic in G'_k ;
 let e be the ω -minimum edge in H_ℓ which
 is open in G'_k and closed in G' ;
 let H_{k+1} be an H -copy in G' that contains e but is

not contained in G'_k ;

else

let H_{k+1} be an H -copy in G' that is not contained
in G'_k and intersects G'_k in at least one edge;

$G'_{k+1} \leftarrow G'_k \cup H_{k+1}$;

$k \leftarrow k + 1$;

To see that the process is well-defined it is enough to observe that every H -copy \hat{H} in G'_k , different from H_0, \dots, H_k , is problematic in G'_k : the construction implies that every edge of such H -copy \hat{H} belongs to some H_i , for $i \in \{0, \dots, k\}$, thus \hat{H} is problematic by the definition.

We need to show that w.h.p. the highest value that the parameter k reaches is bounded by a constant. We first collect some properties of this process. Consider some H -copy H_i , for $i \geq 1$. We distinguish two cases: a) if H_i intersects G'_{i-1} in *exactly* one edge and no other vertices, we call this a *regular* copy, and b) if H_i intersects G'_{i-1} in some subgraph D with $v_D \geq 3$, we call this a *degenerate* copy. Let us denote with $\text{reg}(\ell)$ and $\text{deg}(\ell)$ the number of H -copies H_i , $1 \leq i \leq \ell$, which are regular, resp. degenerate. Furthermore, for $0 \leq i \leq \ell$ we say that the copy H_i is *fully-open at time* ℓ if H_i is a regular copy and no new vertex of H_i , i.e. no vertex of $V(H_i) \setminus V(G'_{i-1})$, is touched by any of the copies H_{i+1}, \dots, H_ℓ . Let us denote with $f_o(\ell)$ the number of fully-open copies at time ℓ .

With these definitions, we are ready to give an overview of the proof. First, we restrict our attention only to the first $\Theta(\log n)$ H -copies chosen by the process. In particular, if a sequence (H_0, \dots, H_k) that generates G' is longer than $k_0 = \Theta(\log n)$, we consider only the subsequence (H_0, \dots, H_{k_0}) . Clearly, if such a subsequence of graphs does not appear in $G_{n,p}$, then the whole sequence does not appear as well. Using a union-bound argument and the first-moment method, we show that w.h.p. no sequence of length at most k_0 (regardless of whether it is the whole sequence or a subsequence) contains more than ξ degenerate copies, for some constant ξ that we determine below. As we will see, this implies that either the whole sequence is of constant length (in which case we are done), or we are in the regime where a subsequence of length k_0 is considered. However, again using a union-bound argument we show that every sequence of length k_0 with at most ξ many degenerate steps is unlikely to appear in $G_{n,p}$, which finishes the proof. We now make these ideas precise.

The following lemma implies that every fully-open copy contains exactly $e_H - 1$ open edges.

Lemma 26 (Lemma 3.4 in [12]). *Let H be a strictly 2-balanced graph and let G be an arbitrary graph. Construct a graph G_H by attaching an H -copy H_e to an edge h_e of G . Then G_H has the property that if \hat{H} is an H -copy in G_H that contains at least one vertex from $V(H_e) \setminus V(h_e)$, then $\hat{H} = H_e$.*

For $\ell \geq 1$, let

$$\Delta(\ell) := |\{i < \ell : H_i \text{ fully-open at time } \ell - 1, \text{ but not at time } \ell\}|.$$

Clearly, a regular copy can ‘destroy’ at most one fully open copy, thus $\Delta(\ell) \leq 1$ if H_ℓ is a regular copy. On the other hand, a degenerate copy intersects one copy H_i in an edge and may touch up to $v_H - 2$ internal vertices of fully-open copies, thus $\Delta(\ell) \leq v_H - 1$ if H_ℓ is a degenerate copy. The following claim is from [12] (Claim 3.5); the only difference is that we here have $e_H - 3$ while in [12] we had $e_H - 2$. (This difference comes from the fact that we now allow one open edge.)

Claim 27. *For any sequence H_i, \dots, H_{i+e_H-3} of consecutive regular copies such that $\Delta(i) = 1$ we have $\Delta(i+1) = \dots = \Delta(i+e_H-3) = 0$.* \square

Similarly, the next claim is proven exactly as Claim 3.6 in [12], with $e_H - 1$ (there) replaced by $e_H - 2$ (here).

Claim 28. *For every $\ell \geq 1$, assuming the process does not stop before adding the ℓ -th copy, we have*

$$f_o(\ell) \geq \text{reg}(\ell) \left(1 - \frac{1}{e_H - 2}\right) - \text{deg}(\ell) \cdot v_H.$$

□

Observe that this bound on $f_o(\ell)$ is only meaningful if $e_H \geq 4$. This is the reason why the proof does not go through for the case of triangles.

If $f_o(\ell) > 0$ for some $\ell \geq 1$, then H_ℓ cannot be the last copy in the process, as there exists at least one H -copy with at least $e_H - 1 \geq 2$ open edges, which cannot be by the definition of the H -core. Furthermore, from Claim 28 we have that after adding L copies, out of which at most ξ are degenerate, there are still at least

$$(L - \xi)(1 - 1/(e_H - 2)) - \xi \cdot v_H \quad (5)$$

fully-open copies at time L . Given some $\xi > 0$ (that we choose below) we can then fix $L = L(\xi, H)$ such that the term in (5) is positive. Note that this implies that $f_0(L) > 0$ for any sequence (H_0, \dots, H_L) having at most ξ degenerate steps.

In a first moment calculation that upper bounds the expected number of sequences (H_0, \dots, H_k) , we have to multiply the number of choices for H_i with the probability that the chosen H -copy is in $G_{n,p}$. For a regular copy where H_i is attached to an open edge, the open edge to which it is attached is given deterministically by the design of our algorithm, provided that $f_o(i-1) > 0$. We just have to choose the edge (and orientation) in the new copy that we attach to it. Thus, this term is bounded by

$$2e_H \cdot n^{v_H-2} \cdot p^{e_H-1} \leq 2e_H \cdot c < \frac{1}{2}, \quad (6)$$

for $0 < c < 1/(4e_H)$. For a regular copy H_i that is either attached to a closed edge or to an open edge and $f_o(i-1) = 0$, the edge to which we attach the regular copy might not be given deterministically so we have to choose an edge to which we attach H_i , which we can do in at most $i \cdot e_H$ ways.

To bound the term for degenerate copies one first easily checks (see inequality 3.3 in [12]) that there exists an $\alpha > 0$ such that

$$(v_H - v_J) - \frac{e_H - e_J}{m_2(H)} < -\alpha, \quad \text{for all } J \subsetneq H \text{ with } v_J \geq 3.$$

Thus, we can bound the case that the copy H_i is a degenerate copy by

$$\sum_{J \subsetneq H, v_J \geq 3} (i \cdot v_H)^{v_J} \cdot n^{v_H - v_J} \cdot p^{e_H - e_J} < (i \cdot v_H \cdot 2^{e_H})^{v_H} \cdot n^{-\alpha}, \quad (7)$$

with room to spare. Given α as above we fix $\xi = \xi(\alpha, H)$ such that $\alpha \cdot \xi > v_H + 1$.

With these preparations at hand we can now finish the proof exactly as in [12].

First, we show that for $k_0 = (v_H + 1) \log n$ the expected number of sequences $(H_0, \dots, H_{k'})$, $k' \leq k_0$, that contain exactly ξ many degenerate copies and the ξ -th degenerate copy is $H_{k'}$, vanishes. We can choose the time of the appearance of the previous $\xi - 1$ degenerate copies in at most $\binom{k'-1}{\xi-1}$ ways. The copy H_0 can be chosen in at most n^{v_H} ways. Furthermore, by the earlier observation, for each regular copy that occurs before step L we can upper-bound the contributing term by $L \cdot e_H/2$, while for every regular copy that occurs after step L we have a factor of $1/2$. Finally, using the estimate in (7) for the degenerate steps, we obtain the

following upper-bound on the expected number of such sequences,

$$\sum_{k' \leq k_0} \binom{k' - 1}{\xi - 1} n^{v_H} [(k_0 \cdot v_H \cdot 2^{e_H})^{v_H} \cdot n^{-\alpha}]^\xi (L \cdot e_H)^L \cdot 2^{-(k' - \xi - L)} = o(1),$$

owing to the choice of ξ . Therefore, w.h.p. no such sequence appears in $G_{n,p}$.

We can assume now that a sequence which generates G' has a property that among the first k_0 copies there are at most ξ degenerate ones. Thus, by the choice of L , for every $L \leq k' \leq k_0$ we have $f_0(k') > 0$. This implies that the sequence either has length at most L or length at least k_0 . It remains to show that the latter is unlikely to happen.

We can easily upper bound the expected number of sequences of length at least k_0 as follows:

$$\sum_{\xi' < \xi} \binom{k_0}{\xi'} n^{v_H} \cdot (L \cdot e_H)^L \cdot 2^{-(k_0 - L)} \leq n^{v_H} \cdot n^{-(v_H + 1)} \cdot o(n) = o(1),$$

where we used estimates similar to the ones in the previous case, together with the fact that degenerate steps contributes a factor of at most $1/2$ (with room to spare). Therefore, w.h.p. the sequence that generates G' cannot be of length k_0 , thus we are only left with the case that such a sequence has constant length. This finishes the proof of Lemma 25.

6. PROOF OF THEOREM 4

Proof of Theorem 4. If $m_2(H) \geq 2$, then H_P satisfies the condition of Theorem 2, and the conclusion of the theorem trivially follows. Therefore, we can assume that $m_2(H) < 2$.

Assume $p \ll n^{-t}$, where $t = \min\{\frac{5}{9}, 1/m_2(H)\}$. If $t = 5/9$, then by Theorem 3 Breaker can prevent Maker from creating a copy of K_3 , and if $t = 1/m_2(H) < \frac{5}{9}$ then by Theorem 2 Breaker can prevent Maker from creating a copy of H . In any case, there exists a subgraph of H_P which Maker cannot create, thus Breaker wins in the H_P -game.

So, let now $p \gg n^{-t}$ and $G \sim G_{n,p}$. We first show that G satisfies w.h.p. the following properties:

- (P1) for every subset $X \subseteq V(G)$ with $|X| \leq 1/(2p)$, it holds that $|N_G(X)| \geq |X|np/4$,
- (P2) G contains a copy of K_5^- , a complete graph on 5 vertices with one arbitrary edge removed, and
- (P3) for every subset $S \subseteq V(G)$ of size $n/20$, the induced subgraph $G[S]$ satisfies statement of Theorem 16, i.e., $G[S] \setminus R$ is Maker's win in the H -game for every $R \subseteq G[S]$ with $e_R \leq \frac{1}{2(\chi(H)-1)} \binom{n/20}{2} p$.

Note that in the property (P3) we are upper-bounding the number of edges of R somewhat generously, comparing to the bound given by Theorem 16. However, for our needs such a bound suffices.

Let $X \subseteq V(G)$ be an arbitrary subset of vertices of size at most $1/(2p)$. Then, for each vertex $v \in V(G) \setminus X$ we have

$$\Pr[v \in N_G(X)] = 1 - (1 - p)^{|X|} \geq |X|p - (|X|p)^2 \geq |X|p/2.$$

(We used the fact that $(1 - a)^b \leq 1 - ab + (ab)^2$ for any positive integer b and $0 < a < 1$.) Since events $v \in N_G(X)$ are independent for different $v \in V(G) \setminus X$, by applying Chernoff's inequality we get

$$\Pr[|N_G(X)| < |X|np/4] = e^{-\Omega(|X|np)}.$$

Now a simple union-bound calculation gives

$$\begin{aligned} \Pr[\exists X \subseteq V(G) : |X| \leq 1/(2p), |N_G(X)| < |X|np/4] \\ \leq \sum_{x=1}^{1/(2p)} \binom{n}{x} e^{-\Omega(xnp)} \leq \sum_{x=1}^{1/(2p)} e^{x \log n - \Omega(xnp)} = o(1), \end{aligned}$$

which proves property **(P1)**. Next, property **(P2)** follows from $m(K_5^-) = 9/5$ and the result of Bollobás [3]. Finally, property **(P3)** follows from Theorem 16 and a simple union-bound calculation,

$$\begin{aligned} \Pr[\exists S \subseteq V(G) : |S| = n/20, G[S] \text{ does not satisfy Theorem 16}] \leq \\ \leq \binom{n}{n/20} \cdot e^{-\Theta(n^2p)} \leq e^{n - \Theta(n^2p)} = o(1). \end{aligned}$$

As $t > \frac{1}{2}$ (since we assumed that $m_2(H) < 2$) and the property of “being Maker’s win” is monotone increasing, without loss of generality we add a technical assumption that $p \ll n^{-1/2}$. In the following we use M to denote Maker’s graph. We split the strategy of Maker into several phases.

Phase 1. Denote with \hat{K} an arbitrary copy of K_5^- in G (existence follows by property **(P2)**). It is not hard to check that playing only on the edges of \hat{K} , Maker can create a copy of K_3 in at most 4 moves (see [18]). Let $K = \{v_1, v_2, v_3\}$ be the vertices of the obtained K_3 -copy.

Phase 2. By property **(P1)**, every vertex has at least $np/4 \gg 1/(np^2)$ incident edges in G . Thus, in the next $8/np^2$ rounds Maker can claim edges such that the set $N_1 = N_M(v_1) \setminus K$ is of size $8/(np^2)$.

Phase 3. By property **(P1)** and $|N_1| \leq 1/(2p)$, we have

$$|N_G(N_1)| \geq |N_1|np/4 \geq 2/p$$

and thus, with room to spare, $|N_G(N_1) \setminus K| \geq 1.9/p$. Therefore, owing to the fact that $1/(np^2) \ll 1/p$, regardless of Breaker’s moves in the past and in the next $1/(2p)$ rounds Maker can claim edges such that the set $N_2 = N_M(N_1) \setminus K$ is of size $1/(2p)$.

Phase 4. Again, by property **(P1)** we have that

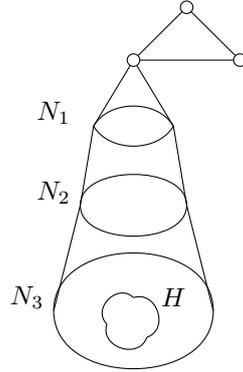
$$|N_G(N_2) \setminus (N_1 \cup K)| \geq |N_2|np/4 - 3 - 8/(np^2) \geq n/9.$$

Since $O(1/p) \ll n$ edges have been played so far, regardless of Breaker’s moves in the past and the next $n/20$ rounds Maker can claim edges such that the set $N_3 = N_M(N_2) \setminus (N_1 \cup K)$ is of size $n/20$.

Phase 5. Maker creates a copy of H in the induced subgraph $G[N_3]$.

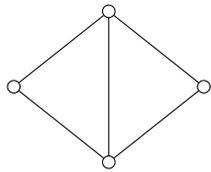
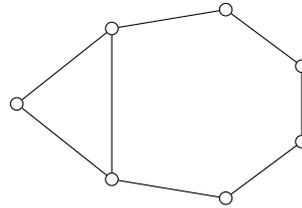
It remains to show that the last step (Phase 5) is indeed possible. First, observe that until this phase only $O(1) + O(1/(np^2)) + O(1/p) + O(n) = o(n^2p)$ rounds have been played. In other words, assuming that n is sufficiently large, less than $\frac{1}{2(\chi(H)-1)} \binom{n/20}{2} p$ rounds have been played to this point. Let $R \subset E(G)$ be the set of Breaker’s edges, and note that by the previous observation we have $|R| \leq \frac{1}{2(\chi(H)-1)} \binom{n/20}{2} p$. Therefore, by property **(P3)** Maker can create a copy of H in $G[N_3] \setminus R$, and by construction of the set N_3 any such copy of H closes a copy of H_P in Maker’s graph (see Figure 2). This completes the proof of Theorem 4. \square

We close this section by mentioning that the phenomena of Theorem 4 do hold for 2-connected graphs as well. If we connect two vertices of a triangle by a path, then the threshold of the resulting graph will also depend on the length of this path. Let C_3^+ and C_6^+ be as defined in Figures 3 and 4.

FIGURE 2. Evolution of Maker's graph in H_p -game.

Adapting the proof of the 0-statement of Theorem 2 one can show that Breaker wins the C_3^+ -game on $G_{n,p}$ whenever $p \leq n^{-1/2-\varepsilon}$ for some $\varepsilon > 0$. In addition, it follows from Theorem 16 that there exists a positive constant C such that w.h.p. Maker has a winning strategy in the C_3^+ -game, provided that $p \geq Cn^{-1/m_2(C_3^+)} = Cn^{-1/2}$.

For C_6^+ it follows from Theorem 3 that Breaker can prevent Maker from obtaining a copy of K_3 (and thus of C_6^+ as well), whenever $p \ll n^{-5/9}$. On the other hand, adapting the ideas of the proof of Theorem 4 one can show that for $p \gg n^{-5/9}$ Maker has a winning strategy.

FIGURE 3. C_3^+ graphFIGURE 4. C_6^+ graph

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