Online Balanced Graph Avoidance Games

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Abstract

We introduce and study online balanced coloring games on the random graph process. The game is played by a player we call Painter. Edges of the complete graph with n vertices are introduced two at the time, in a random order. For each pair of edges, Painter immediately and irrevocably chooses one of the two possibilities to color one of them red and the other one blue. His goal is to avoid creating a monochromatic copy of a small fixed graph F for as long as possible.

We show that the duration of the game is determined by a threshold function $m_H = m_H(n)$ for certain graph-theoretic structures, e.g., cycles. That is, for every graph H in this family, Painter will asymptotically almost surely (a.a.s.) lose the game after $m = \omega(m_H)$ edge pairs in the process. On the other hand, there exists an essentially optimal strategy: if the game lasts for $m = o(m_H)$ moves, Painter can a.a.s. successfully avoid monochromatic copies of H. Our attempt is to determine the threshold function for several classes of graphs.

1 Introduction

The games we study are played by a single player, whom we call Painter. He maintains a balanced 2-coloring in the random graph process, coloring two

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edges at a time in an online fashion. His goal is to avoid creating a monochromatic copy of a fixed graph F for as long as possible.

The precise description of the game's setup, its rules, and of Painter's objectives are as follows. Let $e_1, e_2, \ldots, e_M$ be the edges of K_n where $M = \binom{n}{2}$, and let $\pi \in S_M$ be a permutation of the set $[M]$, chosen uniformly at random. By G_i, $i = 1, \ldots, M$, we denote the graph on n vertices with the edge-set $E(G_i) = \{e_{\pi(1)}, e_{\pi(2)}, \ldots, e_{\pi(i)}\}$. In the ith move of the game, Painter is presented with edges $e_{\pi(2i-1)}$ and $e_{\pi(2i)}$. He then immediately and irrevocably chooses one of the two possibilities to color one of them red and the other one blue. Therefore, after playing the first i moves, Painter has created a balanced 2-coloring of the graph G_{2i}. Note that at move i, he has no knowledge of the order in which the remaining edges will be presented to him in the future.

Let F be a fixed graph. Painter loses the game as soon as he creates a monochromatic copy of F, i.e., Painter loses in the move $\min\{i : G_{2i}$ contains a monochromatic, not necessarily induced copy of $F\}$. His goal is to play as long as possible without losing. It is well-known that for n large enough, every 2-coloring of edges of K_n contains a monochromatic copy of F. Therefore, Painter cannot survive to the end of the game. Assuming that his strategy is fixed, for every graph process, there is an integer i such that Painter loses in his ith move playing on that particular graph process. Since the graph process on which the game is played is chosen uniformly at random, for fixed n and i, we can reason about the probability that Painter loses before his ith move. Note that, generally speaking, Painter can lose the game in two ways. If one of the two edges to be colored closes both a red and a blue copy of F, then he obviously cannot properly color it. We call this a *bichromatic threat.* Also, if both edges to be colored close a monochromatic copy of F of the same color, the game is over. We refer to this as *monochromatic threat.* It is easy to see that these are the only two possibilities for losing.

1.1 Our Results

In this paper, we attempt to determine the maximal number of moves that Painter can a.a.s. (asymptotically almost surely) play without losing. More precisely, we would like to find a threshold function $m_F = m_F(n)$ for which

- there exists a strategy for Painter, such that for any sequence $m \ll m_F$, we have
$$\mathbb{P}\left[\text{Painter loses in the first } m \text{ moves}\right] \to 0 \ ,$$
- regardless of Painter's strategy, for any sequence $m \gg m_F$, we have
$$\mathbb{P}\left[\text{Painter loses in the first } m \text{ moves}\right] \to 1 \ ,$$

as n tends to infinity. Our interests lie in determining this threshold for a number of graph-theoretic structures. Observe that the existence of this threshold is not guaranteed – there may exist a graph F for which there is no such threshold.

In Section 2 we study constraints that imply non-trivial lower and upper bounds on the duration of the balanced graph avoidance game for certain families of graphs. Namely, Proposition 6 gives a lower bound and Proposition 9 an upper bound on the length of the game under certain conditions. As we will show, these bounds are tight in some but not in all cases. However, there are families of graphs for which they give rise to a threshold. This holds for cycles of fixed length.

Theorem 1 *For any fixed integer $\ell \geq 3$, the threshold of the online balanced avoidance game for cycles of length ℓ exists and is given by*

$$m_{C_\ell} = n^{\frac{2\ell}{2\ell-1}}.$$

The game in which Painter's goal is to avoid k-stars S_k is analyzed in Section 3. We can prove the threshold $m_{S_k} = n^{\frac{2k-2}{2k-1}}$ for this game. Moreover, we show that Painter can play the game of stars of all sizes at the same time. In the game of avoiding k-paths P_k, which we mention in Section 4, we have the exact value of m_{P_k} only for $k \leq 4$. For greater k we exhibit some bounds.

1.2 Motivation and Related Work

Friedgut et al. [7] introduced the concept of an online game played on the random graph process. In this game the player colors edges with two colors, one at a time in an online fashion. His goal is to avoid a monochromatic copy of a triangle for as long as possible. Note that one color may be used more frequently than the other by the player.

Extending this result, Marciniszyn, Spöhel, and Steger [9] analyze the game of avoiding monochromatic cliques K_ℓ of any fixed size ℓ, and they exhibit a threshold for the number of moves at which the player loses a.a.s. It turns out that an optimal strategy is to play greedily – using the first color whenever possible, and the second one only to prevent from losing immediately. The colorings obtained by following this strategy are typically unbalanced. A natural question arising is: if the player is forced to keep his coloring balanced, how long can he survive without losing? We try to give an answer to this question by looking at the analogous game in which the coloring of the graph is balanced. As it turns out, several thresholds that we obtain in the balanced game are not the same as for the unbalanced game, showing that the balancedness condition makes a difference. For instance, applying a general criterion from [9] to cycles C_ℓ of length ℓ yields the threshold $n^{1+1/\ell}$ in the unbalanced case, whereas we derive the threshold $n^{1+1/(2\ell-1)}$ from our results for balanced online colorings. Hence, the balanced online cycle avoidance game will end substantially earlier than the unbalanced game.

Another motivation comes from Beck's Chooser-Picker games on graphs [2,3]. During the game, the balanced coloring of the subset of edges of K_n is maintained. In the "misère" version of the game, Chooser wins if at the end of the game (when all the edges are colored) there is no red copy of a fixed graph F. Otherwise, Picker wins. When Picker is playing randomly, this game is quite similar to the balanced avoidance games that we introduce here. A balanced two coloring of the edges of the random graph process is maintained by Chooser, and he colors them two at a time. The only difference is that in our game we investigate a Ramsey-type [10] property, where F has to be avoided in *both* colors. Avoider-Enforcer type of weak positional games (see [3]) on graphs also deals with avoiding a fixed graph F. Two players, Avoider and Enforcer, alternately claim edges of K_n, and Avoider wins if he has not claimed a copy of the graph F to the end of the game.

Generally speaking, studying games on graphs and properties of random graphs in parallel often uncovers surprising connections between thresholds for winning a game on one side, and thresholds for certain properties of random graphs on the other – see [1] and [12], where this phenomenon is pointed out for Maker-Breaker games on graphs. As the reader will see, we also encounter some relationships of this kind while dealing with balanced avoidance games.

1.3 Preliminaries

Our notation follows [8]. We use the well-known symbols O, Ω, Θ, o, and ω in order to express asymptotic properties of sequences. For any two sequences a_n and b_n, we write $a_n \asymp b_n$ if $a_n = \Theta(b_n)$. Similarly, we write $a_n \ll b_n$ or $b_n \gg a_n$ if $a_n \geq 0$ and $a_n = o(b_n)$.

All graphs are labeled, simple, and undirected. For a graph $H = (V, E)$, we abbreviate $|V(H)|$ by v_H or $v(H)$, and $|E(H)|$ by e_H or $e(H)$. A *density measure* δ maps a graph to a non-negative real value. We say that H is *balanced w.r.t.* δ if for all $H' \subseteq H$, we have $\delta(H') \leq \delta(H)$. Moreover, H is *strictly balanced w.r.t.* δ if for all proper subgraphs $H' \subsetneq H$, we have $\delta(H') < \delta(H)$. A well-known density measure is

$$d(H) := \frac{e_H}{v_H} \ .$$

If H is balanced w.r.t. this d, we say it is *balanced in the ordinary sense* or simply *balanced.*

A well-studied property of random graphs is the containment of subgraphs of fixed size. The following theorem of Bollobás [4], which is a generalization of a result of Erdős and Rényi [6] from balanced to arbitrary graphs, determines the threshold for this property.

Theorem 2 *Let H be a nonempty graph. Then the threshold for the property that $G(n,p)$ contains a copy of H is*

$$p_0(H,n) := n^{-1/m(H)} \quad ,$$

where $m(H) := \max_{H' \subseteq H} d(H')$.

We denote the number of copies of a fixed graph H appearing in the random graph $G(n,p)$ by $X(G(n,p),H)$, and let $\mu = \mathbb{E}[X(G(n,p),H)]$. We will repeatedly use the following theorem of Vu [13, Theorem 2.1].

Theorem 3 *If H is balanced in the ordinary sense, ε is a positive constant, and there exists a constant $\alpha > 0$ such that $\mu = \Omega(n^\alpha)$, then we have*

$$\mathbb{P}[X(G(n,p),H) \geq (1+\varepsilon)\mu] \leq \exp\left\{-\Omega\left(n^{\frac{\alpha}{v_H - 1}}\right)\right\} \quad .$$

Another important density measure is

$$d_2(H) := \begin{cases} \frac{e_H - 1}{v_H - 2} & \text{if } v(H) \geq 3 \\ \frac{1}{2} & \text{if } H \cong K_2 \\ 0 & \text{otherwise} \end{cases} \quad ,$$

for any graph H. This measure was introduced by Rödl and Ruciński [11, Theorem 3], who studied Ramsey properties of random graphs. For the sake of simplicity, we present their theorem in a slightly weaker form. A k-edge-coloring stands for an edge coloring of a graph with at most k distinct colors.

Theorem 4 *Let $k \geq 1$ and H be a nonempty graph. Then there exist constants $B = B(H,k)$ and $a = a(H,k)$ such that in every k-edge-coloring of a random graph $G(n,p)$ with*

$$p = p(n) \geq B n^{-1/m_2(H)} \quad ,$$

there are a.a.s. at least $a n^{v_H} p^{e_H}$ monochromatic copies of H, where

$$m_2(H) := \max_{H' \subseteq H : v(H') \geq 3} d_2(H') \quad .$$

The following corollary of Chernoff's well-known inequalities [8, Theorem 2.1] yields exponentially small bounds on the tails of the binomial distribution.

Lemma 5 *Let $X \in \mathrm{Bin}(n,p)$ and $0 < \varepsilon \le 3/2$. Then*

$$\mathbb{P}\left[\left|X - \mathbb{E}[X]\right| \ge \varepsilon \mathbb{E}[X]\right] \le 2\exp\left\{-\frac{\varepsilon^2}{3}\mathbb{E}[X]\right\} \ .$$

2 Bounds on the duration of the game

In this section we prove a lower and an upper bound on the duration of the balanced F-avoidance game that hold under certain conditions. In some cases, e.g., for cycles of fixed length, the bounds match and yield a threshold, but there are classes of graphs for which we do not obtain such sharp results. As we shall prove, the growth rate of the threshold is determined by the following density measure

$$d_b(H) := \begin{cases} \frac{2e_H-1}{2v_H-2} & \text{if } e(H) \ge 1 \\ 0 & \text{otherwise} \end{cases} \ . \tag{1}$$

2.1 Lower Bound

The following proposition gives a lower bound on the duration of the balanced F-avoidance game under certain conditions. The statement holds if all subgraphs of F with one edge less are balanced in the ordinary sense. We define $\mathcal{C}(F)$ as the following family of subgraphs of F:

$$\mathcal{C}(F) := \left\{C \subseteq F : \exists\, e \in E(F) \text{ s.t. } C \text{ is a connected component of } F \setminus \{e\}\right\} \ .$$

Proposition 6 *Let F be a nonempty graph. If every graph $C \in \mathcal{C}(F)$ is balanced, then Painter can a.a.s. play any*

$$N(n) \ll n^{2-1/d_b(F)}$$

moves in the balanced online game avoiding a monochromatic copy of F.

Before we present the proof of Proposition 6, we introduce some notation. Let the random variable $X(G_i, H)$ count the number of subgraphs isomorphic to H in G_i, where G_i is the graph consisting of the first i edges in the random graph process. Let $\mathcal{Q}(H,x)$ denote the family of graphs that contain at least x copies of H. Clearly, $\mathcal{Q}(H,x)$ is a monotone increasing family. We need the following technical lemma, which, generally speaking, expresses the asymptotic equivalence of the models $G(n,p)$ and G_i with respect to $\mathcal{Q}$.

Lemma 7 *For $p = 8N/n^2$ and all $0 \leq i \leq N \leq \binom{n}{2}/2$, we have*

$$\mathbb{P}\left[G_{2i} \in \mathcal{Q}(H,x)\right] \leq \mathbb{P}\left[G(n,p) \in \mathcal{Q}(H,x)\right] + \mathrm{e}^{-\Theta(N)} \quad .$$

PROOF. Observe that each graph G_{2i}, $0 \leq i \leq N$, appearing in the random process is distributed like $G(n, 2i)$, the uniform random graph with exactly $2i$ edges. Since Q is a monotone increasing property, we have

$$\mathbb{P}\left[G_{2i} \in \mathcal{Q}(H,x)\right] \leq \mathbb{P}\left[G(n,2N) \in \mathcal{Q}(H,x)\right]. \tag{2}$$

Applying the law of total probability and the monotonicity of Q, we obtain

$$\begin{aligned}\mathbb{P}\left[G(n,p) \in \mathcal{Q}(H,x)\right] &= \sum_{m=0}^{\binom{n}{2}} \mathbb{P}\left[G(n,m) \in \mathcal{Q}(H,x)\right] \cdot \mathbb{P}\left[e(G(n,p)) = m\right] \\ &\geq \mathbb{P}\left[G(n,2N) \in \mathcal{Q}(H,x)\right] \cdot \mathbb{P}\left[e(G(n,p)) \geq 2N\right] \quad .\end{aligned}$$

It follows that

$$\mathbb{P}\left[G(n,2N) \in \mathcal{Q}(H,x)\right] \leq \frac{\mathbb{P}\left[G(n,p) \in \mathcal{Q}(H,x)\right]}{\mathbb{P}\left[e(G(n,p)) \geq 2N\right]} \quad .$$

For $p = 8N/n^2$, Chernoff bounds (cf. Lemma 5) imply that

$$\mathbb{P}\left[e(G(n,p)) \geq 2N\right] \geq 1 - \mathrm{e}^{-\Theta(N)} \quad .$$

Hence, together with (2), we have

$$\begin{aligned}\mathbb{P}\left[G_{2i} \in \mathcal{Q}(H,x)\right] &\leq \frac{\mathbb{P}\left[G(n,p) \in \mathcal{Q}(H,x)\right]}{1 - \mathrm{e}^{-\Theta(N)}} \\ &= \mathbb{P}\left[G(n,p) \in \mathcal{Q}(H,x)\right] + \mathrm{e}^{-\Theta(N)} \quad .\end{aligned}$$

This concludes the proof of Lemma 7. □

Now we can prove Proposition 6.

Proof of Proposition 6 We have to argue that there exists a strategy for Painter that a.a.s. enables him to avoid monochromatic copies of F in every step of the random process, up to G_{2N}. He plays greedily: if one of the two possibilities to complete a move would create a monochromatic copy of F, then he chooses the other one. Otherwise, he plays arbitrarily.

Let $\mathcal{F}_-$ denote the family of pairwise non-isomorphic subgraphs of F with $e_F - 1$ edges. For $F_- \in \mathcal{F}_-$, we have that $v(F_-) = v(F)$ and $e(F_-) = e(F) - 1$. Since all edges in the random graph process appear independently uniformly at random, the probability of losing the game in one particular step is determined by the number of edges uv that close a monochromatic copy of F_- to F. There are two different configurations that force Painter to create a monochromatic copy of F.

In the first case, a new edge may appear as a vertex pair uv that is covered by both a red copy F_-^r and a blue copy F_-^b with $F_-^r, F_-^b \in \mathcal{F}_-$. But this implies the existence of a graph $F^{(2)}$ in $G(n, 2N)$ consisting of two subgraphs isomorphic to F, which share exactly one edge and possibly more vertices. Suppose

$$N = \frac{n^{2-1/d_b(F)}}{\omega} \ ,$$

where ω tends to infinity as $n \to \infty$ arbitrarily slowly. Then the expected number of copies of $F^{(2)}$ in $G(n, 2N)$ is

$$O\left(n^{v(F^{(2)})}\left(\frac{N}{n^2}\right)^{e(F^{(2)})}\right) = O\left(\omega^{-(2e_F-1)}\right) = o(1) \ .$$

It follows from Markov's inequality that Painter is unlikely to create a copy of $F^{(2)}$ in the first N moves.

However, Painter can create a monochromatic copy of F differently. In this event, two edges v_1v_2 and v_3v_4 that are covered by a monochromatic members of $\mathcal{F}_-$ which have the *same* color show up in the same move. We refer to the pair of edges $\{v_1v_2, v_3v_4\}$ as a threat. An upper bound on the number of threats in the graph can be derived by counting the number of subgraphs isomorphic to a member of $\mathcal{F}_-$, and taking its square. Note that not every such threat is actually dangerous to Painter since we disregard the coloring of the surrounding structure. Thus, we overestimate the risk of losing the game.

Let $p = 8N/n^2$, and $F_- \in \mathcal{F}_-$ such that F_- consists of k balanced components, $F_-^1, \ldots, F_-^k \in \mathcal{C}(F)$. W.l.o.g. for each F_i, $1 \le i \le k$, we have

$$d(F_i) < d_b(F), \tag{3}$$

since otherwise we have $m(F) \ge d(F_i) \ge d_b(F)$, and thus a.a.s. F does not appear in G_{2N} due to Theorem 2. In that case Painter will a.a.s. survive N moves. We denote the expected number of copies of F_- in $G(n, p)$ by $\mu(F_-)$, i.e.,

$$\begin{aligned} \mu(F_-) &:= \mathbb{E}\left[X(G(n,p), F_-)\right] \asymp n^{v_F} p^{e_F - 1} \\ &= n^{v(F_-^1)+\ldots+v(F_-^k)} p^{e(F_-^1)+\ldots+e(F_-^k)} \asymp \prod_{i=1}^{k} \mu(F_-^i) \ . \end{aligned} \tag{4}$$

W.l.o.g. we may assume that $\omega = o(\log(n))$. Then for all $1 \le i \le k$, we have

$$\begin{aligned}
\mu(F_-^i) &= \Omega\left(n^{v(F_-^i)}\left(4n^{-1/d_b(F)}/\log(n)\right)^{e(F_-^i)}\right)\\
&= \Omega\left(n^{v(F_-^i)\left(1-d(F_-^i)/d_b(F)\right)}\log(n)^{-e(F_-^i)}\right) = \Omega\left(n^{\varepsilon_i}\right)
\end{aligned}$$

for a suitable $\varepsilon_i = \varepsilon_i(F_-^i) > 0$, where the last step follows from (3). According to (4) there exists a constant $A = A(F_-) \ge 1$ such that

$$A\mu(F_-) \ge \prod_{i=1}^{k} \mu(F_-^i) \quad .$$

We conclude that

$$\begin{aligned}
\mathbb{P}\left[X(G(n,p),F_-) \ge 2A\mu(F_-)\right] &\le \mathbb{P}\left[\prod_{i=1}^{k} X(G(n,p),F_-^i) \ge \prod_{i=1}^{k} 2^{1/k}\mu(F_-^i)\right]\\
&\le \mathbb{P}\left[\bigvee_{i=1}^{k}\left(X(G(n,p),F_-^i) \ge 2^{1/k}\mu(F_-^i)\right)\right]\\
&\le \sum_{i=1}^{k} \mathbb{P}\left[X(G(n,p),F_-^i) \ge 2^{1/k}\mu(F_-^i)\right]\\
&\le \sum_{i=1}^{k} \exp\left\{-\Omega\left(n^{\varepsilon_i/(v(F_-^i)-1)}\right)\right\} \le \exp\{-\Omega\left(n^{\alpha}\right)\}
\end{aligned}$$

for a suitable constant $\alpha = \alpha(F_-) > 0$. The last line was obtained by application of Theorem 3 with parameters $H \leftarrow F_-^i$, $(1+\varepsilon) \leftarrow 2^{1/k}$, and $\alpha \leftarrow \varepsilon_i$. Note that the expression $\exp\left\{-\Omega\left(n^{\varepsilon_i/(v(F_-^i)-1)}\right)\right\}$ can be replaced by $\exp\{-\Omega\left(n^{\varepsilon_i}\right)\}$ for any $\varepsilon_i > 0$, if $v(F_-^i) = 1$.

Let Z_i be the indicator random variable for the event that both new edges close a monochromatic threat of the same color in step i, and let Z denote its sum over all steps. By the law of total probability, we have

$$\begin{aligned}
\mathbb{P}\left[Z > 0\right] &\le \sum_{i=1}^{N} \mathbb{P}\left[Z_i > 0\right]\\
&\le \sum_{i=1}^{N}\left\{\mathbb{P}\left[Z_i > 0 \,\middle|\, \bigwedge_{F_- \in \mathcal{F}_-} G_{2i-2} \notin \mathcal{Q}(F_-, 2A\mu(F_-))\right]\right.\\
&\qquad \left. + \mathbb{P}\left[\bigvee_{F_- \in \mathcal{F}_-} G_{2i-2} \in \mathcal{Q}(F_-, 2A\mu(F_-))\right]\right\} \quad .
\end{aligned}$$

And owing to Lemma 7 and the preceding calculation, this is at most

$$N \left\{ \frac{\left(\sum_{F_- \in \mathcal{F}_-} 2A\mu(F_-)\right)^2}{\frac{1}{4}\left(\binom{n}{2} - 2N\right)^2} + \sum_{F_- \in \mathcal{F}_-} \left(\mathbb{P}\left[G(n,p) \in \mathcal{Q}(F_-, 2A\mu(F_-))\right] + \mathrm{e}^{-\Theta(N)}\right) \right\}$$

$$\leq N \left\{ O\left(n^{2v_F - 4 - 2(e_F - 1)/d_b(F)}\right) + \sum_{F_- \in \mathcal{F}_-} \left(\mathrm{e}^{-\Omega\left(n^{\alpha(F_-)}\right)} + \mathrm{e}^{-\Theta(N)}\right) \right\}$$

$$\leq o\left(n^{2v_F - 2 - (2e_F - 1)/d_b(F)}\right) + o(1) = o(1),$$

since $N = o\left(n^{2-1/d_b(F)}\right)$ and $2v_F - 2 - (2e_F - 1)/d_b(F) = 0$.

As Painter will not create a monochromatic copy of F in a different way, the statement follows. □

If F is a tree on $t > 1$ vertices, then the removal of an edge yields a forest consisting of two smaller trees T_1 and T_2, which are clearly balanced. Thus, we can rewrite Proposition 6 in the following way.

Corollary 8 *Let F be a tree on $t > 1$ vertices. Then Painter can a.a.s. play any*

$$N(n) = o\left(n^{1-1/(2t-3)}\right)$$

moves in the balanced online game avoiding a monochromatic copy of F.

As we shall prove in Section 3, this bound is tight for stars, but not for paths with four edges.

2.2 Upper bound

We provide an upper bound on the duration of the game under the following assumptions.

Proposition 9 *Let F be a non-empty graph that is strictly balanced w.r.t. d_b and contains a subgraph F_- of F with $e_F - 1$ edges satisfying $d_b(F) \geq m_2(F_-)$. Then, no matter how he plays, Painter will a.a.s. lose the balanced F avoidance game in any*

$$N(n) \gg n^{2-1/d_b(F)}$$

moves.

PROOF. Let F_- be fixed such that $m_2(F_-) \leq d_b(F)$. We switch between the binomial model $G(n,p)$ and the uniform random graph model $G(n,N)$, exploiting their asymptotic equivalence via $p = \Theta(N/n^2)$ [5, Theorem 2.2]. We split the games into two rounds of equal length, $N_1 = N_2 := N/2 \gg n^{2-1/d_b(F)}$ and assume w.l.o.g. that $N \leq n^{2-1/d_b(F)} \log n$. Observe that after the first N_1 moves, N edges have been revealed to Painter. Let $X(G(n,N), F_-)$ denote the number of subgraphs isomorphic to F_- in $G(n,N)$.

Claim 10 *In every 2-edge-coloring of the random graph $G(n,N)$, there are a.a.s.*

$$\Omega\left(\mathbb{E}\left[X(G(n,N),F_-)\right]\right)$$

pairs $uv \in \binom{[n]}{2} \setminus E(G(n,N))$ that complete a monochromatic copy of F_- in the same color, say red, to F.

PROOF. We call an edge *critical*, if it completes an entirely red copy of F_- to F. Theorem 4 yields that a.a.s. the number of (w.l.o.g.) monochromatically red subgraphs of $G(n,N)$ isomorphic to F_- is

$$\Omega\left(\mathbb{E}\left[X(G(n,N),F_-)\right]\right),$$

since

$$N \gg n^{2-1/m_2(F_-)}$$

holds due to the assumptions in Proposition 9. Every such copy induces one critical edge in $G(n,N)$, but we may over-count if there are many pairs of monochromatic copies of F_- that cover the same vertex pair.

If one critical edge $e = uv$ is induced by multiple copies of F_-, then $G(n,N)$ contains a subgraph $(F_-)_H$ of the following structure: $(F_-)_H$ is the union of two graphs isomorphic to F_- such that their intersection complemented with e is a copy of a proper subgraph $H \subsetneq F$. For any graph $(F_-)_H$, we have

$$e\Big((F_-)_H\Big) = e(F_-) + e(F_-) - (e_H - 1) = e(F_-) + e_F - e_H \quad ,$$

and

$$v\Big((F_-)_H\Big) = v(F_-) + v_F - v_H \quad .$$

We denote the number of subgraphs isomorphic to $(F_-)_H$ in $G(n,N)$ by $X(G(n,N),(F_-)_H)$. It follows that

$$\begin{aligned}
\mathbb{E}\left[X(G_{n,N},(F_-)_H)\right] &\asymp n^{v(F_-)+v_F-v_H} p^{e(F_-)+e_F-e_H} \\
&\ll \mathbb{E}\left[X(G_{n,N},F_-)\right] \frac{n^{v_F - e_F/d_b(F)}}{n^{v_H - e_H/d_b(F)}} (\log n)^{e_F} \\
&\ll \mathbb{E}\left[X(G_{n,N},F_-)\right] \quad .
\end{aligned}$$

The last step holds since for every subgraph $H \subseteq F$, we can write

$$v_F - e_F/d_b(F) < v_H - e_H/d_b(F)$$

equivalently as

$$\frac{2e_F - 1}{2v_F - 2} = d_b(F) < \frac{e_F - e_H}{v_F - v_H} = \frac{2e_F - 1 - (2e_H - 1)}{2v_F - 2 - (2v_H - 2)} \ .$$

This inequality holds if

$$\frac{2e_F - 1}{2v_F - 2} > \frac{2e_H - 1}{2v_H - 2} \ ,$$

i.e., the graph F is strictly balanced w.r.t. d_b.

Observe that the number of copies $(F_-)_H$ is a.a.s. bounded from above due to Markov's inequality, which yields that, for every $\varepsilon > 0$, we have

$$\mathbb{P}\left[X(G_{n,N}, (F_-)_H) \geq \varepsilon \mathbb{E}\left[X(G_{n,N}, F_-)\right]\right] = o(1) \ .$$

Since the number of critical edges induced by a fixed occurrence of a graph $(F_-)_H$ is bounded by a constant only depending on F, the multiply counted copies of F_- are a.a.s. of lower order of magnitude than $\mathbb{E}\left[X(G_{n,N}, F_-)\right]$ and thus negligible. Moreover, only a negligible fraction of the critical pairs was actually revealed in $G_{n,N}$ since $\mathbb{E}\left[X(G_{n,N}, F)\right] \ll \mathbb{E}\left[X(G_{n,N}, F_-)\right]$. This concludes the proof of Claim 10. $\square$

Continuing the proof of Proposition 9, we apply the claim to show that a.a.s. the game does not last for more than N_2 moves. Suppose that Painter played his first N_1 moves, and the coloring assigned by Painter to the first $2N_1$ edges is fixed. By Claim 10, there are $M = \Omega\left(X(G(n,N), F_-)\right)$ critical pairs of vertices in $\binom{[n]}{2} \setminus E(G(n, 2N_1))$. If two of these pairs are simultaneously presented to Painter, he loses the game. In every step i, the probability of this event is determined by the number of remaining critical pairs. However, Painter will not lose if only one of the edges presented to him is critical. Moreover, that edge will neutralize a critical pair – it will not be critical any more. On the other hand, Painter can neutralize only a tiny fraction of all critical pairs since we have

$$N_2 \ll M \ . \tag{5}$$

In order to justify (5), observe that, for $e_F \geq 3$, (5) is equivalent to

$$N_2 \gg n^{\left(2(e_F-1)-v_F\right)/\left(e_F-2\right)} \ .$$

As we assumed that $N_2 \gg n^{2-1/d_b(F)}$, (5) holds provided that

$$2 - \frac{1}{d_b(F)} \geq \frac{2(e_F - 1) - v_F}{e_F - 2} \ ,$$

which can be rewritten to

$$3v_F \geq 2e_F + 2 \ . \tag{6}$$

This, however, easily follows from $d_b(F) \geq m_2(F_-)$ since that implies that we have

$$\frac{2e_F - 1}{2v_F - 2} \geq \frac{e_F - 2}{v_F - 2} \ ,$$

which is equivalent to (6) for $v_F \geq 3$. If $e_{F_-} = 1$, the path on two edges P_2 is the only graph that is strictly balanced w.r.t. d_b. In that case we have

$$M = \Omega\left(n^3 \frac{N}{n^2}\right) = \Omega\left(nN\right) \ ,$$

which is clearly substantially greater than N_2.

Let X_i be the random variable indicating that the game was lost in step i of the second round. Since, in every move, the probability of being presented with a pair of critical edges that ends the game is at least

$$\Omega\left(\binom{M}{2} / \binom{\binom{n}{2}}{2}\right) = \Omega\left(\frac{M^2}{n^4}\right) \ ,$$

the probability of surviving the whole second round is bounded from above by

$$\begin{aligned}\left(1 - \Omega\left(\frac{M^2}{n^4}\right)\right)^{N_2} &\leq \exp\left\{-\Omega\left(\frac{M^2 N_2}{n^4}\right)\right\} \\ &\leq \exp\left\{-\Omega\left(\frac{\left(n^{v_F}\left(\frac{N}{n^2}\right)^{(e_F-1)}\right)^2 N}{n^4}\right)\right\} \\ &\leq \exp\left\{-\Omega\left(n^{2v_F - 4e_F} N^{2e_F - 1}\right)\right\} \\ &= \mathrm{e}^{-\omega(1)} = o(1) \ .\end{aligned}$$

This concludes the proof of Proposition 9. □

2.3 The cycle game

We remark that Proposition 9 only applies to families of rather sparse graphs. According to (6), the number of edges in the forbidden graph F must be linear in the number of vertices. However, there exist families of graphs that satisfy the conditions of both Proposition 6 and Proposition 9. In that case the lower bound obtained in Proposition 6 matches the upper bound from Proposition 9 up to a multiplicative constant, giving the exact threshold for several games. In particular, we obtain the threshold for the cycle game.

Proof of Theorem 1 Observe that for a cycle C_ℓ, the only member in the family $\mathcal{F}_-$ is a path $P_{\ell-1}$ with $\ell - 1$ edges. Clearly, this graph is balanced in the ordinary sense, and we have

$$d_b(C_\ell) = \frac{2\ell - 1}{2\ell - 2} > 1 = m_2(P_{\ell-1}).$$

Moreover, C_ℓ is strictly balanced w.r.t. d_b. Hence, the statement follows from Corollary 6 and Proposition 9. $\square$

3 Star Game

Corollary 8 and Proposition 9 together imply the existence of the threshold for the star game.

Corollary 11 *For any fixed integer $k \geq 2$, the threshold for the online balanced avoidance game for k-stars exists and is $m_{S_k} = n^{\frac{2k-2}{2k-1}}$.*

But in this section we go one step further, showing that Painter can simultaneously play the star game optimally for stars of all sizes. This is done in a different, more elementary setting than the one used in the proof of Proposition 6. Roughly speaking, our approach enables us to use the independence and uniformness of choice of the edge coming in the random graph process, while conditioning on a property of the same process. That way we can estimate the numbers of monochromatic stars appearing in the course of game more easily.

Note that Corollary 11 follows directly from the next theorem, by setting $\alpha = \frac{1}{2k-1}$.

Theorem 12 *Let $\alpha > 0$ be a constant, $m = n^{1-\alpha}$, and $k_0 = \lfloor \frac{1}{2}\left(\frac{1}{\alpha} + 1\right) \rfloor$.*

(i) After $m' = \omega(m)$ moves of the balanced online game, for every integer $k \leq k_0$ Painter has created $\omega(m^{2k-1} n^{-2k+2})$ monochromatic stars of size k a.a.s.
(ii) For $m'' = o(m)$, there is a strategy for Painter that enables him to create $o(m^{2k-1} n^{-2k+2})$ monochromatic stars of size k, for all $k \leq k_0$, in the first m'' moves of the balanced online game a.a.s.

When we assume that $m' = \omega(m)$ (part (i) of the statement) or that $m'' = o(m)$ (part (ii)), we actually assume that m' and m'' are concrete functions satisfying these conditions, fixed before the game starts.

Written down in a strict mathematical notation, the statement of the first part of the theorem reads as follows. Let $X_k(m)$ denote the number of monochromatic k-stars in G after m moves. Suppose $\alpha > 0$ is a real constant. Let $\nu : \mathbb{N} \to \mathbb{N}$ be a function with $\frac{\nu(n)}{n} \to \infty$ as n tends to infinity, and let $k \leq k_0$. Then there exists a function $\mu : \mathbb{N} \to \mathbb{N}$ with $\frac{\mu(n)}{n} \to \infty$ as n tends to infinity, such that for every strategy of Painter, we have

$$\lim_{n\to\infty} \mathbb{P}\left[X_k(\nu(n^{1-\alpha})) \geq \mu\left(\left(n^{1-\alpha}\right)^{2k-1} n^{-2k+2}\right)\right] = 1.$$

PROOF. *(i)* After m' moves, every $(2k-1)$-star contains a monochromatic k- star. In the graph $G_{m'}$ (i.e., the random graph $G(n, m')$), we can estimate the number of $(2k-1)$-stars and the number of pairs of $(2k-1)$-stars that contain the same k-star, and the statement of the theorem follows.

(ii) Painter's strategy is the following. Whenever he should color edges v_1v_2 and v_3v_4, he spots the largest monochromatic star that is centered at one of the vertices $v_1,\dots,v_4$ at that moment. There may be more than one star with that property in which case he spots one of them arbitrarily. He colors the edge adjacent to the center of the largest monochromatic star using the color complementary to the color of the star, in order to prevent the monochromatic star from increasing in size. The other edge is colored accordingly.

Let $s_k(m'', n)$ be recursively defined by $s_0(m'', n) = m''$ and $s_k(m'', n) = \frac{m''}{n^2}\left(s_{k-1}(m'', n)\right)^2$. Using $m'' = o(m)$, $m = o(n)$, it directly follows by induction on k that $s_k(m'', n) = o(m^{2k-1} n^{-2k+2})$.

Using again induction on k, we will prove that the probability that after m'' moves there are "too many" different monochromatic k-star centers is small, for all $k < k_0$. More precisely, the inductive statement for $k < k_0$ reads as follows. There exist constants $\gamma_k > 0$ and $n_0 \in \mathbb{N}$, such that for every $n \geq n_0$ we have that

$$\mathbb{P}\left[(\#\text{ of monochromatic } k\text{-star centers after } m'' \text{ moves}) > s_k(m'', n)\right] \leq \mathrm{e}^{-n^{\gamma_k}}.$$

Note that at this point we do not care about the number of monochromatic k-stars centered at any of the vertices we count.

The statement holds for $k = 1$, since we have $m'' = o(m)$ edges and every colored edge is a monochromatic 1-star, so there is not more than $2m''$ monochromatic 1-star centers. Hence, the probability we are interested in is identically zero, and we can choose an arbitrary value for γ_1.

Assume that the statement is true for $k-1$, $k < k_0$. Suppose that in a move the number of monochromatic k-star centers is increased. We distinguish two cases.

Case 1. If one of the edges that was to be colored was not adjacent to any monochromatic $(k-1)$-star, then the other edge was adjacent to both blue and red $(k-1)$-star, meaning that in this move at least one new subgraph of size $2(k-1)+1 = 2k-1$ is created. Therefore, if B is the random variable counting the number of subgraphs of size $2k-1$ of $G_{2m''}$, the number of monochromatic k-star centers created in this fashion is at most B. Using Theorem 3 and Lemma 7 we can show that $B > \frac{1}{8}s_k(m'', n)$ holds only with exponentially small probability, for every $k < k_0$.

Case 2. The other possibility is that each of the edges to be colored is adjacent to a monochromatic $(k-1)$-star of the same color. By C_i we denote the indicator random variable which has value 1 if the number of monochromatic k-star centers is increased in ith move, $i \leq m''$, in this way.

Next, for every move $i \leq m''$ we define the following indicator random variables

$$D_i = [(\#\text{ monochromatic } (k-1)\text{-star centers in } G_{2i}) > s_{k-1}(m'', n)].$$

Finally, we define an auxiliary sequence of indicator random variables C_i', $i \leq m''$. Our goal is to define them in such a way that, on one hand, the value of C_i' is less than the value of C_i only for a "reasonably small" number of graph processes (actually, only when $D_{i-1} = 1$), and on the other hand, they are mutually independent and we can apply Chernoff bounds to their sum.

For every graph process, we look at the set containing all possible pairs of edges of G_{2i-2} that, if they appear in ith step, increase the number of monochromatic k-star centers. Denote this set by $T(G_{2i-2})$. Note that $C_i = 1$ if and only if the pair of edges that is to be colored in ith move is in $T(G_{2i-2})$.

If $|T(G_{2i-2})| \leq (n \cdot s_{k-1}(m'', n))^2$, then we construct the set $T'(G_{2i-2})$ by starting from $T(G_{2i-2})$, and adding another $(n \cdot s_{k-1}(m'', n))^2 - T(G_{2i-2})$ pairs of edges from $E(K_n) \setminus E(G_{2i-2})$, by some arbitrary (but deterministic) rule.

On the other hand, if $|T(G_{2i-2})| > (n \cdot s_{k-1}(m'', n))^2$, we construct the set $T'(G_{2i-2})$ by starting from $T(G_{2i-2})$, and removing $T(G_{2i-2}) - (n \cdot s_{k-1}(m'', n))^2$ pairs of edges from $E(K_n) \setminus E(G_{2i-2})$, by some arbitrary (but deterministic) rule.

Hence, we always have $|T'(G_{2i-2})| = (n \cdot s_{k-1}(m'', n))^2$. We define C_i' to be 1 if and only if the pair of edges that is to be colored in ith move is in $T'(G_{2i-2})$. Crucially, $C_i' < C_i$ only when $D_{i-1} = 1$ and therefore we have

$$\sum_{i=1}^{m''} C_i \leq \sum_{i=1}^{m''} \left(C_i' + D_{i-1}\right).$$

Since we know the exact size of $T'(G_{2i-2})$, for every i we get

$$\mathbb{P}\left[C'_i = 1\right] = \frac{(n \cdot s_{k-1}(m'', n))^2}{\binom{\binom{n}{2}-2i+2}{2}},$$

and this probability does not change if we fix the value of a variable C'_j for any other j. Therefore, the variables $\{C'_i\}_i$ are independent and we can apply Chernoff bounds to get

$$\sum_{i=1}^{m''} C'_i \leq 8m'' \frac{(n \cdot s_{k-1}(m'', n))^2}{n^4} \leq \frac{1}{8} s_k(m'', n),$$

with probability $1 - e^{-n^{\gamma''_k}}$, for some $\gamma''_k > 0$.

From the induction hypothesis, there is a constant $\gamma_{k-1} > 0$ such that the probability that $D_{i-1} = 1$ is at most $e^{-n^{\gamma_{k-1}}}$. Then, $\sum_{i=1}^{m''} D_{i-1} \neq 0$ with probability at most $m'' e^{-n^{\gamma_{k-1}}}$.

Since in one move we create at most 4 new star centers, if we denote the total number of monochromatic k-stars after m'' moves by A, we have

$$\begin{aligned} A &\leq 4B + 4\sum_{i=1}^{m''} C_i \\ &\leq 4B + 4\sum_{i=1}^{m''} C'_i + 4\sum_{i=1}^{m''} D_{i-1} \\ &\leq \frac{1}{2} s_k(m'', n) + \frac{1}{2} s_k(m'', n) + 0, \end{aligned}$$

with probability at least $1 - \left(e^{-n^{\gamma'_k}} + e^{-n^{\gamma''_k}} + m'' e^{-n^{\gamma_{k-1}}}\right)$, and thus also at least $1 - e^{-n^{\gamma_k}}$ for some $\gamma_k > 0$. This completes the induction step, if $k < k_0$. For $k = k_0$, the same holds with probability $1 - o(1)$.

We proved that after m'' moves the number of monochromatic k-star centers is $o(m^{2k-1} n^{-2k+2})$, for all $k \leq k_0$, a.a.s. On the other hand, since $m = \Theta(n^{1-\alpha})$, the order of every connected component of $G_{2m''}$ is bounded by a constant a.a.s. Hence, every vertex is a center for at most constantly many monochromatic k-stars, and after playing m'' moves Painter has created at most $o(m^{2k-1} n^{-2k+2})$ monochromatic k-stars a.a.s. □

4 Path Game

In the path game, the objective of Painter is to avoid creating a monochromatic path P_l on l edges for as long as possible, where $l \geq 0$ is a fixed integer. The game is apparently more complicated to analyze than the star game, as we fail to find the exact value of the threshold for $l \geq 5$. Having that in mind, we will just list our results to illustrate the behavior of the game, omitting the proofs. In order to state the next theorem, we need the following technical lemma.

Lemma 13 *Let $s_0 = 0$ and $s_t = \sum_{i=1}^{t} 2^{\lfloor \log i \rfloor}$ for all integers $t \geq 1$. We have*

(i) $4s_t + 1 = s_{2t+1}$ *for all* $t \geq 0$,
(ii) $2s_{t-1} + 2s_t + 1 = s_{2t}$ *for all* $t \geq 1$,
(iii) $s_t = \Theta(t^2)$.

We now give an upper bound on the asymptotic length of the game, which is tight for P_2 and P_3, but already for P_4 this bound can be improved. We believe that this also holds for $l \geq 5$.

Theorem 14 *Using the notation from Lemma 13, for an integer $l \geq 1$, let $\alpha = 1/s_l > 0$. Then for all $0 \leq k \leq l$, after $m \gg n^{1-\alpha}$ moves, Painter has created at least*

$$n^{1-s_k\alpha}$$

monochromatic paths P_k in both colors red and blue a.a.s.

This theorem yields the following upper bound on the path avoidance game.

Corollary 15 *For every $l > 0$, regardless of his strategy Painter will a.a.s. lose the online balanced P_l-avoidance game in $\omega\left(n^{1-1/s_l}\right)$ moves.*

Since a 2-path is also a 2-star, Corollary 11 implies that the last statement gives the exact value in the case $l = 2$. The same is true for $l = 3$: by combining Corollary 15 with Corollary 8, we get that in this case the threshold is $n^{4/5}$. But already for $l = 4$, Corollary 15 is not tight, as the following theorem shows. Note that $s_4 = 9$, and Corollary 15 consequently yields an upper bound of $n^{8/9}$ on the P_4-avoidance game.

With a refined argument from the proof of this bound, we obtain that the game cannot last for substantially more than $n^{7/8}$ moves, which is provably tight. The proof of the respective lower bound in Theorem 16 is similar to the proof of Theorem 12(ii). However, it involves a rather tedious case analysis. We believe that these methods, as they are, cannot be generalized to paths of arbitrary length.

Theorem 16 *The threshold for the online balanced P_4-avoidance game exists and is $n^{7/8}$.*

5 Conclusion and future work

In this paper we studied the online balanced coloring games on the random graph process. For certain families of graphs, we provided lower and upper bounds on the duration of the game. In particular, we were able to determine the existence and the exact value of the thresholds for the games avoiding cycles, stars, and paths with at most four edges. We also gave an upper bound on the length of the game avoiding paths of arbitrary fixed length.

A natural question that remains unanswered is how to obtain the exact threshold for the game of avoiding paths of length greater than four. Apparently, our methods for proving the case P_4 are not easily generalizable. Another interesting question would be to study the threshold for other graph-theoretic structures, in particular for cliques of size greater than 3.

In both [7] and [9], generalizations of the online games with more than two colors are mentioned and analyzed in some cases. One obvious generalization of the problems that we deal with in the present paper would be the game in which s edges at a time are introduced in the random graph process, where $s \geq 2$ is a fixed integer. Then Painter immediately colors them with s distinct colors. Painter's objective remains to avoid creating a monochromatic copy of F for as long as possible.

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