Many collinear k-tuples with no $k+1$ collinear points

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Abstract

For every $k > 3$, we give a construction of planar point sets with many collinear k-tuples and no collinear $(k+1)$-tuples.

1 Introduction

In the early 60's Paul Erdős asked the following question about point-line incidences in the plane: *Is it possible that a planar point set contains many collinear four-tuples, but it contains no five points on a line?* There are various constructions for n-element point sets with $n^2/6 - O(n)$ collinear triples with no four on a line (see [3] or [10]). However, no similar construction is known for larger numbers.

Let us formulate Erdős' problem more precisely. For a finite set P of points in the plane and $k \geq 2$, let $t_k(P)$ be the number of lines meeting P in exactly k points, and let $T_k(P) := \sum_{k' \geq k} t_{k'}(P)$ be the number of lines meeting P in at least k points. For $r > k$ and n, we define

$$t_k^{(r)}(n) := \max_{\substack{|P|=n \\ T_r(P)=0}} t_k(P).$$

In plain words, $t_k^{(r)}(n)$ is the number of lines containing exactly k points from P, maximized over all n point sets P that do not contain r collinear points. In this paper we are concerned about bounding $t_k^{(k+1)}(n)$ from below for $k > 3$. Erdős conjectured that $t_k^{(k+1)}(n) = o(n^2)$ for $k > 3$ and offered \$100 for a proof or disproof [8] (the conjecture is listed as Conjecture 12 in the problem collection of Brass, Moser, and Pach [2]).

1.1 Earlier Results

This problem was one of Erdős' favorite geometric problems, he frequently talked about it and listed it among the open problems in geometry, see [8, 7, 5, 6, 9]. It is not just a simple puzzle which might be hard to solve, it is related to some deep and difficult problems in other fields. It seems that the key to attack this question would be to understand the group structure behind point sets with many collinear triples. We will not investigate this direction in the present paper, our goal is to give a construction showing that Erdős conjecture, if true, is sharp – for $k > 3$, one can not replace the exponent 2 by $2 - c$, for any $c > 0$.

The first result was due to Kárteszi [13] who proved that $t_k^{(k+1)}(n) \geq c_k n \log n$ for all $k > 3$. In 1976 Grünbaum [11] showed that $t_k^{(k+1)}(n) \geq c_k n^{1+1/(k-2)}$. For some 30 years this was the best bound when Ismailescu [12], Brass [1], and Elkies [4] consecutively improved Grünbaum's bound for $k \geq 5$. However, similarly to Grünbaum's bound, the exponent was going to 1 as k went to infinity.

In what follows we are going to give a construction to show that for any $k > 3$ and $\delta > 0$ there is a threshold $n_0 = n_0(k, \delta)$ such that if $n \geq n_0$ then $t_k^{(k+1)}(n) \geq n^{2-\delta}$. On top of that, we note that each of the collinear k-tuples that we count in our construction has an additional property – the distance between every two consecutive points is the same.

1.2 Notation

For $r > 0$, a positive integer d and $x \in \mathbb{R}^d$, by $B_d(x, r)$ we denote the closed ball in $\mathbb{R}^d$ of radius r centered at x, and by $S_d(x, r)$ we denote the sphere in $\mathbb{R}^d$ of radius r centered at x. When $x = 0$, we will occasionally write just $B_d(r)$ and $S_d(r)$.

For a set $S \subseteq \mathbb{R}^d$, let $N(S)$ denote the number of points from the integer lattice $\mathbb{Z}^d$ that belong to S, i.e., $N(S) := \mathbb{Z}^d \cap S$.

2 A lower bound for $t_k^{(k+1)}(n)$

We will prove bounds for even and odd value of k separately, as the odd case needs a bit more attention.

2.1 k is even

Theorem 1 *For $k \geq 4$ even and $\varepsilon > 0$, there is a positive integer n_0 such that for $n > n_0$ we have $t_k^{(k+1)}(n) > n^{2-\varepsilon}$.*

Proof. We will give a construction of a point set P containing no $k+1$ collinear points, with a high value of $t_k(P)$.

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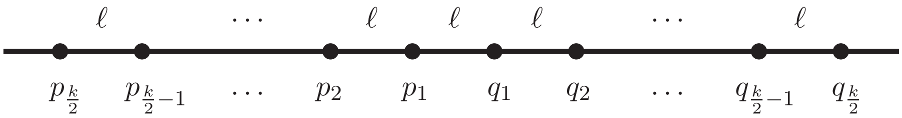

Figure 1: Line s with k points, for k even.

Let d be a positive integer, and let $r_0 > 0$. It is known, see, e.g., [15], that for large enough r_0 we have

$$\begin{aligned} N(B_d(r_0)) &= (1+o(1))V(B_d(r_0)) \\ &= (1+o(1))C_d r_0^d \\ &\geq c_1 r_0^d, \end{aligned}$$

where $C_d = \frac{\pi^{d/2}}{\Gamma((n+2)/2)}$, and $c_1 = c_1(d)$ is a constant depending only on d.

For each integer point from $B_d(r_0)$, the square of its distance to the origin is at most r_0^2. As the square of that distance is an integer, we can apply pigeonhole principle to conclude that there exists r, with $0 < r \leq r_0$, such that the sphere $S_d(r)$ contains at least $1/r_0^2$ fraction of points from $B_d(r_0)$, i.e.,

$$N(S_d(r)) \geq \frac{1}{r_0^2} N(B_d(r_0)) \geq \frac{1}{r_0^2} c_1 r_0^d = c_1 r_0^{d-2}.$$

We now look at unordered pairs of different points from $\mathbb{Z}^d \cap S_d(r)$. The total number of such pairs is at least

$$\binom{N(S_d(r))}{2} \geq \binom{c_1 r_0^{d-2}}{2} \geq c_2 r_0^{2d-4},$$

for some constant $c_2 = c_2(d)$. On the other hand, for every $p, q \in \mathbb{Z}^d \cap S_d(r)$ we know that the Euclidean distance $d(p,q)$ between p and q is at most $2r$, and that the square of that distance is an integer. Hence, there are at most $4r^2$ different possible values for $d(p,q)$. Applying pigeonhole principle again, we get that there are at least

$$\frac{c_2 r_0^{2d-4}}{4r^2} \geq \frac{c_2}{4} r_0^{2d-6}$$

pairs of points from $\mathbb{Z}^d \cap S_d(r)$ that all have the same distance. We denote that distance by ℓ.

Let $p_1, q_1 \in \mathbb{Z}^d \cap S_d(r)$ with $d(p_1, q_1) = \ell$, and let s be the line going through p_1 and q_1. We define $k-2$ points $p_2, \ldots, p_{k/2}, q_2, \ldots, q_{k/2}$ on the line s such that $d(p_i, p_{i+1}) = \ell$ and $d(q_i, q_{i+1}) = \ell$, for all $1 \leq i < k/2$, and such that all k points $p_1, \ldots, p_{k/2}, q_1, \ldots, q_{k/2}$ are different, see Figure 1.

Knowing that p_1 and q_1 are points from $\mathbb{Z}^d$, the way we defined points $p_2, \ldots, p_{k/2}, q_2, \ldots, q_{k/2}$ implies that they have to be in $\mathbb{Z}^d$ as well. If we set $r_i := \sqrt{r^2 + i(i-1)\ell^2}$, for all $i = 1, \ldots, k/2$, then the points p_i and q_i belong to the sphere $S_d(r_i)$, and hence, $p_i, q_i \in \mathbb{Z}^d \cap S_d(r_i)$, for all $i = 1, \ldots, k/2$, see Figure 2.

We define the point set P to be the set of all integer points on spheres $S_d(r_i)$, for all $i = 1, \ldots, k/2$, i.e.,

$$P := \mathbb{Z}^d \cap \left(\cup_{i=1}^{k/2} S_d(r_i) \right).$$

Let $n := |P|$. Obviously, $P \subseteq B_d(r_{k/2})$, so we have

$$n \leq N\left(B_d(r_{k/2})\right) = (1+o(1))V(B_d(r_{k/2})) = c_1 r_{k/2}^d.$$

Plugging in the value of $r_{k/2}$ and having in mind that $\ell < 2r$, we obtain

$$\begin{aligned} n &\leq c_1 \left(\sqrt{r^2 + k/2(k/2-1)4r^2} \right)^d \\ &\leq c_1 \sqrt{k^2+1}^d r^d \\ &\leq c_3 r^d \\ &\leq c_3 r_0^d, \end{aligned}$$

where $c_3 = c_3(d,k)$ is a constant depending only on d and k.

As the point set P is contained in the union of $k/2$ spheres, there are obviously no $k+1$ collinear points in P. On the other hand, every pair of points $p_1, q_1 \in \mathbb{Z}^d \cap S_d(r)$ with $d(p_1, q_1) = \ell$ defines one line that contains k points from P. Hence, the number of lines containing exactly k points from P is

$$t_k(P) \geq \frac{c_2}{4} r_0^{2d-6} \geq \frac{c_2}{4} \frac{1}{c_3^{\frac{2d-6}{d}}} n^{\frac{2d-6}{d}} \geq c_4 n^{\frac{2d-6}{d}},$$

where $c_4 = c_4(d,k)$ is a constant depending only on d and k.

To obtain a point set in two dimensions, we can project our d dimensional point set to an arbitrary (two dimensional) plane in $\mathbb{R}^d$. The vector v along which we project should be chosen so that every two points from our point set are mapped to different points, and every three points that are not collinear are mapped to points that are still not collinear. Obviously, such vector can be found.

For ε given, we can pick d such that $\frac{2d-6}{d} > 2 - \varepsilon$. As we increase r_0, we obtain constructions with n growing to infinity. When n is large enough, the statement of the theorem will hold. □

2.2 k is odd

Theorem 2 *For $k \geq 4$ odd and $\varepsilon > 0$, there is a positive integer n_0 such that for $n > n_0$ we have $t_k^{(k+1)}(n) > n^{2-\varepsilon}$.*

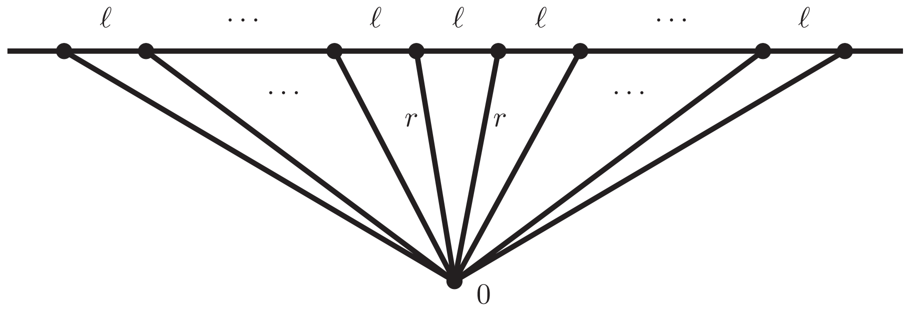

Figure 2: The position of the k points related to the origin, for k even.

The proof for k odd follows the lines of the proof for k even, but requires an additional twist, as we cannot obtain all the points in pairs as before. What we do, roughly speaking, is find the k-tuple of collinear points on $\frac{k+1}{2}$ selected spheres, where the inner most sphere will contain just one point of the k-tuple and each of the other spheres will contain two. Then we make an additional restriction on the point set on the inner most sphere to prevent $k+1$ collinear points from appearing in our construction.

As due to space restrictions we cannot present this proof in full detail, we omit it altogether. It can be found in [14].

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References

[1] Peter Brass. On point sets without k collinear points. In *Discrete geometry*, volume 253 of *Monogr. Textbooks Pure Appl. Math.*, pages 185–192. Dekker, New York, 2003.

[2] Peter Brass, William Moser, and János Pach. *Research problems in discrete geometry.* Springer, New York, 2005.

[3] Stefan A. Burr, Branko Grünbaum, and N. J. A. Sloane. The orchard problem. *Geometriae Dedicata*, 2:397–424, 1974.

[4] Noam D. Elkies. On some points-and-lines problems and configurations. *Period. Math. Hungar.*, 53(1-2):133–148, 2006.

[5] P. Erdős. Some combinational problems in geometry. In *Geometry and differential geometry (Proc. Conf., Univ. Haifa, Haifa, 1979)*, volume 792 of *Lecture Notes in Math.*, pages 46–53. Springer, Berlin, 1980.

[6] Pál Erdős. Néhány elemi geometriai problémáról (on some problems in elementary geometry, in hungarian). *Középiskolai Matematikai Lapok*, 61:49–54, 1980.

[7] Paul Erdős. On some problems of elementary and combinatorial geometry. *Ann. Mat. Pura Appl. (4)*, 103:99–108, 1975.

[8] Paul Erdős. Research problems. *Periodica Mathematica Hungarica*, 15:101–103, 1984.

[9] Paul Erdős and George Purdy. *Some extremal problems in geometry. IV.* Utilitas Math., Winnipeg, Man., 1976.

[10] Z. Füredi and I. Palásti. Arrangements of lines with a large number of triangles. *Proc. Amer. Math. Soc.*, 92(4):561–566, 1984.

[11] Branko Grünbaum. New views on some old questions of combinatorial geometry. In *Colloquio Internazionale sulle Teorie Combinatorie (Rome, 1973), Tomo I*, pages 451–468. Atti dei Convegni Lincei, No. 17. Accad. Naz. Lincei, Rome, 1976.

[12] Dan Ismailescu. Restricted point configurations with many collinear k-tuplets. *Discrete Comput. Geom.*, 28(4):571–575, 2002. Discrete and computational geometry and graph drawing (Columbia, SC, 2001).

[13] Ferenc Kárteszi. Sylvester egy tételéről és Erdős egy sejtéséről. *Középiskolai Matematikai Lapok*, 26:3–10, 1963.

[14] József Solymosi and Miloš Stojaković. Many collinear k-tuples with no $k+1$ collinear points (full paper). *arXiv:1107.0327v1 [math.CO]*, 2011.

[15] Arnold Walfisz. *Gitterpunkte in mehrdimensionalen Kugeln.* Monografie Matematyczne. Vol. 33. Państwowe Wydawnictwo Naukowe, Warsaw, 1957.