Avoider-Enforcer: The Rules of the Game

Dan Hefetz [1]

Institute of Theoretical Computer Science, ETH Zurich, CH-8092 Switzerland

Michael Krivelevich [2]

School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, 69978, Israel

Miloš Stojaković [3]

Department of Mathematics and Informatics, University of Novi Sad, Serbia

Tibor Szabó [4]

Department of Mathematics and Statistics, McGill University, Montreal, Canada

Abstract

An Avoider-Enforcer game is played by two players, called Avoider and Enforcer, on a hypergraph $\mathcal{F} \subseteq 2^X$. The players claim previously unoccupied elements of the board X in turns. Enforcer *wins* if Avoider claims all vertices of some element of $\mathcal{F}$, otherwise Avoider wins. In a more general version of the game a *bias* b is introduced to level up the players' chances of winning; Avoider claims one element of the board in each of his moves, while Enforcer responds by claiming b elements. This traditional set of rules for Avoider-Enforcer games is known to have a shortcoming: it is not bias monotone.

We relax the traditional rules in a rather natural way to obtain bias monotonicity. We analyze this new set of rules and compare it with the traditional ones to conclude some surprising results. In particular, we show that under the new rules the threshold bias for both the connectivity and Hamiltonicity games, played on the edge set of the complete graph K_n, is asymptotically equal to $n/\log n$.

Keywords: Positional games, connectivity, Hamiltonicity.

1 Biased Maker-Breaker games.

In this paper we consider Avoider-Enforcer games. To motivate our investigation we start with a short discussion of their widely studied ancestors, Maker-Breaker games.

Let p and q be positive integers and let $\mathcal{F} \subseteq 2^X$ be a hypergraph over the vertex set X. In a $(p : q)$ *Maker-Breaker game* $\mathcal{F}$, two players, called Maker and Breaker, take turns selecting previously unclaimed vertices of X (with Maker going first). Maker selects p vertices per turn and Breaker selects q vertices per turn. The integers p, q are called the *bias*es of the respective players. The game ends when every element of the board has been claimed by one of the players. Maker wins the game if he claims all the vertices of some winning set; otherwise Breaker wins.

Chvátal and Erdős [3] studied Maker-Breaker games played on the edge set of the complete graph K_n. They have come to realize that natural graph games are often "easily" won by Maker when played in a fair fashion (that is, with $p = q = 1$), so they explored a more general question: What is the largest bias b of Breaker, against which Maker can still win a particular game, if his bias is 1? For such a question to make sense, one would like to have the following property: if the $(1 : b)$ game $\mathcal{F}$ is a Breaker's win for some integer b, then the $(1 : b')$ game $\mathcal{F}$ is also a Breaker's win for any $b' \geq b$. It is easy to see that this holds for any family $\mathcal{F}$. Formally, Maker-Breaker games are *bias monotone*, as Maker wins the $(p : q)$ Maker-Breaker game $\mathcal{F}$ for some hypergraph $\mathcal{F}$ and positive integers p, q, then he also wins the $(p + 1 : q)$ and the $(p : q-1)$ games (the analogous statement for Breaker's win holds as well).

For a family $\mathcal{F}$ of sets, let the *threshold bias* $b_{\mathcal{F}}$ be the non-negative integer for which Maker has a winning strategy in the $(1 : b)$ game $\mathcal{F}$ if and only if $b < b_{\mathcal{F}}$. Note that $b_{\mathcal{F}}$ is well-defined for any (monotone increasing) family $\mathcal{F}$

[1] Email: dan.hefetz@inf.ethz.ch
[2] Email: krivelev@post.tau.ac.il
[3] Email: smilos@inf.ethz.ch
[4] Email: szabo@math.mcgill.ca

(unless $\mathcal{F} = \emptyset$ or $\mathcal{F}$ contains a hyperedge of size at most one).

Chvátal and Erdős [3] have initiated the study of the biased graph games like "connectivity" and "Hamiltonicity", where the families of winning sets are the family $\mathcal{T} = \mathcal{T}(n) \subseteq 2^{E(K_n)}$ of all n-vertex connected graphs and the family $\mathcal{H} = \mathcal{H}(n) \subseteq 2^{E(K_n)}$ of all n-vertex Hamiltonian graphs, respectively. They proved that $b_{\mathcal{T}} = \Theta\left(\frac{n}{\log n}\right)$, and Gebauer and Szabó [5] recently showed that

$$b_{\mathcal{T}} = (1 + o(1))\,\frac{n}{\log n}.$$

Beck [1] has shown that for the Hamiltonicity game we have $b_{\mathcal{H}} = \Theta\left(\frac{n}{\log n}\right)$. The current best estimates are

$$(\log 2 - o(1))\,\frac{n}{\log n} \leq b_{\mathcal{H}} \leq (1 + o(1))\,\frac{n}{\log n},$$

the lower bound is due to Krivelevich and Szabó [8].

2 Biased Avoider-Enforcer games.

Avoider-Enforcer games are the *misère* version of Maker-Breaker games. Generally speaking, a *misère* game is played according to its conventional rules, except that it is played to "lose". The only difference to Maker-Breaker game rules is that Avoider wins the game if he does *not* claim all the vertices of any hyperedge of $\mathcal{F}$; otherwise Enforcer wins.

Similarly to Maker-Breaker games, one would like to define for every family $\mathcal{F}$ the Avoider-Enforcer threshold bias $f_{\mathcal{F}}$ as the non-negative integer for which Enforcer wins the $(1 : b)$ game $\mathcal{F}$ if and only if $b < f_{\mathcal{F}}$. Somewhat surprisingly, unlike for Maker-Breaker games, such a threshold *does not exist* in general for Avoider-Enforcer games (see [6]).

In order to overcome the non-monotonicity of Avoider-Enforcer games and, as a consequence, the lack of a well-defined threshold bias, we offer a modification of the rules of Avoider-Enforcer games. We refer to the new rules as *monotone rules*, while the original set of rules will be referred to as *strict rules*. In this new setting of Avoider-Enforcer games everything remains the same as before except that we allow both players to claim *more* elements per turn than their respective bias. It is easy to see that Avoider-Enforcer games with these rules *are* bias monotone. Hence, one can define the threshold bias $f^{mon}_{\mathcal{F}}$ of the monotone game $\mathcal{F}$ as the non-negative integer for which Enforcer has a winning strategy in the $(1 : b)$ game if and only if $b \leq f^{mon}_{\mathcal{F}}$.

Our relaxation of the rules of Avoider-Enforcer games is inspired by the seemingly plausible assumption that "taking more edges cannot possibly help a player in an Avoider-Enforcer game". The presumed analogy to Maker-Breaker games further supports the idea of monotone rules, since the analogous relaxation of the rules of Maker-Breaker games does not change the outcome of the game – it is known that allowing a player to claim less edges than his respective bias in a Maker-Breaker game cannot help him.

One may wonder about the relationship between a biased Avoider-Enforcer game played according to the strict rules and the same game played according to the monotone rules. Is it true that our relaxation of the rules has no significant effect, other than making the game bias-monotone? Unexpectedly, the results we obtain are strikingly different, even for such a natural graph game as connectivity, which is even bias monotone under the strict rules.

Let k be a positive integer and let $\mathcal{D}_k \subseteq 2^{E(K_n)}$ denote the hypergraph containing the edge sets of all graphs on n vertices with minimum degree at least k. The main result of our paper is the following theorem.

Theorem 2.1 *If $b \geq \frac{n-1}{\log(n-2)-1}$ and n is sufficiently large, then Avoider has a winning strategy in the monotone $(1:b)$ game $\mathcal{D}_1$. Therefore,*

$$f^{mon}_{\mathcal{D}_1} \leq (1+o(1))\frac{n}{\log n}.$$

This theorem coupled with Theorem 1.5 of [6] exemplifies that in the connectivity game Avoider *does benefit* from having the possibility of taking *more* than one edge per move. As proved in [6], when playing according to the strict rules, Avoider can only win if the bias of Enforcer is at least as large as $\lfloor \frac{n-1}{2} \rfloor + 1$, so Avoider will have less than $n-1$ edges at the end. On the other hand, when playing according to the monotone rules, Avoider can avoid building a connected graph even if Enforcer's bias is as small as $\Theta\left(\frac{n}{\log n}\right)$.

Combined with the results of [8], Theorem 2.1 also has the following important corollary. Let $\mathcal{C}_k \subseteq 2^{E(K_n)}$ denote the hypergraph containing the edge sets of all k-connected spanning subgraphs of K_n.

Corollary 2.2

$$f^{mon}_{\mathcal{D}_k}, f^{mon}_{\mathcal{T}}, f^{mon}_{\mathcal{C}_k}, f^{mon}_{\mathcal{H}} = (1+o(1))\frac{n}{\log n}.$$

Note that for some of these games, such as "Hamiltonicity" or "k-connectivity", where $k \geq 2$, currently we do *not* have such tight results for the Maker-Breaker version.

3 Concluding remarks and open problems

A natural question one may ask is: *Which set of rules is "better" than the other?*

The advantage of monotone rules is the existence of a threshold bias for every game. Moreover, some of the obtained results concerning the threshold bias of the monotone Avoider-Enforcer game tend to show great similarity to their Maker-Breaker analogues.

The benefit of the strict rules lies in their applicability to Maker-Breaker games (see, e.g., [7]) or to discrepancy type games (see, e.g., [2,4]). In these applications, in order to provide a strategy for Maker or for Breaker, one defines an auxiliary Avoider-Enforcer game which models the original Maker-Breaker game, and uses the winning strategy of Avoider or Enforcer in the auxiliary game. Clearly, in this situation the monotone rules are useless.

References

[1] J. Beck, Random graphs and positional games on the complete graph, *Annals of Discrete Math.*, 28 (1985), 7–13.

[2] J. Beck, **Combinatorial Games: Tic-Tac-Toe Theory**, Cambridge University Press, 2008.

[3] V. Chvátal and P. Erdős, Biased positional games, *Annals of Discrete Math.*, 2 (1978), 221–228.

[4] A. Frieze, M. Krivelevich, O. Pikhurko and T. Szabó, The game of JumbleG, *Combinatorics, Probability and Computing*, 14 (2005), 783–793.

[5] H. Gebauer and T. Szabó, Asymptotic random graph intuition for the biased connectivity game, to appear in *Random Structures Algorithms*.

[6] D. Hefetz, M. Krivelevich and T. Szabó, Avoider-Enforcer games, *Journal of Combinatorial Theory, Ser. A.*, 114 (2007), 840–853.

[7] D. Hefetz, M. Krivelevich, M. Stojaković and T. Szabó, Planarity, colorability and minor games, *SIAM Journal on Discrete Mathematics*, 22 (2008), 194-212.

[8] M. Krivelevich and T. Szabó, Biased positional games and small hypergraphs with large covers, *Electronic Journal of Combinatorics*, 15 (1) (2008), publ. R70.