Digitalizing line segments

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Abstract

We introduce a novel and general approach for digitalization of line segments in the plane that satisfies a set of axioms naturally arising from Euclidean axioms. In particular, we show how to derive such a system of digital segments from any total order on the integers. As a consequence, using a well-chosen total order, we manage to define a system of digital segments such that all digital segments are, in Hausdorff metric, optimally close to their corresponding Euclidean segments, thus giving an explicit construction that resolves the main question of [1].

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1 Introduction

One of the most fundamental challenges in digital geometry is to define a "good" digital representation of a geometric object. Of course, the meaning of the word "good" here heavily depends on particular conditions we may impose. Looking at the problem of digitalization in the plane, the goal is to find a set of points on the integer grid $\mathbb{Z}^2$ that approximates well a given object. The topology of the grid $\mathbb{Z}^2$ is commonly defined by the graph whose vertices are all the points of the grid, and each point is connected by an edge to each of the four points that are either horizontally or vertically adjacent to it.

Knowing that a straight line segment is one of the most basic geometric objects and a building block for many other objects, defining its digitalization in a satisfying manner is vital. Hence, it is no wonder that this has been a hot scientific topic in the last few decades, see [3] for a recent survey and [2] for related work, dealing with the problem of representing objects in digital geometry without causing topological and combinatorial inconsistencies.

For any pair of points p and q in the grid $\mathbb{Z}^2$ we want to define the digital line segment $S(p,q)$ connecting them, that is, $\{p,q\} \subseteq S(p,q) \subseteq \mathbb{Z}^2$. Chun et al. in [1] put forward the following four axioms that arise naturally from properties of line segments in Euclidean geometry.

(S1) *Grid path property:* For all $p,q \in \mathbb{Z}^2$, $S(p,q)$ is the vertex set of a path from p to q in the grid graph.

(S2) *Symmetry property:* For all $p,q \in \mathbb{Z}^2$, we have $S(p,q) = S(q,p)$.

(S3) *Subsegment property:* For all $p,q \in \mathbb{Z}^2$ and every $r \in S(p,q)$, we have $S(p,r) \subseteq S(p,q)$.

(S4) *Prolongation property:* For all $p,q \in \mathbb{Z}^2$, there exists $r \in \mathbb{Z}^2$, such that $r \notin S(p,q)$ and $S(p,q) \subseteq S(p,r)$.

First, note that (S3) is not satisfied by the usual way a computer visualizes a segment. A natural definition of the digital straight segment between $p = (p_x, p_y)$ and $q = (q_x, q_y)$, where $p_x \leq q_x$ and $0 \leq q_y - p_y < q_x - p_x$ is $\left\{\left(x, \left\lfloor (x-p_x)\frac{q_y-p_y}{q_x-p_x} + p_y + 0.5\right\rfloor\right) : p_x \leq x \leq q_x\right\}$. This does not satisfy (S1), but it could be easily fixed by a slight modification of the definition. Still, it also does not satisfy (S3), for example, for $p = (0,0), r = (1,0), q = (4,1)$, the subsegment from r to q is not contained in the segment from p to q.

Even though the set of axioms (S1)-(S4) seems rather natural, there are still some fairly exotic examples of digital segment systems that satisfy all four of them. For example, let us fix a double spiral $\mathcal{D}$ centered at an arbitrary point of $\mathbb{Z}^2$, traversing all the points of $\mathbb{Z}^2$. As it is a spanning path of the grid graph, we can set $S(p, q)$ to be the path between p and q on $\mathcal{D}$, for every $p, q \in \mathbb{Z}^2$. It is easy to verify that this system satisfies axioms (S1)-(S4).

Another condition was introduced in [1] to enforce the monotonicity of the segments, ruling out pathological examples like the one above. Here, we phrase this monotonicity axiom differently, but still, the system of axioms (S1)-(S5) remains equivalent to the one given in [1].

(S5) *Monotonicity property:* If both $p, q \in \mathbb{Z}^2$ lie on a line that is either horizontal or vertical, then the whole segment $S(p, q)$ belongs to the line.

We call a system of digital line segments that satisfies the system of axioms (S1)-(S5) a *consistent digital line segments system (CDS)*. It is straightforward to verify that every CDS also satisfies the following three conditions.

(c1) If the slope of the line going through p and q is non-negative, then the slope of the line going through any two points of $S(p, q)$ is non-negative. The same holds for non-positive slopes.

(c2) For all $p, q \in \mathbb{Z}^2$, the grid-parallel box spanned by points p and q contains $S(p, q)$.

(c3) If the intersection of two digital segments contains two points $p, q \in \mathbb{Z}^2$, then their intersection also contains the whole digital segment $S(p, q)$.

We give a simple example of a CDS, where the segments follow the boundary of the grid-parallel box spanned by the endpoints. Let $p, q \in \mathbb{Z}^2$ be two points with coordinates $p = (p_x, p_y)$ and $q = (q_x, q_y)$. If $p_y \leq q_y$, we define $S(p, q) = S(q, p) = \{(x, p_y) : \min\{p_x, q_x\} \leq x \leq \max\{p_x, q_x\}\} \cup \{(q_x, y) : p_y \leq y \leq q_y\}\}$. If $p_y > q_y$, we swap the points p and q, and define the segment as in the previous case.

It can be verified that this way we defined a CDS, but the digital segments in this system visually still do not resemble well the Euclidean segments.

One of the standard ways to measure how close a digital segment is to a Euclidean segment is to use the Hausdorff distance. We denote by $\overline{pq}$ the Euclidean segment between p and q, and by $|\overline{pq}|$ the Euclidean length of $\overline{pq}$. For two plane objects A and B, by $H(A, B)$ we denote their Hausdorff distance.

The main question raised in [1] was if it is possible to define a CDS such that a Euclidean segment and its digitalization have a reasonably small Hausdorff distance. More precisely, the goal is to find a CDS satisfying the following condition.

(H) *Small Hausdorff distance property:* For every $p, q \in \mathbb{Z}^2$, we have that $H(\overline{pq}, S(p, q)) = O(\log |\overline{pq}|)$.

Note that in the CDS example we gave, the Hausdorff distance between a Euclidean segment of length n and its digitalization can be as large as $n/\sqrt{2}$.

While this question was not resolved in [1], a clever construction of a system of digital rays *emanating from the origin of* $\mathbb{Z}^2$ that satisfy (S1)-(S5) and (H) was presented. Moreover, it was shown using Schmidt's theorem [4] that already for rays emanating from the origin, the log-bound imposed in condition (H) is the best bound we can hope for, directly implying the following theorem.

Theorem 1.1 [1] *There exists a constant $c > 0$, such that for any CDS and any $d > 0$, there exist $p, q \in \mathbb{Z}^2$ with $|\overline{pq}| > d$ and $H(\overline{pq}, S(p, q)) > c \log |\overline{pq}|$.*

In this paper, we introduce a novel and general approach for the construction of a CDS. Namely, for any total order $\prec$ on $\mathbb{Z}$, we show how to derive a CDS from $\prec$. (By total order we always mean a strict total order.) As a consequence, we manage to define a CDS that satisfies (H), deriving it from a specially chosen order on $\mathbb{Z}$, and thus giving the explicit construction that resolves the main question of [1].

Theorem 1.2 *There is a CDS that satisfies condition (H).*

Note that Theorem 1.1 ensures that such a CDS is optimal up to a constant factor in terms of the Hausdorff distance from the Euclidean segments.

2 Digital line segments derived from a total order on $\mathbb{Z}$

Let $\prec$ be a total order on $\mathbb{Z}$. We are going to define a CDS $\mathcal{S}_\prec$, deriving it from $\prec$.

Let $p, q \in \mathbb{Z}^2$, $p = (p_x, p_y)$ and $q = (q_x, q_y)$. If $p_x > q_x$, we swap p and q. Hence, from now on we may assume that $p_x \leq q_x$.

If $p_y \leq q_y$, then $S_\prec(p, q)$ is defined as follows. We start at the point $p = (p_x, p_y)$ and we repeatedly go either up or to the right, collecting the points from $\mathbb{Z}^2$, until we reach q. Note that the sum of the coordinates $x + y$ increases by 1 in each step. In total we have to make $q_x + q_y - p_x - p_y$ steps and in exactly $q_y - p_y$ of them we have to go up. The decision whether to go up or to the right is made as follows: if we are at the point (x, y) for which $x + y$ is among the $q_y - p_y$ greatest elements of the interval $[p_x + p_y, q_x + q_y - 1]$ according to $\prec$, we go up, otherwise we go to the right. We will refer to this interval as the *segment interval.*

If $p_y > q_y$, that is, if p is the top-left and q the bottom-right corner of the grid-parallel box spanned by p and q, then we define $S_\prec(p,q)$ as the mirror reflection of $S_\prec((-q_x,q_y),(-p_x,p_y))$ over the y-axis.

Theorem 2.1 *$\mathcal{S}_\prec$, defined as above, is a CDS.*

Example. Suppose $p=(0,0)$ and $q=(2,2)$. Their segment interval consists of four numbers, $0,1,2,3$. If $\prec$ is the natural order on $\mathbb{Z}$, then the two greatest elements of the segment interval are 2 and 3. Since $0+0$ is not one of these, at $(0,0)$ we go right, to $(1,0)$. At $(1,0)$ we again go to right, to $(2,0)$, from there to $(2,1)$ (since $2+0$ is one of the greater elements) and finally to $(2,2)$. In fact, it can be easily seen that using the natural order on $\mathbb{Z}$ we get the CDS that we already mentioned, the one that always follows the boundary of the box spanned by the endpoints.

3 Digital segments with small Hausdorff distance to Euclidean segments

For integers k and $l \geq 2$, let $|k|_l$ denote the number of times k is divisible by l, that is,

$$|k|_l = \sup\{m : l^m \mid k\}.$$

We define a total order on $\mathbb{Z}$ as follows. Let $a \prec b$ if and only if there exists a non-negative integer i such that $|a-i|_2 < |b-i|_2$, and for all $j \in \{0,\dots,i-1\}$ we have $|a-j|_2 = |b-j|_2$. In plain words, for two integers a and b, we say that the one that contains a higher power of 2 is greater under $\prec$. In case of a tie, we repeatedly subtract 1 from both a and b, until at some point one of them contains a higher power of 2 than the other. Thus, for example, $-1 \prec -5 \prec 3 \prec -3 \prec 5 \prec 1 \prec -2 \prec 6 \prec -6 \prec 2 \prec -4 \prec 4 \prec 0$.

Note that if we take the elements of an interval of the form $(-2^n, 2^n)$ in $\prec$-decreasing order and we apply the function $0.5 - x2^{-n-1}$ to them, then we get the first few elements of the Van der Corput sequence [5].

We can show that using this total order to define the system of digital line segments $\mathcal{S}_\prec$, as described in the previous section, we obtain a CDS which satisfies condition (H), thus proving Theorem 1.2.

At first sight it may be surprising to observe that all the segments emanating from the origin in our construction coincide with the ones given in the construction of digital rays in [1]. However, it is not a coincidence, as the construction from [1] also relies on the same total order on integers.

4 Higher dimensions

It is natural to ask whether there is a CDS in more than two dimensions. The definition of a CDS directly carries over to the higher dimensional spaces – we consider $\mathbb{Z}^d$, for a fixed integer $d \geq 3$, with the usual graph structure, that is, two points p and q are adjacent if they differ in exactly one coordinate by exactly one. The axioms (S1)-(S4) stay the same, and the monotonicity axiom (S5) now reads as follows: If for $p = (p_1, \ldots, p_d), q = (q_1, \ldots, q_d) \in \mathbb{Z}^d$ there is an i such that $p_i = q_i$, then for all $r \in S(p, q)$ we have $r_i = p_i = q_i$.

It is not hard to see that we can again derive a consistent system from an arbitrary total order $\prec$ on $\mathbb{Z}$ if we consider only segments that have positive slope in all coordinates; the only difference is that now we have to cut the segment interval $[p_1 + \ldots + p_d, q_1 + \ldots + q_d - 1]$ into d parts, according to $\prec$.

Theorem 4.1 *The definition above yields a CDS of segments with positive slope in all coordinates.*

Of course, we can use the same construction to define segments for all the remaining slope types. However, unlike in the 2-dimensional case, putting them all together in an attempt to construct a complete CDS fails, as the axiom (S3) is violated. We are curious if this approach to construction can be modified to yield a CDS.

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