Winning fast in biased Maker-Breaker games

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Abstract

We study the biased $(1 : b)$ Maker-Breaker positional games, played on the edge set of the complete graph on n vertices, K_n. Given Breaker's bias b, possibly depending on n, we determine the bounds for minimal number of moves, depending on b, in which Maker can win in each of the two standard graph games, the Perfect Matching game and the Hamilton Cycle game.

1 Introduction

In a Maker-Breaker positional game, a finite set X and a family $\mathcal{E}$ of subsets of X are given, and two players, Maker and Breaker, alternate in claiming unclaimed elements of X until all the elements are claimed, with Breaker going first. In the standard setting, each player claims exactly one element of X. Maker wins if he claims all elements of a set from $\mathcal{E}$, and Breaker wins otherwise. The set X is referred to as the *board*, and the elements of $\mathcal{E}$ as the *winning sets*. As Maker-Breaker positional games are finite games of perfect information and no chance moves, we know that in every game one of the players has a winning strategy. More on various aspects of positional game theory can be found in the monograph of Beck [1] and in the recent monograph [9].

We are interested in positional games on graphs, where the board X is the edge set of a graph, and we will mostly deal with games played on the edge set of the complete graph $E(K_n)$. Three standard positional games that we will look at here are *Connectivity* game, where Maker wants to claim a spanning tree, *Perfect Matching* game, where the winning sets are all perfect matchings of the base graph, and *Hamilton Cycle* game, where Maker's goal is to claim a Hamilton cycle.

Once the order n of the base graph gets large, it turns out that Maker can win in each of the three mentioned games in a straightforward fashion. But

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our curiosity does not end there, as there are several standard approaches to make the setting more interesting to study. One of them is the so-called *biased games*, where Breaker is given more power by being allowed to claim more than one edge per move. The other approach we focus on is the *fast win* of Maker, where the question we want to answer is not just if Maker can win, but also how fast he can win.

Given a positive integer b, in the $(1 : b)$ biased game Breaker claims b edges in each move, while Maker claims a single edge. The parameter b is called the bias. Maker-Breaker games are "bias monotone", i.e. if Breaker wins the $(1 : b)$ game, for some value of b, then he also wins the $(1 : b + 1)$ game. Due to the bias monotonicity of Maker-Breaker games, it is straightforward to conclude that for any positional game there is some value $b_0 = b_0(n)$ such that Maker wins the game for all $b < b_0(n)$, while Breaker wins for $b \geq b_0(n)$ (see [9] for details). We call $b_0(n)$ the *threshold bias* for that game.

The biased games were first introduced and studied by Chvátal and Erdős in [2], and some thirty years later the papers of Gebauer and Szabó [7] and Krivelevich [11] finally located the leading term of the thresholds for the games of Connectivity, Perfect Matching and Hamilton Cycle, which turned out to be $n/\ln n$ for all three games.

Moving on to the concept of fast winning, when we know that Maker can win an unbiased game, a natural question that we can ask is – what is the minimum number of moves for Maker to win? Questions of this type appeared frequently, often as subproblems, in classical papers on positional games, and the concept of fast Maker's win was further formalized in [8]. It is not hard to see that Maker can win the unbiased Connectivity game as fast as the size of the winning set allows, in $n - 1$ moves. For the other two games it takes him a bit longer (one move longer, to be more precise) – he can win unbiased Perfect Matching game in $n/2 + 1$ moves (for n even) [8], and unbiased Hamilton cycle game in $n + 1$ moves [10], and in both cases that is the best he can do. We note that some research has also been done on fast Maker's win in unbiased k-Connectivity, Perfect Matching and Hamilton Cycle games played on the edge set of a random graph, see [3].

Knowing how fast Maker can win, and how to win fast, is important, as this often helps us when looking at other positional games. Indeed, there are numerous examples where a player's winning strategy may call for building a certain structure *quickly* before proceeding to another task. Also, one of very few tools that proved to be useful when tackling the so-called strong positional games are the fast Maker's winning strategies, see [4, 5, 6].

2 Results

Our goal here is to combine the two presented concepts – the biased games and the fast winning, looking into the possibilities for Maker to *win fast in biased games.* In other words, given a game $\mathcal{G}$ and a bias b such that Maker can win

$(1:b)$ biased game, we want to know in how many moves he can win the game. One obvious lower bound for the duration of the game is the size of the smallest winning set, and that is $n-1$ for Connectivity, $\frac{n}{2}$ for Perfect Matching, and n for Hamilton Cycle.

It is not hard to see that in Connectivity game Maker does not ever need to close a cycle, and therefore, even in the biased game, whenever he can win he can do so in exactly $n-1$ moves. This imposes a natural question whether Maker can do the same in case of other two aforementioned games, i.e. to win in optimal number of moves whenever he has a winning strategy. It turns out that for the biased games of Perfect Matching and Hamilton Cycle this is not possible, as Maker needs additional moves to finish building the desired graphs. Hence, we are also interested in seeing how the increase in bias influences the duration of the games.

The following theorem gives fast Maker's win in Perfect Matching game for most of the range of biases for which Maker can win, up to the (order of the) threshold for Maker's win in the game.

Theorem 2.1 *There exist $\delta > 0$ and $C > 0$ such that for every $b \leq \frac{\delta n}{100 \ln n}$, Maker can win the $(1:b)$ Perfect Matching game played on $E(K_n)$ within $\frac{n}{2} + Cb \ln b$ moves, for large enough n.*

Moving on to Hamilton Cycle game, winning quickly becomes more difficult for Maker. In this game, the strategies of Maker change as bias grows, unlike in Perfect Matching game, where Maker can use the same strategy on almost whole range of biases for which he can win. The following two theorems give results for fast Maker's win in Hamilton Cycle game for different bias ranges. The first one is more powerful, but it applies only for small values of bias, while the second one covers a wider range of bias.

Theorem 2.2 *There exists $C > 0$ such that for $b \geq 2$ and $b = o\left(\frac{\ln n}{\ln \ln n}\right)$, Maker can win the $(1:b)$ Hamilton Cycle game game played on $E(K_n)$ within $n + Cb^2 \ln b$ moves, for large enough n.*

Theorem 2.3 *There exist $\delta > 0$ and $C > 0$, such that for $b \leq \delta \sqrt{\frac{n}{\ln^5 n}}$ Maker can win the $(1:b)$ Hamilton Cycle game played on $E(K_n)$ within $n + Cb^2 \ln^5 n$ moves, for large enough n.*

Finally, when the bias is large, we can apply the following result of Krivelevich [11], as it provides Maker with a win within $14n$ moves in Hamilton Cycle game, and thus also in Perfect Matching game.

Theorem 2.4 ([11], Theorem 1) *Maker can win the $(1:b)$ Hamilton Cycle game played on $E(K_n)$ in at most $14n$ moves, for every $b \leq \left(1 - \frac{30}{\ln^{1/4} n}\right) \frac{n}{\ln n}$, for large enough n.*

On the other hand, we are curious about finding Breaker's strategies in both Perfect Matching and Hamilton Cycle games for preventing Maker from winning in optimal number of moves. The following theorem shows that Breaker can postpone Maker's win, and that we can move away from the obvious lower bound in both games.

Theorem 2.5 *In* $(1 : b)$ *Maker-Breaker game, for every bias* b*, Breaker can postpone Maker's win*

(i) in Perfect Matching game for at least $\frac{n}{2} + \frac{b}{4}$ *moves,*

(ii) in Hamilton Cycle game for at least $n + \frac{b}{2}$ *moves.*

3 Concluding remarks

If the number of moves Maker needs to play in order to win in Perfect Matching game is denoted by $p(b)$, on the whole range of biases between 1 and $(1 - o(1))n/\ln n$ we have that $\frac{b}{4} \leq p(b) - \frac{n}{2} \leq O(b \ln b)$, as given by Theorem 2.1, Theorem 2.4 and Theorem 2.5 (i).

In Hamilton Cycle game, if we denote the number of moves Maker needs to play in order to win by $h(b)$, then Theorem 2.2, Theorem 2.3, Theorem 2.4 and Theorem 2.5 (ii) provide non-trivial upper and lower bounds for the whole range of biases between 1 and $(1 - o(1))n/\ln n$. If we look at the value $h(b) - n$ and express both the upper and lower bounds as functions of b, the lower bound on the whole range is $\frac{b}{2}$, while the upper bound varies between $b^{1+\varepsilon}$ and $b^{7+\varepsilon}$, for any $\varepsilon > 0$. In particular, for b a constant, both upper and lower bounds are a constant.

Finding the right order of magnitude of both $p(b) - \frac{n}{2}$ and $h(b) - n$ remains an open problem, and we are particularly curious if they are linear in b.

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