Bichromatic Triangle Games

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Abstract

We study a combinatorial game called Bichromatic Triangle Game, defined as follows. Two players $\mathcal{R}$ and $\mathcal{B}$ construct a triangulation on a given planar point set V. Starting from no edges, players $\mathcal{R}$ and $\mathcal{B}$ take turns drawing one edge that connects two points in V. Player $\mathcal{R}$ uses color red and player $\mathcal{B}$ uses color blue. The first player who completes one empty monochromatic triangle is the winner. We show that either player can force a tie in the Bichromatic Triangle Game when the points of V are in convex position and also in the case when there is exactly one inner point in the set V. As an easy consequence of those results, we obtain that the outcome of the Bichromatic Complete Triangulation Game (a version of the Bichromatic Triangle Game in which players draw edges until they complete a triangulation) is also a tie for the same two cases regarding set V.

Keywords: Combinatorial games on graphs, games on triangulations, bichromatic triangle game, polynomial algorithm

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[1] Grant 2008/06508-8 from *Fundação de Amparo à Pesquisa do Estado de São Paulo*. Partially supported by *Conselho Nacional de Desenvolvimento Científico e Tecnológico* Project 475064/2010-0. Email: `manic.gordana@ufabc.edu.br`

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[3] Partly supported by *Ministry of Science and Environmental Protection, Republic of Serbia*, and *Provincial Secretariat for Science, Province of Vojvodina*. Email: `smilos@inf.ethz.ch`

1 Introduction

Games on triangulations belong to the more general area of combinatorial games on graphs, which usually involve two players. We consider games with perfect information (i.e., there is no hidden information in contrast to many card games) and each of the players plays optimally (i.e., both players do their best to win). Any such game without ties has two possible outcomes: whoever moves first can force the opponent to lose, or else whoever moves second can force the opponent to lose, no matter how the other player moves throughout the game. Such forcing procedures are called *winning strategies*. However, combinatorial games with two players may have a third outcome: one of the players can force a tie. For more information on combinatorial game theory we refer the reader to [2], [3] and [4].

Let $V \subseteq \mathbb{R}^2$ be a set of points in the plane with no three collinear points. A *triangulation* of V is a simplicial decomposition of its convex hull whose vertices are precisely the points in V. Aichholzer et al. [1] consider several combinatorial games involving the vertices, edges (straight line segments), and faces (triangles) of some triangulation. The goal for each game is to characterize who wins the game and design efficient algorithms to compute a winning strategy, or else to characterize which player can force a tie and determine his defense strategy.

Aichholzer et al. [1] conjecture that for many of the games they consider even determining the outcome may be NP-hard for general triangulations, that is, when there is no predetermined condition for the points of V apart from not having any three collinear points. Therefore, they focus their attention to special classes of triangulations where positive results are possible, for example when points from V are in convex position. We say that the points of a set V are in *convex position* when they are the vertices of some convex polygon.

Aichholzer et al. [1] leave as an open problem to characterize the outcomes of the Bichromatic Triangle Game and the Bichromatic Complete Triangulation Game. We first define the Bichromatic Complete Triangulation Game. Two players $\mathcal{R}$ and $\mathcal{B}$ construct a triangulation on a given point set V. Starting from no edges, players $\mathcal{R}$ and $\mathcal{B}$ play in turns by drawing one edge in each move (with $\mathcal{R}$ making the first move). In each move the chosen edge is not allowed to cross any of the previously drawn edges. Player $\mathcal{R}$ uses color red and player $\mathcal{B}$ uses color blue. Each time a player completes one or more empty monochromatic triangles, the player wins the corresponding number of points and it is again his turn (an extra move). A triangle is said to be *empty* when it contains no points from V in its interior. Once the triangulation is complete, the game stops and the player who owns more points is the winner. Bichromatic Triangle Game starts as Bichromatic Complete Triangulation Game, but has a different stopping condition. Namely, the first player who completes one empty monochromatic triangle is the winner. We show here that $\mathcal{B}$ can force a tie in the Bichromatic Triangle Game when the points in V are in convex position, and also when there is exactly one inner point in V (we say that $v \in V$ is an *inner point* of V if v belongs to the interior of the convex hull of V). As an easy consequence of the results we obtained for the Bichromatic Triangle Game, we have that the outcome of the Bichromatic Complete Triangulation Game is also a tie if the points of V are in convex position and in the case when there is exactly one inner point in the set V.

Combinatorial games keep attracting the interest of mathematicians and computer scientists because they have applications to modeling several real-life applications and also because they reveal mathematical properties of the underlying structures, in our case, of planar triangulations [1].

1.1 Notation

Let $V \subseteq \mathbb{R}^2$ be a set of points in the plane. For $u, v \in V$, denote by uv the line segment with endpoints u and v, which we loosely call an *edge*. We conveniently identify the set of all such line segments with $\binom{V}{2}$. A *configuration* of the Bichromatic Triangle Game is a triple $(V, E_{\mathcal{R}}, E_{\mathcal{B}})$, where $V \subseteq \mathbb{R}^2$, $E_{\mathcal{R}}, E_{\mathcal{B}} \subseteq \binom{V}{2}$ are the sets of edges drawn by $\mathcal{R}$ and $\mathcal{B}$ respectively, and furthermore, no two edges in $E_{\mathcal{R}} \cup E_{\mathcal{B}}$ are allowed to cross. A *free edge* with respect to a configuration $(V, E_{\mathcal{R}}, E_{\mathcal{B}})$ is an edge in $\binom{V}{2} \setminus (E_{\mathcal{R}} \cup E_{\mathcal{B}})$ that does not cross any of the edges in $E_{\mathcal{R}} \cup E_{\mathcal{B}}$. Given a configuration $\mathscr{C} = (V, E_{\mathcal{R}}, E_{\mathcal{B}})$ and a set of points $W \subseteq V$, we define the *induced configuration* $\mathscr{C}[W] = (W, E'_{\mathcal{R}}, E'_{\mathcal{B}})$ where $E'_{\mathcal{R}} = \binom{W}{2} \cap E_{\mathcal{R}}$ and $E'_{\mathcal{B}} = \binom{W}{2} \cap E_{\mathcal{B}}$. Two configurations are said to be *independent* if they share precisely one edge which has already been taken by one of the players in both configurations. Given points $x, y, z \in \mathbb{R}^2$, with the sequence (x, y, z) being in clockwise order and non-collinear, we denote by $\widehat{xyz}$ the open subset of points $w \in \mathbb{R}^2$ such that the sequences (x, y, w) and (w, y, z) are both in clockwise order and both non-collinear. Let V be a set of points in the plane in convex position. Suppose the points of V are $v_1, \ldots, v_n$ in clockwise order. For $x = v_i, y = v_j$, we denote by $\overset{\frown}{xy}$ the set of points $\{v_i, v_{i+1}, \ldots, v_j\}$, where index addition is taken modulo n. Moreover, for $x, y \in V$, if $x = v_i$ and $y = v_{i+1}$, we say that x and y are *consecutive* in V.

2 Our Contributions

Lemma 2.1 *Suppose $\mathscr{C} = (V, E_{\mathcal{R}}, E_{\mathcal{B}})$ is a configuration of the Bichromatic Triangle Game where V is a set of points in convex position and all drawn* *edges, i.e., edges in $E_{\mathcal{R}} \cup E_{\mathcal{B}}$, are between consecutive points in V. If $|E_{\mathcal{R}}| < 2$, then $\mathcal{B}$ can force a tie in $\mathscr{C}$ regardless of which player is next to make a move.*

Proof. The proof is by induction on $|V|$. If $|V| \leq 3$, the statement holds. Hence, suppose $|V| = n > 3$ and suppose the statement is true for $|V| < n$. We denote by r the cardinality of $E_{\mathcal{R}}$.

If $\mathcal{R}$ is next to play, let uv be the edge that $\mathcal{R}$ draws, and let $\mathscr{C}'$ be the resulting configuration, i.e., $\mathscr{C}' = (V, E_{\mathcal{R}} \cup \{uv\}, E_{\mathcal{B}})$. This move divided $\mathscr{C}$ into two independent configurations $\mathscr{C}_1 = \mathscr{C}'[\overrightarrow{\frown}{uv}]$ and $\mathscr{C}_2 = \mathscr{C}'[\overrightarrow{\frown}{vu}]$ with uv belonging to both configurations (one of $\mathscr{C}_1$, $\mathscr{C}_2$ consists of a single edge if u and v are consecutive in V). In order for $\mathcal{B}$ to force a tie in $\mathscr{C}$, he must force a tie both in $\mathscr{C}_1$ and in $\mathscr{C}_2$. If $r = 1$, we may assume, without loss of generality, that $\mathscr{C}_1$ has two red edges, and that $\mathscr{C}_2$ has one red edge (including edge uv in both cases). Note that a tie can be easily forced when the configuration has at most three points, so we may assume that $\mathscr{C}_1$ has at least four points. Now $\mathcal{B}$ picks a free edge xy in $\mathscr{C}_1$ with $\mathscr{C}_1[\overrightarrow{\frown}{xy}]$ and $\mathscr{C}_1[\overrightarrow{\frown}{yx}]$ each containing a single red edge. The existence of such an edge is guaranteed by the properties of $\mathscr{C}_1$, i.e., it has at least four points, its vertices are in convex position, and its drawn edges are between consecutive points (note that xy and uv may have a common vertex). After $\mathcal{B}$ draws xy we obtain a configuration $\mathscr{C}_1'$ which is divided into two independent configurations $\mathscr{C}_1'[\overrightarrow{\frown}{xy}]$ and $\mathscr{C}_1'[\overrightarrow{\frown}{yx}]$. By the induction hypothesis, $\mathcal{B}$ can force a tie in both $\mathscr{C}_1'[\overrightarrow{\frown}{xy}]$ and $\mathscr{C}_1'[\overrightarrow{\frown}{yx}]$ and hence $\mathcal{B}$ can force a tie in $\mathscr{C}_1$. As for $\mathscr{C}_2$, it is either trivial i.e., a single edge, or $\mathcal{B}$ can force a tie in it by the induction hypothesis. If, however, $r = 0$ and u and v are not consecutive in V, then both $\mathscr{C}_1$ and $\mathscr{C}_2$ have one red edge, and by the induction hypothesis $\mathcal{B}$ can force a tie in $\mathscr{C}_1$ and in $\mathscr{C}_2$. Finally, if $r = 0$ and u and v are consecutive in V, then we set $\mathscr{C} = \mathscr{C}'$ and apply the argument from the next paragraph.

If $\mathcal{B}$ is next to play he can choose to draw any free edge that does not connect two consecutive points from V, dividing $\mathscr{C}$ into two independent configurations $\mathscr{C}_1$ and $\mathscr{C}_2$. By the induction hypothesis $\mathcal{B}$ can force a tie both in $\mathscr{C}_1$ and in $\mathscr{C}_2$. Hence, $\mathcal{B}$ can force a tie in $\mathscr{C}$. □

Theorem 2.2 *Player $\mathcal{B}$ can force a tie in the Bichromatic Triangle Game when the points in V are in convex position.*

Proof. The first edge that $\mathcal{R}$ draws divides the initial configuration $\mathscr{C}$ into two independent configurations (one possibly trivial). By Lemma 2.1, $\mathcal{B}$ can force a tie in both of these configurations, and hence, $\mathcal{B}$ can force a tie in $\mathscr{C}$.□

Theorem 2.3 *Player $\mathcal{B}$ can force a tie in the Bichromatic Triangle Game when the set of points V has exactly one inner point.*

Proof. We analyze every possible move of $\mathcal{R}$ and determine what $\mathcal{B}$'s response should be. Every time player $\mathcal{B}$ draws an edge, we divide the problem of forcing a tie into smaller independent subproblems. This process may be repeated more than once until we fall into one of the following three types of subproblems in which player $\mathcal{B}$ can force a tie. The first type of subproblems are configurations with a set of points in convex position with all drawn edges being between consecutive points and with at most one red edge. Player $\mathcal{B}$ can force a tie in those subproblems by Lemma 2.1. The second type of subproblems are trivial configurations on four points with some drawn edges. The third type of configurations are of a special kind, which we analyze separately. It is also part of $\mathcal{B}$'s defense strategy to make its move in the same subproblem as player $\mathcal{R}$'s last move, unless there are no more free edges to be drawn in that particular subproblem. In the later case, either the game is finished or $\mathcal{B}$ makes a move in any subproblem which still has free edges.

Let x be the unique inner point of V. Initially we have the configuration $(V, \emptyset, \emptyset)$ and it is $\mathcal{R}$'s turn. We have two possibilities: (i) $\mathcal{R}$ draws an edge ux for some $u \neq x$, or (ii) $\mathcal{R}$ draws an edge uv for $u, v \neq x$.

If (i) happens, let r be the line passing through x that is perpendicular to the segment ux, and let s be the line containing the segment ux. Note that r divides the plane into two half-planes, and there must exist at least one point v in the half-plane not containing u. Among all possible choices, pick v as close as possible to the line s. Now $\mathcal{B}$ draws xv and we are left with subproblems $(\overset{\frown}{uv} \cup \{x\}, \{ux\}, \{xv\})$ and $(\overset{\frown}{vu} \cup \{x\}, \{ux\}, \{xv\})$. One of them is convex, and the other is handled by the Special Subproblem below.

Special Subproblem. *Player $\mathcal{B}$ can force a tie in the configuration $(W, \{ux\}, \{xv\})$, where x is the unique inner point of W, points u and v are consecutive in W, and no points are contained in $\widehat{vux}$.* Indeed, consider the points of $W - x$ in clockwise order, and let w be the first point after v. If player $\mathcal{B}$ can draw uw, he does such a move and we are left with a trivial subproblem on four points, and one or more convex subproblems (see Fig. 1(a); the dashed red edges are just examples of possible moves of $\mathcal{R}$ that do not prevent $\mathcal{B}$ from playing uw). Otherwise, $\mathcal{R}$ has drawn either uw or vz or xz for some $z \in \overset{\frown}{wu}$, with $z \neq u, w$. If $\mathcal{R}$ drew uw then $\mathcal{B}$ draws xw (see Fig. 1(b)), and if $\mathcal{R}$ drew vz then $\mathcal{B}$ draws uz (see Fig. 1(c)). In both cases, the problem is split into a subproblem on four points, and one or more convex subproblems. If $\mathcal{R}$ drew xz, then $\mathcal{B}$ draws uz and we are left with two subproblems $(\overset{\frown}{zu}, \emptyset, \{uz\})$ and $(\overset{\frown}{vz} \cup \{x\}, \{xz\}, \{xv\})$ (see Fig. 1(d)). While the first subproblem is always convex (or just one edge), the second subproblem may not be convex. However, if $(\overset{\frown}{vz} \cup \{x\}, \{xz\}, \{xv\})$ is not convex, it is a smaller instance of Special Subproblem, which can be handled by induction. As for the case (ii), the analysis is omitted here due to space limitations. □

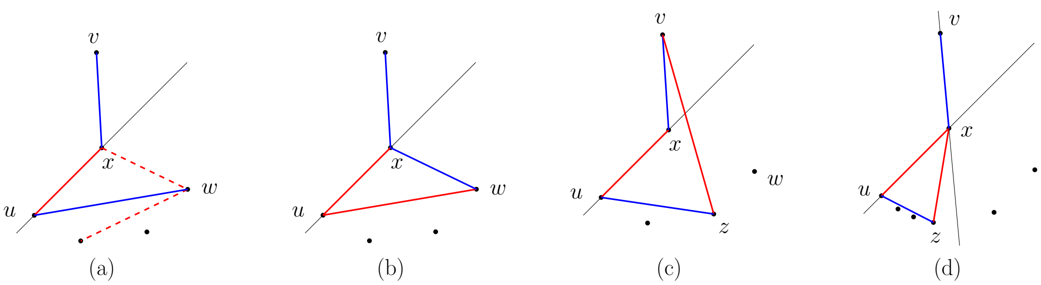

Fig. 1. Cases mentioned in the previous paragraph.

The proof that player $\mathcal{R}$ as well can force a tie when V has zero inner points is symmetric to that of Theorem 2.2, and for the case when V has one inner point the proof is even simpler than the proof of Theorem 2.3.

3 Summary and Remarks

We consider the Bichromatic Triangle and the Bichromatic Complete Triangulation Games and show that either player can force a tie in these games when the given points are in convex position or when there is exactly one inner point. Natural open questions that arise are to design polynomial algorithms to compute winning strategies or to determine which player can force a tie in these games in the general case, i.e., when there is no predetermined conditions for the given set of points, or else to show that the problem of characterizing the outcome of any of these games in the general case is NP-hard.

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