A threshold for the Maker-Breaker clique game*

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Abstract

We study the Maker-Breaker k-clique game played on the edge set of the random graph $G(n,p)$. In this game, two players, Maker and Breaker, alternately claim unclaimed edges of $G(n,p)$, until all the edges are claimed. Maker wins if he claims all the edges of a k-clique; Breaker wins otherwise. We determine that the threshold for the graph property that Maker can win this game is at $n^{-\frac{2}{k+1}}$, for all $k > 3$, thus proving a conjecture from [10]. More precisely, we conclude that there exist constants $c, C > 0$ such that when $p > Cn^{-\frac{2}{k+1}}$ the game is Maker's win a.a.s., and when $p < cn^{-\frac{2}{k+1}}$ it is Breaker's win a.a.s.

For the triangle game, when $k = 3$, we give a more precise result, describing the hitting time of Maker's win in the random graph process. We show that, with high probability, Maker can win the triangle game exactly at the time when a copy of K_5 with one edge removed appears in the random graph process. As a consequence, we are able to give an expression for the limiting probability of Maker's win in the triangle game played on the edge set of $G(n,p)$.

1 Introduction

Let X be a finite set and let $\mathcal{F} \subseteq 2^X$ be a family of subsets of X. In the positional game $(X, \mathcal{F})$, two players take turns in claiming one previously unclaimed element of X. The set X is called the "board", and the members of $\mathcal{F}$ are referred to as the "winning sets". In a *Maker-Breaker* positional game, the two players are called Maker and Breaker. Maker wins the game if he occupies all elements of some winning set; Breaker wins otherwise. We will assume that Maker starts the game. A game $(X, \mathcal{F})$ is said to be a *Maker's win* if Maker has a strategy that ensures his win against any strategy of Breaker; otherwise it is a *Breaker's win*. Note that $\mathcal{F}$ alone determines whether the game is Maker's win or Breaker's win.

A well-studied class of positional games are the *games on graphs*, where the board is the set of edges of a graph. The winning sets in this case are usually representatives of some graph theoretic structure. The first game studied in this area was the connectivity game, a generalization of the well-known Shannon switching game, where Maker's goal is to claim a spanning connected graph by the end of the game. We denote the game by $(E(K_n), \mathcal{T})$. Another important game is the Hamilton cycle game $(E(K_n), \mathcal{H})$, where $\mathcal{H} = \mathcal{H}_n$ consists of the edge sets of all Hamilton cycles of K_n.

*Part of this work was done during a visit of Miloš Stojaković to Amsterdam, supported by European Science Foundation (ESF), activity entitled 'Games for Design and Verification'. We gratefully acknowledge the support.

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In the clique game the winning sets are the edge sets of all k-cliques, for a fixed integer $k \geq 3$. We denote this game with $(E(K_n), \mathcal{K}_k)$. Note that the size of the winning sets is fixed and does not depend on n, which distinguishes it from the connectivity game and the Hamilton cycle game. A simple Ramsey argument coupled with the strategy stealing argument (see [1] for details) ensures Maker's win if n is large.

All three games that we introduced are straightforward Maker's wins when n is large enough. This is however not the end of the story. We present two general approaches to even out the odds, giving Breaker more power – *biased games* and *random games*.

Biased games. Biased games are a widely studied generalization of positional games, introduced by Chvátal and Erdős in [4]. Given two positive integers a and b and a positional game $(X, \mathcal{F})$, in the *biased* $(a : b)$ *game* Maker claims a previously unclaimed elements of the board in one move, while Breaker claims b previously unclaimed elements. The rules determining the outcome remain the same. The games we introduced initially are $(1 : 1)$ games, also referred to as *unbiased games*.

Now, if an unbiased game $(X, \mathcal{F})$ is a Maker's win, we choose to play the same game with $(1 : b)$ bias, increasing b until Breaker starts winning. Formally, we want to answer the following question: What is the largest integer $b_{\mathcal{F}}$ for which Maker can win the biased $(1 : b_{\mathcal{F}})$ game? This value is called the *threshold bias* of $\mathcal{F}$.

For the connectivity game, it was shown by Chvátal and Erdős [4] and Gebauer and Szabó [6] that the threshold bias is $b_{\mathcal{T}} = (1+o(1))\frac{n}{\log n}$. The result of Krivelevich [8] gives the leading term of the threshold bias for the Hamilton cycle game, $b_{\mathcal{H}} = (1+o(1))\frac{n}{\log n}$. In the k-clique game, Bednarska and Łuczak [2] found the order of the threshold bias, $b_{\mathcal{K}_k} = \Theta(n^{\frac{2}{k+1}})$.

2 Our results

Random games. An alternative way to give Breaker more power in a positional game, introduced by the second author and Szabó in [10], is to randomly thin out the board before the game starts, thus eliminating some of the winning sets.

For games on graphs, given a game $\mathcal{F}$ that is Maker's win when played on $E(K_n)$, we want to find the *threshold probability* $p_{\mathcal{F}}$ so that, if the game is played on $E(G(n,p))$, an almost sure Maker's win turns into an almost sure Breaker's win, that is,

$$Pr[\mathcal{F} \text{ played on } E(G(n,p)) \text{ is Breaker's win}] \to 1 \text{ for } p = o(p_{\mathcal{F}}),$$

and

$$Pr[\mathcal{F} \text{ played on } E(G(n,p)) \text{ is Maker's win}] \to 1 \text{ for } p = \omega(p_{\mathcal{F}}),$$

when $n \to \infty$. Such a threshold $p_{\mathcal{F}}$ exists, as "being Maker's win" is clearly a monotone increasing graph property.

The threshold probability for the connectivity game was determined to be $\frac{\log n}{n}$ in [10], and shown to be sharp. As for the Hamilton cycle game, the order of magnitude of the threshold was given in [9]. Using a different approach, it was proven in [7] that the threshold is $\frac{\log n}{n}$ and it is sharp. Finally, as a consequence of a hitting time result, Ben-Shimon et al. [3] closed this question by giving a very precise description of the low order terms of the limiting probability.

Moving to the clique game, it was shown in [10] that for every $k \geq 4$ and every $\varepsilon > 0$ we have $n^{-\frac{2}{k+1}-\varepsilon} \leq p_{\mathcal{K}_k} \leq n^{-\frac{2}{k+1}}$. Moreover, it was proved that there exist a constant $C > 0$ such that for $p \geq Cn^{-\frac{2}{k+1}}$ Maker wins the k-clique game on $G(n,p)$ a.a.s. The threshold for the triangle game was determined to be $p_{\mathcal{K}_3} = n^{-\frac{5}{9}}$, showing that the behavior of the triangle game is different from the k-clique game for $k \geq 4$, as $\frac{9}{5} < \frac{3+1}{2} = 2$.

Our main result is the following theorem. It gives a lower bound on the threshold for the k-clique game, when $k \geq 4$, which matches the upper bound from [10] up to the leading constant.

Theorem 2.1 *Let $k \geq 4$. There exists a constant $c > 0$ such that for $p \leq cn^{-\frac{2}{k+1}}$ Breaker wins the Maker-Breaker k-clique game played on the edge set of $G(n,p)$ a.a.s.*

The threshold probability for the k-clique game for $k \geq 4$ was conjectured to be $p_{\mathcal{K}_k} = n^{-\frac{2}{k+1}}$ in [10]. The previous theorem resolves this conjecture in the affirmative. Summing up the results of Theorem 2.1 and Theorem 19 from [10], we now have the following.

Corollary 2.2 *Let $k \geq 4$ and consider the Maker-Breaker k-clique game on the edge set of $G(n,p)$. There exist constants $c, C > 0$ such that the following hold:*

(i) *If $p \geq Cn^{-\frac{2}{k+1}}$, then Maker wins a.a.s.;*

(ii) *If $p \leq cn^{-\frac{2}{k+1}}$, then Breaker wins a.a.s.*

A result of this type is sometimes called a "semi-sharp threshold" in the random graphs literature.

Hitting time of Maker's win. We look at the same collection of positional games on graphs, now played in a slightly different random setting. Let V be a set of cardinality n, and let π be a permutation of the set $\binom{V}{2}$. If by G_i we denote the graph on the vertex set V whose edges are the first i edges in the permutation π, $G_i = (V, \pi^{-1}([i]))$, then we say that $\tilde{G} = \{G_i\}_{i=0}^{\binom{n}{2}}$ is a *graph process*. Given a monotone increasing graph property $\mathcal{P}$ and a graph process $\tilde{G}$, we define the hitting time of $\mathcal{P}$ with $\tau(\tilde{G}; \mathcal{P}) = \min\{t : G_t \in \mathcal{P}\}$. If π is chosen uniformly at random from the set of all permutations of the set $\binom{V}{2}$, we say that $\tilde{G}$ is a *random graph process*. Such processes are closely related to the model of random graph we described above.

Given a positional game, our general goal is to describe the hitting time of the graph property "Maker's win" in a typical graph process. For a game $\mathcal{F}$, by $\mathcal{M}_{\mathcal{F}}$ we denote the graph property "Maker wins $\mathcal{F}$". It was shown in [10] that in the connectivity game (with the technical assumption that Breaker is the first to play), for a random graph process $\tilde{G}$, we have $\tau(\tilde{G}; \mathcal{M}_{\mathcal{T}}) = \tau(\tilde{G}; \delta_2)$, where δ_ℓ is the graph property "minimum degree at least ℓ". Recently, Ben-Shimon et al. [3] resolved the same question for the Hamilton cycle game, obtaining $\tau(\tilde{G}; \mathcal{M}_{\mathcal{H}}) = \tau(\tilde{G}; \delta_4)$. Note that inequality in one direction for both of these equalities holds trivially.

Moving on to the clique game, we denote the property "the graph contains $K_5 - e$ as a subgraph" with $\mathcal{G}_{\mathcal{K}_5^-}$. We are able to show the following hitting time result for Maker's win in the triangle game.

Theorem 2.3 *For a random graph process $\tilde{G}$, the hitting time for Maker's win in the triangle game is asymptotiaclly almost surely the same as the hitting time for appearance of $K_5 - e$, i.e., $\tau(\tilde{G}; \mathcal{M}_{\mathcal{K}_3}) = \tau(\tilde{G}; \mathcal{G}_{\mathcal{K}_5^-})$ a.a.s.*

By considering the number of copies of $K_5 - e$, we are able to give a precise expression for the probability that Maker wins the triangle game on $G(n,p)$.

Corollary 2.4 *Let $p = p(n)$ be an arbitrary sequence of numbers $\in [0,1]$ and let us write $x = x(n) = p \cdot n^{\frac{2}{k+1}}$. Then*

$$\lim_{n\to\infty} Pr[\textit{Maker wins the triangle game on } G(n,p)] = \begin{cases} 0 & \textit{if } x \to 0, \\ 1 - e^{-c^5/3} & \textit{if } x \to c \in \mathbb{R}, \\ 1 & \textit{if } x \to \infty. \end{cases}$$

3 Conclusion and open problems

Random graph intuition. Chvátal and Erdős [4] observed the following paradigm, which is referred to as the *random graph intuition* in positional game theory. As it turns out for many standard games on graphs, the inverse of the threshold bias $b_{\mathcal{F}}$ in the game played on the complete graph is "closely related" to the probability threshold for the appearance of a member of $\mathcal{F}$ in $G(n,p)$. Another parameter that is often "around" is the threshold probability $p_{\mathcal{F}}$ for Maker's win when played on $G(n,p)$. As we saw, for the two games mentioned in the introduction, the connectivity game and the Hamilton cycle game, all three parameters are exactly equal to $\frac{\log n}{n}$.

In the k-clique game, for $k \geq 4$, the threshold bias is $b_{\mathcal{K}_k} = \Theta(n^{\frac{2}{k+1}})$ and the threshold probability for Maker's win is the inverse (up to the leading constant), $p_{\mathcal{K}_k} = n^{-\frac{2}{k+1}}$, supporting the random graph intuition. But, the threshold probability for appearance of a k-clique in $G(n,p)$ is not at the same place, it is $n^{-\frac{2}{k-1}}$. And in the triangle game there is even more disagreement, as all three parameters are different – they are, respectively, $n^{\frac{1}{2}}$, $n^{-\frac{5}{9}}$ and n^{-1}. Now, more than thirty years after Chvátal and Erdős formulated the paradigm, there is still no general result that would make it more formal. We are curious to the reasons behind the total agreement between the three thresholds in the connectivity game and the Hamilton cycle game, partial disagreement in k-clique game for $k \geq 4$, and the total disagreement in the triangle game.

Random clique game vs. biased clique game. Our Corollary 2.2 gives two constants $c > 0$ and $C > 0$, stating that the probability threshold for Maker's win in the k-clique game on $G(n,p)$ for $k \geq 4$ is between $cn^{-\frac{2}{k+1}}$ and $Cn^{-\frac{2}{k+1}}$. In a way, with this result, the game played on the random graph catches up with the biased k-clique game played on the complete graph, as a result of Bednarska and Łuczak [2] guarantees the existence of constants $c' > 0$ and $C' > 0$, such that the bias threshold for this game is between $c'n^{\frac{2}{k+1}}$ and $C'n^{\frac{2}{k+1}}$, for all $k \geq 3$. Both pairs of constants, c, C and c', C', are quite far apart. Also, in both games, the best known strategy for Maker's exploits the same derandomized random strategy approach, proposed in [2].

We know much more for the triangle game on the random graph, as Corollary 2.4 gives the threshold probability quite accurately, and it turns out to be a coarse threshold. The reason for such different behavior (compared to $k > 3$) may lie behind the fact that $K_3 = C_3$.

A more precise result for the k-clique game when $k \geq 4$? As we saw, we can say a lot about the threshold probability for the triangle game, the connectivity game and the Hamilton cycle game when the game is played on the random graph. We do not know that much about the k-clique game, when $k \geq 4$, and it would be very interesting to see what happens between the bounds given in Corollary 2.2. Also, a graph-theoretic description of the hitting time of Maker's win on the random graph process would be of great importance, as currently we know very little about Maker's winning strategy at the threshold. What we know is that we cannot hope for a result analogous to Theorem 2.3 – the reason for Maker's win cannot be the appearance of a fixed graph, as we know that Breaker wins on every typical (fixed) subgraph of the random graph on the probability threshold. Hence, Maker's optimal strategy must be of "global nature", taking into account a non-constant part of the random graph to win the game. Having that in mind we propose the following conjecture.

Conjecture 3.1 *For every $k \geq 4$ there exists a $c = c(k)$ such that for any fixed $\varepsilon > 0$:*

(i) *If $p \leq (c - \varepsilon)n^{-\frac{2}{k+1}}$, then Breaker wins the k-clique game on $G(n, p)$ a.a.s;*

(ii) *If $p \geq (c + \varepsilon)n^{-\frac{2}{k+1}}$, then Maker wins the k-clique game a.a.s.*

It might be possible to apply the celebrated results of Friedgut [5] to get something slightly weaker than the conjecture. We have however not been able to make such an argument stick.

***H* game.** A natural extension of the clique game is the H game, where Maker's goal is to claim a copy of a given graph H. For the game played on the random graph, we know much less in this case. Some progress has been made in [9]. Apart from results about the threshold probability and hitting time results, a characterization of all graphs H for which the threshold probability in the game on the random graph is equal to the inverse of the threshold bias in the game on the complete graph would be of considerable importance.

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