A sharp threshold for the Hamilton cycle Maker-Breaker game

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October 8, 2008

Abstract

We study the Hamilton cycle Maker-Breaker game, played on the edges of the random graph $G(n,p)$. We prove a conjecture from [13], asserting that the property that Maker is able to win this game, has a sharp threshold at $\frac{\log n}{n}$. Our theorem can be considered a game-theoretic strengthening of classical results from the theory of random graphs: not only does $G(n,p)$ almost surely admit a Hamilton cycle for $p = (1+\varepsilon)\frac{\log n}{n}$, but Maker is able to build one while playing against an adversary.

1 Introduction

Let X be a finite set and let $\mathcal{F} \subseteq 2^X$ be a family of subsets. In the positional game $(X, \mathcal{F})$, two players take turns in claiming one previously unclaimed element of X. The set X is called the "board". In a *Maker/Breaker-type* positional game, the two players are called Maker and Breaker and the members of $\mathcal{F}$ are referred to as the "winning sets". Maker wins the game if he occupies all elements of some winning set; otherwise Breaker wins. We will always assume that Maker starts the game. We say that a game $(X, \mathcal{F})$ is a *Maker's win* if Maker has a strategy that ensures his win in this game against any strategy of Breaker, otherwise the game is a *Breaker's win*. Note that X and $\mathcal{F}$ alone determine whether the game is a Maker's win or a Breaker's win.

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The study of graph games, that is, games whose board is the edge set of a (complete) graph, was initiated by Lehman [11]. In particular, he proved that Maker can easily win the $(E(K_n), \mathcal{T}_n)$ game, where the family $\mathcal{T}_n$ consists of the edge sets of all spanning trees of K_n. Here by "easily" we mean that Maker can win on a much smaller board: for him it is sufficient that the game is played on the edge set of any n-vertex graph that contains two edge-disjoint spanning trees minus an edge. In fact, this is a characterization of the boards on which the spanning tree game is a Maker's win.

In another classical game Maker's goal is to build a Hamilton cycle. Chvátal and Erdős [5] proved that Maker can win the game $(E(K_n), \mathcal{H}_n)$ for sufficiently large n, where $\mathcal{H}_n$ consists of the edge sets of all Hamilton cycles of K_n. Though Maker's winning strategy they have presented is not as simple as the one for the spanning tree game, one should not be surprised by the result itself, as by the end of the game Maker owns many, roughly $n^2/4$, edges. Had the board contained less edges, Maker's job would have clearly been a harder one. However, there is very little known about those graphs whose edge set allows Maker to build a Hamilton cycle. This is in contrast with the spanning tree game, where Lehman's Theorem provides a full characterization. In the present paper we make progress in terms of a probabilistic characterization of the Hamilton cycle game.

Following [13], we give Breaker more power by randomly making the board smaller before the game starts. In particular, we are interested in games played on the random graph $G(n, p)$. For many of the well-studied games, such as $\mathcal{T}_n$ and $\mathcal{H}_n$ above, Maker wins on the edge set of $G(n, 1)$, while any (non-trivial) game played on the edge set of $G(n, 0)$ is a Breaker's win. The *threshold probability* $p_{\mathcal{F}}$ for a sequence of games on graphs $\mathcal{F} = \mathcal{F}_n$ is defined to be the probability at which an almost sure Breaker's win turns into an almost sure Maker's win, that is,

$$Pr[\mathcal{F}_n \text{ played on } E(G(n,p)) \text{ is Breaker's win}] \to 1 \text{ for } p = o(p_{\mathcal{F}}),$$

and

$$Pr[\mathcal{F}_n \text{ played on } E(G(n,p)) \text{ is Maker's win}] \to 1 \text{ for } p = \omega(p_{\mathcal{F}}),$$

when $n \to \infty$. Such a threshold $p_{\mathcal{F}}$ exists [4], as being Maker's win is clearly a monotone increasing graph property.

It was observed in [13] that the threshold probability of the spanning tree game is $\frac{\log n}{n}$; moreover, this threshold is sharp and coincides with the threshold of connectivity, determined by Erdős and Rényi in 1960. The threshold probability for the perfect matching game was also determined, and for the Hamilton cycle game it was proved that the threshold probability satisfies $\frac{\log n}{n} \le p_{\mathcal{H}_n} \le \frac{\log n}{\sqrt{n}}$. It was conjectured that the threshold should be the same as the threshold probability of hamiltonicity in $G(n, p)$. This was verified in [12], where a strategy that enables Maker to build a Hamilton cycle on $G(n, 5.4\frac{\log n}{n})$ almost surely, was given. However, this method does not seem to extend for a proof of the sharpness of the threshold. Using a different approach, we prove that the threshold is indeed sharp and even obtain some bound on the second order term.

Theorem 1 *There exists a constant $\ell > 0$ such that the Hamilton cycle game on $G(n, \frac{\log n + (\log\log n)^\ell}{n})$ is almost surely a Maker's win.*

It should be stressed that the probabilistic part of the statement refers to the creation of a random board and not to the strategies of the players; once a graph G is generated according to the $G(n,p)$ model, the game is completely deterministic and is either Maker's win or Breaker's win.

Theorem 1 is obviously very close to being best possible, as $G(n, \frac{\log n + 3\log\log n - \omega(1)}{n})$, where the $\omega(1)$ term tends to infinity with n arbitrarily slowly, almost surely has at least three vertices of degree at most three, and so Breaker easily wins by forcing Maker to build a graph with minimum degree at most one.

It is instructive to compare Theorem 1 with known results on Hamiltonicity of random graphs. Komlós and Szemerédi [10], and Bollobás [3] proved independently that if $p = p(n) = \frac{\log n + \log\log n + \omega(1)}{n}$, where $\omega(1)$ is any function tending to infinity arbitrarily slowly with n, then $G(n,p)$ is almost surely Hamiltonian; this estimate is easily seen to be essentially tight. Thus, Theorem 1 can be considered as a strengthening (with a slightly weaker bound on the error term) of the above stated classical result: for $p(n) = (1+\varepsilon)\frac{\log n}{n}$ not only does $G(n,p)$ admit a Hamilton cycle a.s., but Maker is a.s. able to build one while playing against an adversary.

The rest of the paper is organized as follows: in Section 2 we mention some known results and notation that will be used later on. In Section 3 we prove Theorem 1. Finally, in Section 4 we present some open problems.

2 Preliminaries

For the sake of simplicity and clarity of presentation, we do not make a particular effort to optimize the constants obtained in our proofs. We also omit floor and ceiling signs whenever these are not crucial. Most of our results are asymptotic in nature and whenever necessary we assume that n is sufficiently large. Throughout the paper, log stands for the natural logarithm. Our graph-theoretic notation is standard and follows that of [6]. In particular, we use the following.

For a graph G, let $V(G)$ and $E(G)$ denote its sets of vertices and edges respectively; and let $v(G) = |V(G)|$ and $e(G) = |E(G)|$. For a set $A \subseteq V(G)$, let $E_G(A)$ denote the set of edges of G with both endpoints in A, and let $e_G(A) = |E_G(A)|$. For disjoint sets $A, B \subseteq V(G)$, let $E_G(A,B)$ denote the set of edges of G with one endpoint in A and the other in B, and let $e_G(A,B) = |E_G(A,B)|$. For a set $S \subseteq V(G)$, let $N_G(S) = \{u \in V(G) \setminus S : \exists v \in S, \{u,v\} \in E(G)\}$ denote the set of neighbors of S in $V(G) \setminus S$. For a vertex $w \in V(G) \setminus S$ let $d_G(w,S) = |\{u \in S : \{u,w\} \in E(G)\}|$ denote the number of vertices of S that are adjacent to w in G. We abbreviate $d_G(w, V \setminus \{w\})$ to $d_G(w)$ which denotes the degree of w in G. The minimum vertex degree in G is denoted by $\delta(G)$. Often, when there is no risk of confusion, we omit G from the notation above.

The following classical theorem, due to Erdős and Selfridge [7], is a useful sufficient condition for Breaker's win in the $(X, \mathcal{F})$ game.

Theorem 2 *Let $(X, \mathcal{F})$ be an arbitrary hypergraph. If*

$$\sum_{A \in \mathcal{F}} 2^{-|A|} < \frac{1}{2},$$

then Breaker has a winning strategy for the $(X, \mathcal{F})$ game.

In order to prove that Maker's graph will indeed be Hamiltonian, we will use the following theorem.

Theorem 3 *[8] Let $12 \leq d \leq e^{\sqrt[3]{\log n}}$ and let $G = (V, E)$ be a graph on n vertices satisfying the following two properties:*

- *For every $S \subseteq V$, if $|S| \leq \frac{n \log \log n \log d}{d \log n \log \log \log n}$, then $|N(S)| \geq d|S|$;*
- *There is an edge in G between any two disjoint subsets $A, B \subseteq V$ with $|A|, |B| \geq \frac{n \log \log n \log d}{4130 \log n \log \log \log n}$.*

Then G is Hamiltonian, for sufficiently large n.

In the proof of Theorem 1 we will use the following Chernoff type bounds.

Lemma 4 *[1, Lemma 2.1] If $X \sim Bin(n, p)$ and $k \geq np$ then $Pr(X \geq k) \leq (enp/k)^k$.*

Note that the bound given in Lemma 4 is especially useful when k is "much larger" than np.

Lemma 5 *[9, Corollary 2.3] Let $X \sim Bin(n, p)$ and let $0 < \varepsilon < 1$. Then $Pr[|X - np| > \varepsilon np] \leq 2 \exp\{-\frac{\varepsilon^2}{3} np\}$.*

Lemma 6 *[9, Theorem 2.10 and Corollary 2.4] If X is a random variable with hypergeometric distribution, $c > 1$, and $x \geq c \cdot \mathbb{E}(X)$, then $Pr(X \geq x) \leq e^{-c'x}$, where $c' = \log c - 1 + 1/c$.*

3 Building a Hamilton cycle one-on-one

In this section we are going to prove Theorem 1, by describing Maker's strategy, and then proving that it is a winning one.

Let

$$p = p(n) = \frac{\log n + (\log\log n)^{\ell}}{n} \ ,$$

and let

$$k = k(n) = \frac{cn \log\log n}{4130 \log n},$$

where $c = \ell - 4$. The value of the constant c, and thus also of ℓ, will be assumed to be sufficiently large throughout the proof. We make no effort to optimize this constant; for our calculations, $c = 3 \cdot 10^{17}$ is large enough.

Before proving Theorem 1, we state and prove some auxiliary lemmas.

Lemma 7 *The random graph $G = G(n,p) = (V,E)$ satisfies the following properties a.s.:*

(P1) $\delta(G) \geq (\log\log n)^{c+2}$*;*

(P2) *Every subset $A \subseteq V$ of cardinality $|A| \leq 4500k/(\log\log n)^c$ spans at most $|A|(\log\log n)^{c+1}$ edges in G;*

(P3) *For every two disjoint subsets $A, B \subseteq V$ of cardinality $|A| \leq 4500k/(\log\log n)^c$ and $|B| = |A|(\log\log n)^c$, the number of edges between A and B does not exceed $|A|(\log\log n)^{c+2}/600$;*

(P4) *For every two disjoint subsets A, B of V of cardinality $k/1000 \leq |A|, |B| \leq k$, we have $0.999|A||B|p \leq e(A,B) \leq 1.001|A||B|p$.*

Proof Properties **P1 - P4** follow by standard first moment calculations and standard bounds on the tail of the binomial distribution.

P1: Let X be a random variable that counts the number of vertices of degree at most $(\log\log n)^{c+2}$. Then

$$\begin{aligned}
\mathbb{E}(X) &\leq n \sum_{i=0}^{(\log\log n)^{c+2}} \binom{n-1}{i} p^i (1-p)^{n-1-i} \\
&\leq n \sum_{i=0}^{(\log\log n)^{c+2}} n^i \left(\frac{(1+o(1))\log n}{n}\right)^i \exp\left\{-p\left(n - 2(\log\log n)^{c+2}\right)\right\} \\
&\leq n \sum_{i=0}^{(\log\log n)^{c+2}} ((1+o(1))\log n)^i e^{-\log n} e^{-(\log\log n)^{c+4}} e^{3\log n (\log\log n)^{c+2}/n} \\
&\leq e^{(1+o(1))(\log\log n)^{c+3} - (\log\log n)^{c+4}} \\
&= o(1).
\end{aligned}$$

It follows by Markov's inequality that $Pr[X > 0] = o(1)$.

P2: Let $A \subseteq V$ be any subset of size $1 \leq a \leq 4500k/(\log\log n)^c$. Let X_A be a random variable that counts the number of edges with both endpoints in

A. Then $X_A \sim Bin(\binom{a}{2}, p)$ and thus $\mathbb{E}(X_A) = \binom{a}{2}p$. In order to bound the probability that X_A is much larger than its expectation, we use Lemma 4. Then we have

$$\begin{aligned}
& Pr[\exists A \subseteq V \text{ with } 1 \leq a \leq 4500k/(\log\log n)^c \text{ and } X_A \geq a(\log\log n)^{c+1}] \\
\leq & \sum_{a=1}^{4500k/(\log\log n)^c} \binom{n}{a} Pr[X_A \geq a(\log\log n)^{c+1}] \\
\leq & \sum_{a=1}^{4500k/(\log\log n)^c} \left[\frac{en}{a}\left(\frac{e\binom{a}{2}p}{a(\log\log n)^{c+1}}\right)^{(\log\log n)^{c+1}}\right]^a \\
\leq & \sum_{a=1}^{4500k/(\log\log n)^c} \left[\frac{en}{a}\left(\frac{a\log n}{n}\right)^{(\log\log n)^{c+1}}\right]^a \\
\leq & \sum_{a=1}^{4500k/(\log\log n)^c} \left[\exp\left\{1+\log(n/a) - (\log\log n)^{c+1}\left(\log(n/a) - \log\log n\right)\right\}\right]^a \\
= & \; o(1),
\end{aligned}$$

where the last equality follows from the upper bound on a.

P3: Let $A \subseteq V$ be any subset of cardinality $1 \leq a \leq 4500k/(\log\log n)^c$ and let B be any subset of $V \setminus A$ of cardinality $b = a(\log\log n)^c$. Let X_{AB} be a random variable that counts the number of edges with one endpoint in A and the other in B. Then $X_{AB} \sim Bin(ab, p)$ and thus $\mathbb{E}(X_{AB}) = abp = a^2p(\log\log n)^c$. Let E denote the event *"there exist two disjoint subsets $A, B \subseteq V$, of sizes $1 \leq |A| = a \leq 4500k/(\log\log n)^c$ and $|B| = b = a(\log\log n)^c$, such that $e(A,B) > a(\log\log n)^{c+2}/600$"*. Using Lemma 4 we get

$$\begin{aligned}
Pr[E] \leq & \sum_{a=1}^{4500k/(\log\log n)^c} \binom{n}{a}\binom{n}{b} Pr[X_{AB} \geq a(\log\log n)^{c+2}/600] \\
\leq & \sum_{a=1}^{4500k/(\log\log n)^c} \left[\frac{en}{a}\left(\frac{en}{b}\right)^{(\log\log n)^c}\left(\frac{eabp}{a(\log\log n)^{c+2}/600}\right)^{(\log\log n)^{c+2}/600}\right]^a \\
\leq & \sum_{a=1}^{4500k/(\log\log n)^c} \left[\frac{en}{a}\left(\frac{en}{b}\right)^{(\log\log n)^c}\left(\frac{a\log n}{n}\right)^{(\log\log n)^{c+2}/600}\right]^a \\
\leq & \sum_{a=1}^{4500k/(\log\log n)^c} [\exp\{1+\log(n/a) + (\log\log n)^c\left(1+\log(n/a)\right) \\
& -\frac{(\log\log n)^{c+2}}{600}\left(\log(n/a) - \log\log n\right)\}]^a \\
= & \; o(1),
\end{aligned}$$

where the last equality follows from the upper bound on a.

P4: Let $A, B \subseteq V$ be any two disjoint subsets of sizes $k/1000 \leq a, b \leq k$ respectively. Let X_{AB} be a random variable that counts the number of edges with one endpoint in A and the other in B. Then $X_{AB} \sim Bin(ab, p)$ and thus $\mathbb{E}(X_{AB}) = abp$. Let E denote the event: *"there exist two disjoint subsets $A, B \subseteq V$, of sizes $k/1000 \leq a, b \leq k$ respectively, such that $|e(A, B) - abp| > 0.001abp$"*. Applying Lemma 5 we obtain

$$
\begin{aligned}
Pr[E] &\leq \sum_{a=k/1000}^{k} \sum_{b=k/1000}^{k} \binom{n}{a}\binom{n}{b} Pr[|X_{AB} - abp| > 0.001abp] \\
&\leq \sum_{a=k/1000}^{k} \sum_{b=k/1000}^{k} \left(\frac{en}{a}\right)^a \left(\frac{en}{b}\right)^b 2\exp\left\{-\frac{abp}{3 \cdot 10^6}\right\} \\
&\leq k^2 \left(\frac{en}{k/1000}\right)^{2k} 2\exp\left\{-\frac{k^2 p}{3 \cdot 10^{12}}\right\} \\
&\leq k^2 \left(\exp\left\{2\log\log n - \frac{c}{3 \cdot 4130 \cdot 10^{12}} \log\log n\right\}\right)^k \\
&= o(1),
\end{aligned}
$$

where the last equality follows by choosing c to be sufficiently large. $\square$

Lemma 8 *Let $G = (V, E)$ be a graph, satisfying property* **P4***; then, for every pair of disjoint subsets $A, B \subseteq V$ of cardinality $|A| = |B| = k$, there exist subsets $A_1 \subseteq A$ and $B_1 \subseteq B$ such that*

- $e(A_1, B_1) \geq 0.197k^2 p$;
- *Every $a \in A_1$ satisfies $0.4kp \leq d(a, B_1) \leq 1.1kp$;*
- *Every $b \in B_1$ satisfies $0.4kp \leq d(b, A_1) \leq 1.1kp$.*

Proof Let $A, B \subseteq V$ be any two disjoint subsets of cardinality k each, and let

$$
\begin{aligned}
X &= \{a \in A : d(a, B) \geq 1.1kp\}, \\
Y &= \{b \in B : d(b, A) \geq 1.1kp\}.
\end{aligned}
$$

It follows by property **P4** that $|X|, |Y| < k/1000$. Indeed, otherwise we would have

$$|X| \cdot 1.1kp \leq e(X, B) \leq 1.001|X||B|p,$$

a contradiction; a similar argument applies to Y. Let $A_0 = A \setminus X$, $B_0 = B \setminus Y$, and let H_0 be the bipartite subgraph of G with bipartition (A_0, B_0). By the aforementioned upper bound on $|X|$ and $|Y|$ we have, $0.999k \leq |A_0|, |B_0| \leq k$. It follows by property **P4** that

$$|E(H_0)| \geq 0.999|A_0||B_0|p \geq (0.999)^3 k^2 p \geq 0.997k^2 p.$$

Now, start with $H_1 = H_0$, and keep deleting vertices of degree at most $0.4kp$ until there are none left. Altogether we delete at most $0.4kp(|A_0| + |B_0|) \leq$

$0.8k^2p$ edges, so the resulting graph H_1 has at least $0.197k^2p$ edges and all degrees are between $0.4kp$ and $1.1kp$; denote its parts by A_1 and B_1. This concludes the proof of the lemma. $\square$

Lemma 9 *Let $G = (V, E)$ be a graph on n vertices satisfying properties* **P1** *and* **P4***; then E can be split into two disjoint sets $E = E_1 \cup E_2$ such that, denoting $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$, we have*

- $\delta(G_1) \geq (\log\log n)^{c+2}/100$*;*
- *For every two disjoint subsets A, B of V, such that $|A| = |B| = k$, we have*
$$e_{G_2}(A, B) \geq 0.1k^2p\ .$$

Proof For each vertex $v \in V$ we choose exactly $(\log\log n)^{c+2}/100$ of the edges incident with it, uniformly at random and independently of the choices for other vertices. We put an edge $e = \{u, v\}$ in E_1 if and only if e is chosen at u or v (or both). Set $E_2 = E \setminus E_1$. Clearly, $\delta(G_1) \geq (\log\log n)^{c+2}/100$.

Let A, B be disjoint subsets of V of cardinality k, and let $A_1 \subseteq A$, $B_1 \subseteq B$, be the subsets whose existence is guaranteed by Lemma 8.

For every $a \in A_1$, denote by X_a the random variable that counts the number of edges from a to B_1, that were chosen by a. Then $\{X_a : a \in A_1\}$ are independent hypergeometric random variables with parameters $d_G(a)$, $d(a, B_1)$ and $(\log\log n)^{c+2}/100$ (that is, exactly $(\log\log n)^{c+2}/100$ elements of $\{\{a, u\} \in E : u \in V\}$ are chosen uniformly at random without replacement, and we count how many of them are in the set $\{\{a, u\} \in E : u \in B_1\}$). Hence by **P1**

$$\mathbb{E}[X_a] = \frac{d(a, B_1)\frac{(\log\log n)^{c+2}}{100}}{d_G(a)} \leq \frac{d(a, B_1)}{100} < 0.02kp\,,$$

and thus, applying Lemma 6 we obtain

$$Pr[X_a \geq 0.03kp] < \exp\{-0.07 \cdot 0.03kp\} < \exp\left\{-\frac{kp}{600}\right\}.$$

Hence, the probability that there are at least $0.01k$ vertices in A_1 such that at each one we chose at least $0.03kp$ edges leading to B_1, is at most

$$\binom{|A_1|}{0.01k} \exp\left\{-\frac{kp}{600} \cdot 0.01k\right\} \leq \exp\left\{-\frac{k^2p}{7 \cdot 10^4}\right\}.$$

Similarly, the probability that there are at least $0.01k$ vertices in B_1 such that at each one we chose at least $0.03kp$ edges leading to A_1 each, is at most

$$\binom{|B_1|}{0.01k} \exp\left\{-\frac{kp}{600} \cdot 0.01k\right\} \leq \exp\left\{-\frac{k^2p}{7 \cdot 10^4}\right\}.$$

Thus, the probability that there exist subsets $A, B \subseteq V$ of cardinality k, for which there are at least $0.01k$ vertices of A_1 such that at each one we chose at least $0.03kp$ edges leading to B_1, or there are at least $0.01k$ vertices of B_1 such that at each one we chose at least $0.03kp$ edges leading to A_1, is at most

$$\begin{aligned}\binom{n}{k}^2 2\exp\left\{-\frac{k^2p}{7\cdot 10^4}\right\} &\leq \left(\frac{en}{k}\right)^{2k} 2\exp\left\{-\frac{k^2p}{7\cdot 10^4}\right\} \\ &\leq \left[\exp\left\{2\log\log n - \frac{kp}{7\cdot 10^4}\right\}\right]^k \\ &= o(1),\end{aligned}$$

where the last equality follows by choosing c to be sufficiently large.

It follows that there exists a choice of edges for the vertices, such that for every $A, B \subseteq V$ the total number of edges chosen between A_1 and B_1 is at most

$$0.01k \cdot 1.1kp + |A_1| \cdot 0.03kp + 0.01k \cdot 1.1kp + |B_1| \cdot 0.03kp < 0.09k^2p\ .$$

Hence, for such a choice

$$\begin{aligned} e_{G_2}(A,B) &\geq e_{G_2}(A_1,B_1) \\ &= e_G(A_1,B_1) - e_{G_1}(A_1,B_1) \\ &\geq 0.197k^2p - 0.09k^2p \\ &> 0.1k^2p. \end{aligned}$$

This concludes the proof of the lemma. □

Lemma 10 *Let H be a graph of minimum degree d; then in a Maker-Breaker game played on H, Maker can build a spanning graph M with minimum degree at least $\lfloor d/4 \rfloor$.*

Proof Let H^* be the graph, obtained from H by adding a new vertex v^* and connecting it to every vertex of odd degree in H (if all the degrees in H are even, then set $H^* = H$). Since all the degrees in H^* are even, it admits a Eulerian orientation $\overrightarrow{H^*}$. For every $v \in V(H)$, let $E(v) = \{\{v,u\} \in E(H) : \overrightarrow{(v,u)} \in E(\overrightarrow{H^*})\}$. Clearly, $|E(v)| \geq \lfloor d_H(v)/2 \rfloor$ and the sets $E(v), v \in V(H)$ are pairwise disjoint. Using an obvious pairing strategy, Maker can claim at least $\lfloor |E(v)|/2 \rfloor$ edges from every set $E(v)$. □

Lemma 11 *Let $G = (V,E)$, where $|V| = n$, be a graph that satisfies properties* **P2**, **P3**. *Let M_1 be a spanning subgraph of G of minimum degree $\delta(M_1) \geq \frac{(\log\log n)^{c+2}}{500}$. Then in M_1, every subset $A \subseteq V$ of cardinality $1 \leq |A| \leq 4500k/(\log\log n)^c$, satisfies $|N_{M_1}(A)| \geq |A|(\log\log n)^c$.*

Proof Let A be a subset of V of cardinality $1 \leq |A| \leq 4500k/(\log\log n)^c$. By property **P2** we have $e_{M_1}(A) \leq e_G(A) \leq |A|(\log\log n)^{c+1}$, and thus there are at least

$$\delta(M_1)|A| - 2e_{M_1}(A) > \frac{|A|(\log\log n)^{c+2}}{600}$$

edges leaving A in M_1. Thus $|N_{M_1}(A)| \geq |A|(\log\log n)^c$, as otherwise we get a contradiction with property **P3**. $\Box$

We are now ready to describe Maker's strategy. From Lemma 7 we have that $G(n,p)$ satisfies properties **P1** - **P4** a.s.; hence, from now on, we assume that the board is the edge-set of a graph G that satisfies all these properties. Before the game starts, Maker splits the board into two parts, $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$, as in Lemma 9. He plays two separate games in parallel, one on E_1 and the other on E_2. Whenever Breaker claims some edge of $E_i,\ i = 1, 2$, Maker plays in E_i as well (except for maybe once if Breaker has claimed the last edge of E_i). Let M denote the graph built by Maker and let $M_1 = (V, E(M) \cap E_1)$, $M_2 = (V, E(M) \cap E_2)$.

The game on E_1 is played according to Lemma 10 and so at the end of the game, Maker's graph M_1 will have minimum degree at least $\frac{(\log\log n)^{c+2}}{500}$. Since G satisfies properties **P2** and **P3**, it follows by Lemma 11 that every $A \subseteq V$ of cardinality at most $4500k/(\log\log n)^c$ satisfies $|N_{M_1}(A)| \geq |A|(\log\log n)^c$.

Playing on E_2, Maker's goal is to claim an edge between every two disjoint subsets $A, B \subseteq V$ of cardinality k. To do that, he can simply adopt the role of Breaker in a different game, played on the hypergraph $\mathcal{H}$ whose set of vertices is E_2 and whose hyperedges are the edge sets of all bipartite subgraphs of G_2 with both parts of size k. Using Theorem 2, we get

$$\begin{aligned}
\sum_{D \in \mathcal{H}} 2^{-|D|} &\leq \binom{n}{k}^2 2^{-0.1k^2 p} \\
&\leq \left(\frac{en}{k}\right)^{2k} e^{-(0.1\log 2)k^2 p} \\
&\leq \exp\{k(2\log\log n - (0.1\log 2)kp)\} \\
&= o(1),
\end{aligned}$$

where the last equality follows since c is sufficiently large.

It follows that Maker can build a graph $M = (V, E(M_1) \cup E(M_2))$, for which

- Every subset $A \subseteq V$ of cardinality at most $4500k/(\log\log n)^c$ satisfies
$$|N_M(A)| \geq |A|(\log\log n)^c;$$
- There is an edge between every two disjoint subsets $A, B \subseteq V$ of cardinality $|A| = |B| = k$.

Thus, the Hamiltonicity of M follows from Theorem 3 by substituting $d = (\log\log n)^c$, for sufficiently large n. $\Box$

4 Open problems

In [13] it was conjectured that, provided that Breaker starts the game, essentially the only reason for Maker to lose the Hamilton cycle game on $G(n, p)$ is that the graph has a vertex of degree 3. One can formulate this conjecture precisely in the model of the random graph process: the hitting time of the event that Maker has a winning strategy in the Hamilton cycle game is *equal* to the hitting time of the event that the minimum degree of the graph is 4 (provided that Breaker starts the game).

A less ambitious goal would be to prove that the second order term of the threshold probabilities coincide asymptotically.

Conjecture 1 *Suppose that Breaker starts the game. If* $p \geq \frac{\log n + 3 \log\log n + \omega(1)}{n}$*, where the* $\omega(1)$ *term tends to infinity with* n *arbitrarily slowly, then a.s. Maker wins the Hamilton cycle game on* $G(n, p)$*.*

If true, then this statement could be viewed as a game theoretic analog of the famous theorem of Komlós and Szemerédi [10] and of Bollobás [3] on the Hamiltonicity of the random graph.

Min-degree-game. We think that the problem considered in Lemma 10 is very interesting in its own right. In Lemma 10, it was proved that, playing on the edge set of any graph with minimum degree d, Maker can build a graph with minimum degree at least $\lfloor d/4 \rfloor$. For $d \gg \log n$ it is known (see, e.g. [2]) that Maker can build a graph with minimum degree $d/2 - o(d)$. It would be very interesting to decide whether this is true for constant d as well. Let m_d be the largest integer such that, playing on any graph with minimum degree d, Maker can build a graph with minimum degree at least m_d. Lemma 10 then asserts that $m_d \geq \lfloor d/4 \rfloor$ holds for every d.

Problem 1 *Determine* m_d*.*

Acknowledgments

The authors would like to thank the referees for their valuable comments, which improved the quality of this paper.

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