Jacobian smoothing Brown's method for NCP

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Abstract. Jacobian smoothing Brown's method for nonlinear complementarity problems (NCP) is studied in this paper. This method is a generalization of classical Brown's method. It belongs to the class of Jacobian smoothing methods for solving semismooth equations. Local convergence of the proposed method is proved in the case of strictly complementary solution of NCP. Furthermore, a localy convergent hybrid method for general NCP is introduced. Some numerical experiments are also presented.

Key words. Nonlinear complementarity problem, semismooth system, Brown's method

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1 Introduction

Nonlinear complementarity problems (NCP) arise from mathematical models of many real problems in economy, engineering, structural analysis and mechanics. The concept of complementarity is related to modelling problems which appear in technical processes.

Reformulation of NCP to the system of nonlinear equations is the first step in solving NCP. The obtained nonsmooth systems are usually solved by iterative methods based on some generalization of methods for smooth systems, so a large class of numerical methods has been developed in recent years.

Brown's method for solving smooth systems is considered in many papers, for example see Brown [1], Frommer [2], Ge et al [3], Milaszewicz [4] etc. It is a variation of Newton's method which incorporates Gaussian elimination process. In this paper we propose a generalization of the classical Brown method for smooth systems to nonsmooth systems obtained by NCP reformulation. Our motivation is based on practical application of the method. The notation which might appear is complex. The proofs are also complicated from the technical point of view, but in spite of that practical realization of the method is not so complicated.

This new method belongs to the class of Jacobian smoothing methods, which is a large group of iterative methods for solving semismooth systems (see Chen [5], Kanzow and Pieper [6], Krejić and Rapajić [7], Li and Fukushima [8]). The main characteristic of these methods is the fact that the nonsmooth function is replaced by the smooth operator. Such methods try to solve the mixed Newton equation. This equation combines the original semismooth function with the Jacobian of its smooth operator.

The remainder of the paper is organized as follows. In Section 2 we collect some background and preliminary properties about Fischer-Burmeister reformulation of NCP and its smooth approximation. The algorithm and convergence result of Jacobian smoothing Brown's method are described in Section 3. We define hybrid method and analyze its convergence in Section 4. Some numerical experiments are presented in Section 5.

2 Preliminaries

Some words about notation are needed. The distance between given matrix $A \in R^{n,n}$ and nonempty set of matrices $\mathcal{A} \subset R^{n,n}$ is denoted by $dist(A, \mathcal{A}) = \inf_{B \in \mathcal{A}} \|A - B\|$. Vectors $\mathbf{e}^i$, $i = 1, \ldots n$ represent the canonical base of R^n. The Jacobian of a continuously differentiable mapping $F : R^n \to R^n$ at $\mathbf{x}$ is denoted by $F'(\mathbf{x})$.

Let $F : R^n \to R^n$ be a smooth mapping, $F_i(\mathbf{x}) : R^n \to R$ and $F(\mathbf{x}) = (F_1(\mathbf{x}), F_2(\mathbf{x}), \ldots, F_n(\mathbf{x}))^\top$. Nonlinear complementarity problem (NCP) consists of finding a vector $\mathbf{x} \in R^n$ such that

$$\mathbf{x} \geq 0,\ F(\mathbf{x}) \geq 0,\ \mathbf{x}^\top F(\mathbf{x}) = 0.$$

NCP can be transformed to the semismooth system of nonlinear equations as given in Fischer [9]

$$\Phi(\mathbf{x}) = 0, \quad \Phi : R^n \to R^n, \tag{1}$$

$$\Phi(\mathbf{x}) = (\Phi_1(\mathbf{x}), \Phi_2(\mathbf{x}), \ldots, \Phi_n(\mathbf{x}))^\top$$

where

$$\Phi_i(\mathbf{x}) = \phi(x_i, F_i(\mathbf{x})), \quad i = 1, ..., n$$

is defined by Fischer-Burmeister function $\phi : R^2 \to R$

$$\phi(a, b) = \sqrt{a^2 + b^2} - a - b. \tag{2}$$

For a smoothing parameter $\mu > 0$, Kanzow [10] defined the related smoothing problem

$$\Phi_\mu(\mathbf{x}) = 0, \quad \Phi_\mu : R^n \to R^n,$$

$$\Phi_\mu(\mathbf{x}) = (\Phi_1(\mathbf{x}, \mu), \Phi_2(\mathbf{x}, \mu), \ldots, \Phi_n(\mathbf{x}, \mu))^\top$$

where

$$\Phi_i(\mathbf{x}, \mu) = \phi_\mu(x_i, F_i(\mathbf{x})), \quad i = 1, ..., n$$

is defined by function $\phi_\mu : R^2 \to R$

$$\phi_\mu(a, b) = \sqrt{a^2 + b^2 + 2\mu} - a - b. \tag{3}$$

Function $\Phi_\mu : R^n \to R^n$ is smooth for any fixed $\mu > 0$.

The B-subdifferential of function Φ at $\mathbf{x}$ is defined by

$$\partial_B \Phi(\mathbf{x}) = \{\lim_{\mathbf{x}^k \to \mathbf{x}} \Phi'(\mathbf{x}^k) : \mathbf{x}^k \in D_\Phi\},$$

where D_Φ is the set where Φ is differentiable. The convex hull of B-subdifferential

$$\partial\Phi(\mathbf{x}) = conv \partial_B \Phi(\mathbf{x})$$

is called the generalized Jacobian of Φ at $\mathbf{x}$ in the sense of Clark [11].

In this paper we use a kind of generalized Jacobian of Φ, called C-subdifferential of Φ, denoted by $\partial_C \Phi$ and defined as

$$\partial_C \Phi(\mathbf{x}) = \partial\Phi_1(\mathbf{x}) \times \partial\Phi_2(\mathbf{x}) \times \ldots \times \partial\Phi_n(\mathbf{x}),$$

where $\partial\Phi_i(\mathbf{x})$ is the generalized gradient of Φ_i at $\mathbf{x}$

$$\partial\Phi_i(\mathbf{x}) = conv\{\lim_{\mathbf{x}^k \to x} \Phi_i'(\mathbf{x}^k) : \mathbf{x}^k \in D_{\Phi_i}\}$$

and D_{Φ_i} is the set where Φ_i is differentiable.

It is well known that all elements of the set $\partial_C \Phi(\mathbf{x})$ have the form

$$\partial_C \Phi(\mathbf{x}) = D_a(\mathbf{x}) + D_b(\mathbf{x}) F'(\mathbf{x}),$$

where
$D_a(\mathbf{x}) = diag(a_1(\mathbf{x}), \ldots, a_n(\mathbf{x}))$ and $D_b(\mathbf{x}) = diag(b_1(\mathbf{x}), \ldots, b_n(\mathbf{x}))$ are diagonal matrices with elements

$$a_i(\mathbf{x}) = \frac{x_i}{\sqrt{x_i^2 + F_i^2(\mathbf{x})}} - 1, \quad b_i(\mathbf{x}) = \frac{F_i(\mathbf{x})}{\sqrt{x_i^2 + F_i^2(\mathbf{x})}} - 1,$$

when $(x_i, F_i(\mathbf{x})) \neq (0, 0)$ and

$$a_i(\mathbf{x}) = \xi_i - 1, \quad b_i(\mathbf{x}) = \rho_i - 1, \quad (\xi_i, \rho_i) \in R^2, \quad \|(\xi_i, \rho_i)\| \leq 1,$$

for $(x_i, F_i(\mathbf{x})) = (0, 0)$.

Let $\mathbf{x}^*$ be the solution of NCP. Since NCP is equivalent to system (1), $\mathbf{x}^*$ is also the solution of (1). Let us denote

$$\Phi^0(\mathbf{x}) = \lim_{\mu \to 0} \Phi'_\mu(\mathbf{x}). \tag{4}$$

The properties of Φ_μ are analyzed in Kanzow and Pieper [6]. It is shown that

$$\lim_{\mu \to 0} dist\left(\Phi'_\mu(\mathbf{x}), \partial_C \Phi(\mathbf{x})\right) = 0$$

i.e.

$$\Phi^0(\mathbf{x}) \in \partial_C \Phi(\mathbf{x})$$

for any $\mathbf{x} \in R^n$, so the function Φ_μ has the Jacobian consistency property. Semismoothness of Φ and Jacobian consistence property of Φ_μ imply that

$$\lim_{\mathbf{h} \to 0} \frac{\|\Phi(\mathbf{x}+\mathbf{h}) - \Phi(\mathbf{x}) - \Phi^0(\mathbf{x}+\mathbf{h})\mathbf{h}\|}{\|\mathbf{h}\|} = 0, \tag{5}$$

which is given in Chen [5]. Next Lemma also follows from Chen [5].

Lemma 1 *[5] Function Φ_μ has the Jacobian consistency property. If all elements $V_\mathbf{x} \in \partial_C \Phi(\mathbf{x})$ are nonsingular, then there are an open ball $\mathcal{N}(\mathbf{x}, r)$ and* *a positive constant M such that for any $\mathbf{y} \in \mathcal{N}(\mathbf{x}, r)$, $\Phi^0(\mathbf{y})$ is nonsingular and*

$$\left\| \Phi^0(\mathbf{y})^{-1} \right\| \leq M.$$

Furthermore, there are $M_1 \geq M$ and $\mu_1 > 0$ such that for any $\mathbf{y} \in \mathcal{N}(\mathbf{x}, r)$ and $\mu \in (0, \mu_1)$, $\Phi'_\mu(\mathbf{y})$ is nonsingular and

$$\left\| \Phi'_\mu(\mathbf{y})^{-1} \right\| \leq M_1.$$

3 The algorithm and convergence result

In this section we define a new algorithm for NCP and prove its local convergence. We were motivated by Brown's method for smooth systems. As mentioned before, nonsmooth systems obtained by the reformulation of NCP, can be solved applying smoothing methods, so we make a generalization of classical Brown's method which belongs to the class of Jacobian smoothing methods.

For any $\mu > 0$ function Φ_μ is continuously differentiable with the Jacobian

$$\Phi'_\mu(\mathbf{x}) := \Phi'(\mathbf{x}, \mu) = \begin{bmatrix} \frac{\partial \Phi_1(\mathbf{x},\mu)}{\partial x_1} & \frac{\partial \Phi_1(\mathbf{x},\mu)}{\partial x_2} & \cdots & \frac{\partial \Phi_1(\mathbf{x},\mu)}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial \Phi_n(\mathbf{x},\mu)}{\partial x_1} & \frac{\partial \Phi_n(\mathbf{x},\mu)}{\partial x_2} & \cdots & \frac{\partial \Phi_n(\mathbf{x},\mu)}{\partial x_n} \end{bmatrix}. \tag{6}$$

We introduce some notation necessary for describing the algorithm.

Let

$$\bar{\mathbf{x}}^i = (x_i, x_{i+1}, ..., x_n)^\top \in R^{n-i+1},$$

$$\bar{\mathbf{x}}^{*,i} = (x_i^*, x_{i+1}^*, ..., x_n^*)^\top \in R^{n-i+1},$$

$$\bar{\mathbf{x}}^{k,i} = (x_i^k, x_{i+1}^k, ..., x_n^k)^\top \in R^{n-i+1} \quad \text{and}$$

$$\bar{\mathbf{e}}^i = (1, 0, 0, ..., 0)^\top \in R^{n-i+1},$$

for $i = 1, 2, ..., n$.

For a given smoothing parameter $\mu_k > 0$, vector $\mathbf{x}^k = (x_1^k, x_2^k, ..., x_n^k)^\top \in R^n$ and index i, $i = 1, 2, ..., n$, we successively define some functions based on Fischer-Burmeister function (2) and its smooth operator (3),

$$\tilde{\phi}_i(\bar{\mathbf{x}}^i) = \phi(x_i, F_i(s_1, s_2, ..., s_{i-2}, s_{i-1}, \bar{\mathbf{x}}^i)), \tag{7}$$

$$\tilde{\phi}_i(\bar{\mathbf{x}}^i, \mu_k) = \phi_{\mu_k}(x_i, F_i(s_1, s_2, ..., s_{i-2}, s_{i-1}, \bar{\mathbf{x}}^i)), \tag{8}$$

where $s_l = s_l(\bar{\mathbf{x}}^{l+1})$, for $l = 1, 2, ..., i-1$ and

$$s_i(\bar{\mathbf{x}}^{i+1}) = x_i^k - \left(\frac{\partial \tilde{\phi}_i}{\partial x_i}\bigg|_{\bar{\mathbf{x}}^{k,i}} \right)^{-1} \left[\sum_{j=i+1}^{n} \left(\frac{\partial \tilde{\phi}_i}{\partial x_j}\bigg|_{\bar{\mathbf{x}}^{k,i}} \right) (x_j - x_j^k) + \tilde{\phi}_i(\bar{\mathbf{x}}^{k,i}) \right], \tag{9}$$

where $\frac{\partial \tilde{\phi}_i}{\partial x_j}\Big|_{\bar{\mathbf{x}}^{k,i}}$, $\quad j = i, i+1, ..., n$ are the partial derivatives of smooth functions $\tilde{\phi}_i(\bar{\mathbf{x}}^i, \mu_k)$ at $\bar{\mathbf{x}}^{k,i}$.

Under the assumption that $\partial \tilde{\phi}_i / \partial x_i \neq 0$, functions s_i are continuous functions for fixed μ_k, functions $\tilde{\phi}_i(\bar{\mathbf{x}}^i)$ are continuous and semismooth, while $\tilde{\phi}_i(\bar{\mathbf{x}}^i, \mu_k)$ are continuously differentiable functions for given $\mu_k > 0$.

Let us describe the new method for NCP.

Algorithm 1: Jacobian smoothing Brown's method (JSB)

S0: Let $\mathbf{x}^0 \in R^n$ and a sequence $\{\mu_k\} > 0$ be given, $k := 0$.

S1: Compute

$$x_n^{k+1} = x_n^k - \left(\frac{\partial \tilde{\phi}_n}{\partial x_n}\bigg|_{x_n^k} \right)^{-1} \tilde{\phi}_n(x_n^k),$$

then for $i = n-1, n-2, ..., 1$ do

$$x_i^{k+1} = s_i(\bar{\mathbf{x}}^{k+1,i+1}),$$

where $s_i(\bar{\mathbf{x}}^{i+1})$ is defined by (9).

S2: Set $k := k+1$ and return to step S1. ♣

The matrix formulation of JSB method is

$$U(\mathbf{x}^k, \mu_k)(\mathbf{x}^{k+1} - \mathbf{x}^k) = -m(\mathbf{x}^k),$$

where

$$U(\mathbf{x}^k, \mu_k) = \begin{bmatrix} u_{11}^{k,\mu_k} & u_{12}^{k,\mu_k} & \cdots & u_{1n}^{k,\mu_k} \\ 0 & u_{22}^{k,\mu_k} & \cdots & u_{2n}^{k,\mu_k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn}^{k,\mu_k} \end{bmatrix} \tag{10}$$

is upper triangular matrix with elements

$$u_{ij}^{k,\mu_k} = \left.\frac{\partial \tilde{\phi}_i}{\partial x_j}\right|_{\bar{\mathbf{x}}^{k,i}} \quad \text{for} \quad i = 1, 2, ..., n, \quad j = i, i+1, ..., n, \tag{11}$$

and components of the vector $m(\mathbf{x}^k)$ are

$$m_i(\mathbf{x}^k) = \tilde{\phi}_i(\bar{\mathbf{x}}^{k,i}), \quad i = 1, 2, ..., n.$$

More important, the u_{ij}^{k,μ_k} have precisely the same value as the corresponding elements obtained by triangularization of the Jacobian matrix $\Phi'_{\mu_k}(\mathbf{x}^k)$ using Gaussian elimination with partial pivoting. If $\Phi'_{\mu_k}(\mathbf{x}^k)$ is nonsingular then there exists a permutation matrix P such that

$$P\Phi'_{\mu_k}(\mathbf{x}^k) = LU(\mathbf{x}^k, \mu_k).$$

From now on, without the loss of generality we suppose that this transformation with P is already done, so we assume that

$$\Phi'_{\mu_k}(\mathbf{x}^k) = LU(\mathbf{x}^k, \mu_k).$$

In this section we are going to prove the local convergence of the Jacobian smoothing Brown's method, which will be established in the case of strictly complementary solution of NCP. Before that we should state some necessary Definitions and Lemmas.

Definition 1 *The solution $\mathbf{x}^*$ of NCP is a strictly complementary solution if*

$$x_i^* + F_i(\mathbf{x}^*) > 0$$

holds for every $i = 1, 2, ..., n$.

If $\mathbf{x}^*$ is a strictly complementary solution of NCP, then there exists a neighbourhood $\mathcal{N}(\mathbf{x}^*, \varepsilon)$ in which function Φ is differentiable.

Definition 2 *Let $\mathbf{x}^*$ be a strictly complementary solution of NCP. For $\mathbf{x}^k \in \mathcal{N}(\mathbf{x}^*, \varepsilon)$ we define a matrix*

$$U^0(\mathbf{x}^k) = \lim_{\mu_k \to 0} U(\mathbf{x}^k, \mu_k),$$

$$U^0(\mathbf{x}^k) = \begin{bmatrix} u_{11}^{k,0} & u_{12}^{k,0} & \cdots & u_{1n}^{k,0} \\ 0 & u_{22}^{k,0} & \cdots & u_{2n}^{k,0} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn}^{k,0} \end{bmatrix},$$

where

$$u_{ij}^{k,0} = \lim_{\mu_k \to 0} u_{ij}^{k,\mu_k}, \tag{12}$$

and u_{ij}^{k,μ_k}, $i = 1, 2, ..., n$, $j = i, i+1, ..., n$ *are given with (11).*

We also define a vector

$$\mathbf{u}_i^0 := (u_{ii}^{k,0}, u_{i,i+1}^{k,0}, ..., u_{in}^{k,0})^\top \in R^{n-i+1}.$$

The Jacobian matrix $\Phi'_\mu(\mathbf{x})$ (6) can be presented in the following way

$$\Phi'_\mu(\mathbf{x}) = D_a(\mathbf{x}, \mu) + D_b(\mathbf{x}, \mu) F'(\mathbf{x}),$$

where
$D_a(\mathbf{x}, \mu) = diag(a_1(\mathbf{x}, \mu), \ldots, a_n(\mathbf{x}, \mu))$, $D_b(\mathbf{x}, \mu) = diag(b_1(\mathbf{x}, \mu), \ldots, b_n(\mathbf{x}, \mu))$ are diagonal matrices with elements

$$a_i(\mathbf{x}, \mu) = \frac{x_i}{\sqrt{x_i^2 + F_i^2(\mathbf{x}) + 2\mu}} - 1, \quad b_i(\mathbf{x}, \mu) = \frac{F_i(\mathbf{x})}{\sqrt{x_i^2 + F_i^2(\mathbf{x}) + 2\mu}} - 1 \tag{13}$$

for $i = 1, 2, ..., n$.

In order to prove the convergence of JSB method we need the following Lemmas.

Lemma 2 *Let* $\mathbf{x}^*$ *be a strictly complementary solution of NCP. Then there exists a neighbourhood* $\mathcal{N}(\mathbf{x}^*, \varepsilon)$ *such that for* $\mathbf{x}^k \in \mathcal{N}(\mathbf{x}^*, \varepsilon)$ *and related matrix* $U^0(\mathbf{x}^k)$ *the following relations are satisfied*

$$\mathbf{u}_i^0 \in \partial \tilde{\phi}_i(\bar{\mathbf{x}}^{k,i}) \quad for \quad i = 1, 2, ..., n,$$

where $U^0(\mathbf{x}^k)$ *and* $\mathbf{u}_i^0$ *are given in Definition 2.*

Proof. Let $l_i(\bar{\mathbf{x}}^{k,i}) = F_i(s_1, s_2, ..., s_{i-1}, \bar{\mathbf{x}}^{k,i})$. Using (2), (3), (7) and (8) it follows that

$$\tilde{\phi}_i(\bar{\mathbf{x}}^{k,i}) = \sqrt{(x_i^k)^2 + l_i^2(\bar{\mathbf{x}}^{k,i})} - x_i^k - l_i(\bar{\mathbf{x}}^{k,i}),$$

$$\tilde{\phi}_i(\bar{\mathbf{x}}^{k,i},\mu_k) = \sqrt{(x_i^k)^2 + l_i^2(\bar{\mathbf{x}}^{k,i}) + 2\mu_k} - x_i^k - l_i(\bar{\mathbf{x}}^{k,i}),$$

hold for $i = 1, 2, ..., n$. Since $\mathbf{x}^*$ is a strictly complementary solution of NCP, there exists a neighbourhood $\mathcal{N}(\mathbf{x}^*, \varepsilon)$ in which Φ is differentiable. From this and (12) the following equalities are valid:

$$\begin{aligned}
u_{ii}^{k,0} &= \lim_{\mu_k\to 0} u_{ii}^{k,\mu_k} = \lim_{\mu_k\to 0}\left(\frac{\partial\tilde{\phi}_i}{\partial x_i}\bigg|_{\bar{\mathbf{x}}^{k,i}}\right) \\
&= \lim_{\mu_k\to 0}[\left(\frac{x_i^k}{\sqrt{(x_i^k)^2 + l_i^2(\bar{\mathbf{x}}^{k,i}) + 2\mu_k}} - 1\right) \\
&\quad + \left(\frac{l_i(\bar{\mathbf{x}}^{k,i})}{\sqrt{(x_i^k)^2 + l_i^2(\bar{\mathbf{x}}^{k,i}) + 2\mu_k}} - 1\right)\left(\frac{\partial l_i}{\partial x_i}\bigg|_{\bar{\mathbf{x}}^{k,i}}\right)] \\
&= \frac{x_i^k}{\sqrt{(x_i^k)^2 + l_i^2(\bar{\mathbf{x}}^{k,i})}} - 1 + \left(\frac{l_i(\bar{\mathbf{x}}^{k,i})}{\sqrt{(x_i^k)^2 + l_i^2(\bar{\mathbf{x}}^{k,i})}} - 1\right)\left(\frac{\partial l_i}{\partial x_i}\bigg|_{\bar{\mathbf{x}}^{k,i}}\right)
\end{aligned}$$

for $j = i$, and

$$\begin{aligned}
u_{ij}^{k,0} &= \lim_{\mu_k\to 0} u_{ij}^{k,\mu_k} = \lim_{\mu_k\to 0}\left(\frac{\partial\tilde{\phi}_i}{\partial x_j}\bigg|_{\bar{\mathbf{x}}^{k,i}}\right) \\
&= \lim_{\mu_k\to 0}\left(\frac{l_i(\bar{\mathbf{x}}^{k,i})}{\sqrt{(x_i^k)^2 + l_i^2(\bar{\mathbf{x}}^{k,i}) + 2\mu_k}} - 1\right)\left(\frac{\partial l_i}{\partial x_j}\bigg|_{\bar{\mathbf{x}}^{k,i}}\right) \\
&= \left(\frac{l_i(\bar{\mathbf{x}}^{k,i})}{\sqrt{(x_i^k)^2 + l_i^2(\bar{\mathbf{x}}^{k,i})}} - 1\right)\left(\frac{\partial l_i}{\partial x_j}\bigg|_{\bar{\mathbf{x}}^{k,i}}\right)
\end{aligned}$$

for $j = i+1, i+2, ..., n$.

Then, Definition 2 implies

$$\begin{aligned}
\mathbf{u}_i^0 &:= (u_{ii}^{k,0}, u_{i,i+1}^{k,0}, ..., u_{in}^{k,0})^\top \\
&= \lim_{\mu_k\to 0}(u_{ii}^{k,\mu_k}, u_{i,i+1}^{k,\mu_k}, ..., u_{in}^{k,\mu_k})^\top \\
&= \lim_{\mu_k\to 0}(\frac{\partial\tilde{\phi}_i}{\partial x_i}, \frac{\partial\tilde{\phi}_i}{\partial x_{i+1}}, ..., \frac{\partial\tilde{\phi}_i}{\partial x_n})^\top\bigg|_{\bar{\mathbf{x}}^{k,i}} \\
&= \lim_{\mu_k\to 0}[(\frac{x_i^k}{\sqrt{(x_i^k)^2 + l_i^2(\bar{\mathbf{x}}^{k,i}) + 2\mu_k}} - 1)\bar{\mathbf{e}}^i
\end{aligned}$$

$$+(\frac{l_i(\bar{\mathbf{x}}^{k,i})}{\sqrt{(x_i^k)^2 + l_i^2(\bar{\mathbf{x}}^{k,i}) + 2\mu_k}} - 1)l_i'(\bar{\mathbf{x}}^{k,i})] \tag{14}$$

$$= \quad (\frac{x_i^k}{\sqrt{(x_i^k)^2 + l_i^2(\bar{\mathbf{x}}^{k,i})}} - 1)\bar{\mathbf{e}}^i + (\frac{l_i(\bar{\mathbf{x}}^{k,i})}{\sqrt{(x_i^k)^2 + l_i^2(\bar{\mathbf{x}}^{k,i})}} - 1)l_i'(\bar{\mathbf{x}}^{k,i})$$

for $i = 1, 2, ..., n$, where $\bar{\mathbf{e}}^i = (1, 0, ..., 0)^\top \in R^{n-i+1}$, and $l_i'(\bar{\mathbf{x}}^{k,i}) \in R^{n-i+1}$ is a vector with components $\frac{\partial l_i}{\partial x_j}\Big|_{\bar{\mathbf{x}}^i = \bar{\mathbf{x}}^{k,i}}$ for $j = i, i+1, ..., n$.

On the other hand, since Φ is differentiable function in $\mathcal{N}(\mathbf{x}^*, \varepsilon)$, it follows that

$$\partial\tilde{\phi}_i(\bar{\mathbf{x}}^{k,i}) = (\frac{x_i^k}{\sqrt{(x_i^k)^2 + l_i^2(\bar{\mathbf{x}}^{k,i})}} - 1)\bar{\mathbf{e}}^i + (\frac{l_i(\bar{\mathbf{x}}^{k,i})}{\sqrt{(x_i^k)^2 + l_i^2(\bar{\mathbf{x}}^{k,i})}} - 1)l_i'(\bar{\mathbf{x}}^{k,i}) \tag{15}$$

holds for $i = 1, 2, ..., n$.

It is clear that (14) and (15) imply

$$\mathbf{u}_i^0 \in \partial\tilde{\phi}_i(\bar{\mathbf{x}}^{k,i}) \quad \text{for} \quad i = 1, 2, ..., n. \qquad \square$$

Lemma 3 *a) If $\Phi'_{\mu_k}(\mathbf{x}^k)$ is nonsingular matrix and $\|\Phi'_{\mu_k}(\mathbf{x}^k)^{-1}\| \leq M$ then $U(\mathbf{x}^k, \mu_k)$ is nonsingular, $\|U(\mathbf{x}^k, \mu_k)^{-1}\| \leq M_1$ and $\|U(\mathbf{x}^k, \mu_k)\| \leq M_2$.*
b) If $\|\Phi^0(\mathbf{x}^k)^{-1}\| \leq M_3$ then $\|U^0(\mathbf{x}^k)^{-1}\| \leq M_4$ and $\|U^0(\mathbf{x}^k)\| \leq \widetilde{M}_2$, where $U^0(\mathbf{x}^k)$ is given in Definition 2.

Proof. a) Since $\Phi'_{\mu_k}(\mathbf{x}^k)$ is nonsingular and $U(\mathbf{x}^k, \mu_k)$ is obtained by the triangularization of the Jacobian matrix $\Phi'_{\mu_k}(\mathbf{x}^k)$ then

$$\Phi'_{\mu_k}(\mathbf{x}^k) = LU(\mathbf{x}^k, \mu_k), \tag{16}$$

and $U(\mathbf{x}^k, \mu_k)$ is also nonsingular. The boundedness of $\Phi'_{\mu_k}(\mathbf{x}^k)^{-1}$ and (16) imply that $U(\mathbf{x}^k, \mu_k)^{-1}$ is bounded i.e.

$$\|U(\mathbf{x}^k, \mu_k)^{-1}\| \leq M_1.$$

Since $\Phi^0(\mathbf{x}^k) \in \partial_C\Phi(\mathbf{x}^k)$ and $\partial_C\Phi(\mathbf{x}^k)$ is a compact set, there follows that

$$\|\Phi^0(\mathbf{x}^k)\| \leq \bar{M}_2. \tag{17}$$

Relation (16) and boundedness of $\Phi'_{\mu_k}(\mathbf{x}^k)^{-1}$ imply that $\Phi'_{\mu_k}(\mathbf{x}^k)$ and $U(\mathbf{x}^k, \mu_k)$ are bounded i.e.

$$\|U(\mathbf{x}^k, \mu_k)\| \leq M_2.$$

b) Since $\|\Phi^0(\mathbf{x}^k)^{-1}\| \le M_3$ and from the definition of $U^0(\mathbf{x}^k)$ and the fact that $U(\mathbf{x}^k, \mu_k)$ can be obtained by the triangularization of $\Phi'_{\mu_k}(\mathbf{x}^k)$, it follows that $U^0(\mathbf{x}^k)$ and $U^0(\mathbf{x}^k)^{-1}$ are bounded. □

In the same way as in Brown [1], the iteration process (Algorithm 1) can be formalized by writing the method in terms of the iteration function $G = (G_1, G_2, ..., G_n)^\top$, beginning with a starting iteration $\mathbf{x}^0$ and a sequence of positive numbers $\{\mu_k\}$ as

$$\mathbf{x}^{k+1} = G(\mathbf{x}^k), \quad k = 0, 1, ...,$$

where the iterative function G has the form

$$G_i(x_1, ..., x_n) = x_i - \left(\frac{\partial \tilde{\phi}_i(\bar{\mathbf{x}}^i, \mu_k)}{\partial x_i}\right)^{-1} \left[\sum_{j=i+1}^{n} \frac{\partial \tilde{\phi}_i(\bar{\mathbf{x}}^i, \mu_k)}{\partial x_j}(G_j - x_j) + \tilde{\phi}_i(\bar{\mathbf{x}}^i)\right], \tag{18}$$

for $i = 1, ..., n$, and functions $\tilde{\phi}_i(\bar{\mathbf{x}}^i)$ and $\tilde{\phi}_i(\bar{\mathbf{x}}^i, \mu_k)$ are given with (7) and (8). Functions $s_1, s_2, ..., s_{i-1}, \quad i = 1, 2, ..., n$ are themselves functions of x_j and are obtained recursively by substitution in the system

$$s_l = x_l - \left(\frac{\partial \tilde{\phi}_l(\bar{\mathbf{x}}^l, \mu_k)}{\partial x_l}\right)^{-1} \left[\sum_{j=l+1}^{n} \frac{\partial \tilde{\phi}_l(\bar{\mathbf{x}}^l, \mu_k)}{\partial x_j}(s_j - x_j) + \tilde{\phi}_l(\bar{\mathbf{x}}^l)\right], l = 1, ..., i-1 \tag{19}$$

and $s_n = x_n$ for completeness.

Lemma 4 *Any fixed point $\mathbf{x}^*$ of the iterative function G defined by (18) and (19) is a solution of NCP.*

Proof. Since $\mathbf{x}^* = G(\mathbf{x}^*)$ i.e. $x_i^* = G_i(\mathbf{x}^*)$, $i = 1, ..., n$, it follows from (18) that

$$\tilde{\phi}_i(\bar{\mathbf{x}}^{*,i}) = 0, \quad i = 1, ..., n, \tag{20}$$

so

$$\phi(x_i^*, F_i(s_1, s_2, ..., s_{i-2}, s_{i-1}, \bar{\mathbf{x}}^{*,i})) = 0, \; i = 1, ..., n. \tag{21}$$

Using (19) and (20) we have $s_l = x_l^*$ for $l = 1, ..., i-1$, so from (21) there follows

$$\phi(x_i^*, F_i(x_1^*, ..., x_n^*)) = 0,$$

which implies that $\mathbf{x}^*$ is a solution of the system $\Phi(x) = 0$ and also the solution of NCP.

Now we establish local superlinear convergence of JSB method.

Theorem 1 *Let $\mathbf{x}^*$ be the solution of the system $\mathbf{x} = G(\mathbf{x})$ which is a strictly complementary solution of NCP and $\Phi'(\mathbf{x}^*)$ is nonsingular matrix. Then there exist positive constants ε, μ such that for $\|\mathbf{x}^0 - \mathbf{x}^*\| \leq \varepsilon$ and a sequence of positive numbers $\{\mu_k\} \leq \mu$ which satisfies $\lim_{k\to\infty} \mu_k = 0$, it follows that the sequence $\{\mathbf{x}^k\}$ generated by JSB method is well defined and converges r-superlinearly to $\mathbf{x}^*$.*

Proof. The solution $\mathbf{x}^*$ is a strictly complementary solution of NCP, so $\partial_C\Phi(\mathbf{x}^*) = \{\Phi'(\mathbf{x}^*)\}$. Since function Φ_μ satisfies the Jacobian consistence property, i.e. $\Phi^0(\mathbf{x}^*) \in \partial_C\Phi(\mathbf{x}^*)$ and $\Phi'(\mathbf{x}^*)$ is nonsingular, then Lemma 1 implies that there exist a neighbourhood $\mathcal{N}_0(\mathbf{x}^*, \varepsilon_0) = \{\mathbf{x} \in R^n,\ \|\mathbf{x} - \mathbf{x}^*\| \leq \varepsilon_0\}$ and a constant $M > 0$ such that for any $\mathbf{x} \in \mathcal{N}_0(\mathbf{x}^*, \varepsilon_0)$ hold that $\Phi^0(\mathbf{x})$ is nonsingular and $\|\Phi^0(\mathbf{x})^{-1}\| \leq M$.

Since $\mathbf{x}^*$ is a strictly complementary solution of NCP, there exists a neighbourhood $\mathcal{N}_1(\mathbf{x}^*, \varepsilon_1) = \{\mathbf{x} \in R^n,\ \|\mathbf{x} - \mathbf{x}^*\| \leq \varepsilon_1\}$ such that function Φ is differentiable for $x \in \mathcal{N}_1(\mathbf{x}^*, \varepsilon_1)$. Let δ_k be a sequence of positive numbers such that

$$\lim_{k\to\infty} \delta_k = 0. \tag{22}$$

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and

$$\mathcal{N}(\mathbf{x}^*, \varepsilon) = \{\mathbf{x} \in R^n,\ \|\mathbf{x} - \mathbf{x}^*\| \leq \varepsilon\}.$$

From Definition 2

$$U^0(\mathbf{x}^k) = \lim_{\mu_k \to 0} U(\mathbf{x}^k, \mu_k)$$

follows that for given δ_k and $\mathbf{x}^k \in \mathcal{N}(\mathbf{x}^*, \varepsilon)$ there exists $\mu_k > 0$ such that

$$|u_{ii}^{k,0} - u_{ii}^{k,\mu_k}| \leq \delta_k \quad \text{for} \quad i = 1, 2, ..., n. \tag{23}$$

Since for $\mathbf{x}^k \in \mathcal{N}(\mathbf{x}^*, \varepsilon)$ holds $\|\Phi^0(\mathbf{x}^k)^{-1}\| \leq M$, Lemma 3 implies

$$\|U^0(\mathbf{x}^k)^{-1}\| \leq M_3 \tag{24}$$

and

$$\|U^0(\mathbf{x}^k)\| \leq M_2. \tag{25}$$

Upper triangular structure of $U^0(\mathbf{x}^k)$ and (24) imply that $(u_{ii}^{k,0})^{-1}$ are bounded for $i = 1, 2, ..., n$. From this fact, (23) and Perturbation Lemma we obtain

$$|(u_{ii}^{k,\mu_k})^{-1}| \le M_1 \quad \text{for} \quad i = 1, 2, ..., n. \tag{26}$$

From (25) there follows

$$|u_{ij}^{k,0}| \le M_2 \quad \text{for} \quad i = 1, 2, ..., n, \quad j \ge i. \tag{27}$$

Compactness of $\partial_C \Phi(\mathbf{x}^k)$ and $\Phi^0(\mathbf{x}^k) \in \partial_C \Phi(\mathbf{x}^k)$ imply $\|\Phi^0(\mathbf{x}^k)\| \le M_4$, so from definition of $\Phi^0(\mathbf{x}^k)$ there follows $\|\Phi'_{\mu_k}(\mathbf{x}^k)\| \le \bar{M}_4$. The boundeness of $\Phi'_{\mu_k}(\mathbf{x}^k)$ and the fact that $U(\mathbf{x}^k, \mu_k)$ can be obtained by triangularization of $\Phi'_{\mu_k}(\mathbf{x}^k)$, imply

$$\|U(\mathbf{x}^k, \mu_k)\| \le M_5,$$

so

$$|u_{ij}^{k,\mu_k}| \le M_5 \quad \text{for} \quad i = 1, 2, ..., n, \quad j \ge i. \tag{28}$$

From semismoothness of $\tilde{\phi}_i$, (5) and Lemma 2, there follows

$$|\tilde{\phi}_i(\bar{\mathbf{x}}^{k,i}) - \tilde{\phi}_i(\bar{\mathbf{x}}^{*,i}) - \mathbf{u}_i^0(\bar{\mathbf{x}}^{k,i} - \bar{\mathbf{x}}^{*,i})| = o(\|\bar{\mathbf{x}}^{k,i} - \bar{\mathbf{x}}^{*,i}\|) \tag{29}$$

for $i = 1, 2, ..., n$. Since $\tilde{\phi}_i$ are semismooth they are locally Lipschitzian, so

$$|\tilde{\phi}_i(\bar{\mathbf{x}}^{k,i}) - \tilde{\phi}_i(\bar{\mathbf{x}}^{*,i})| \le L\|\bar{\mathbf{x}}^{k,i} - \bar{\mathbf{x}}^{*,i}\| \tag{30}$$

holds for $i = 1, 2, ..., n$.

We have to prove

$$|x_i^{k+1} - x_i^*| \le o(|x_i^k - x_i^*|) + \sum_{j=i+1}^{n} c_j |x_j^k - x_j^*| \tag{31}$$

for all $i = n, n-1, ..., 2, 1$, where $c_j > 0$, $j = i+1, ..., n$.

Firstly, we will prove by induction that the inequality

$$|x_i^{k+1} - x_i^k| \le C|x_i^k - x_i^*| + \sum_{j=i+1}^{n} c_j |x_j^k - x_j^*| \tag{32}$$

holds for every $i = n, n-1, ..., 2$, where $C > 0, c_j > 0, j = i+1, ..., n$.

Using Algorithm 1, (20), (26) and (30), for $i = n$ follows

$$\begin{aligned}
|x_n^{k+1} - x_n^k| &\leq \left| \left(\frac{\partial \tilde{\phi}_n}{\partial x_n}\bigg|_{x_n^k} \right)^{-1} \right| \cdot \left| \tilde{\phi}_n(x_n^k) \right| \\
&= |(\frac{\partial \tilde{\phi}_n}{\partial x_n}\bigg|_{x_n^k})^{-1}| \cdot |\tilde{\phi}_n(x_n^k) - \tilde{\phi}_n(x_n^*)| \\
&\leq L|(u_{nn}^{k,\mu_k})^{-1}| \cdot |x_n^k - x_n^*| \\
&\leq M_1 L |x_n^k - x_n^*| \\
&= C|x_n^k - x_n^*|,
\end{aligned}$$

where $C = M_1 L$.

The induction hypothesis is that

$$|x_l^{k+1} - x_l^k| \leq C|x_l^k - x_l^*| + \sum_{j=l+1}^{n} c_j |x_j^k - x_j^*| \tag{33}$$

holds for $l = n, n-1, ..., i+1$.

Now, we prove that (33) holds for $l = i$. The Algorithm 1, (20), (26), (28) and (30) imply

$$\begin{aligned}
|x_i^{k+1} - x_i^k| &\leq |(\frac{\partial \tilde{\phi}_i}{\partial x_i}\bigg|_{\bar{\mathbf{x}}^{k,i}})^{-1}| \cdot |\tilde{\phi}_i(\bar{\mathbf{x}}^{k,i}) + \sum_{j=i+1}^{n} (\frac{\partial \tilde{\phi}_i}{\partial x_j}\bigg|_{\bar{\mathbf{x}}^{k,i}})(x_j^{k+1} - x_j^k)| \\
&= |(u_{ii}^{k,\mu_k})^{-1}| \cdot |\tilde{\phi}_i(\bar{\mathbf{x}}^{k,i}) - \tilde{\phi}_i(\bar{\mathbf{x}}^{*,i}) + \sum_{j=i+1}^{n} u_{ij}^{k,\mu_k}(x_j^{k+1} - x_j^k)| \\
&\leq |(u_{ii}^{k,\mu_k})^{-1}| \left[|\tilde{\phi}_i(\bar{\mathbf{x}}^{k,i}) - \tilde{\phi}_i(\bar{\mathbf{x}}^{*,i})| + \sum_{j=i+1}^{n} |u_{ij}^{k,\mu_k}||x_j^{k+1} - x_j^k| \right] \\
&\leq M_1 \left[L\|\bar{\mathbf{x}}^{k,i} - \bar{\mathbf{x}}^{*,i}\| + M_5 \sum_{j=i+1}^{n} |x_j^{k+1} - x_j^k| \right] \\
&\leq M_1 \left[L \sum_{j=i}^{n} |x_j^k - x_j^*| + M_5 \sum_{j=i+1}^{n} |x_j^{k+1} - x_j^k| \right] \\
&= M_1 \left[L|x_i^k - x_i^*| + L \sum_{j=i+1}^{n} |x_j^k - x_j^*| + M_5 \sum_{j=i+1}^{n} |x_j^{k+1} - x_j^k| \right]
\end{aligned}$$

$$
\begin{aligned}
&= M_1 L|x_i^k - x_i^*| + M_1 L \sum_{j=i+1}^{n} |x_j^k - x_j^*| \\
&\quad + M_1 M_5 \sum_{j=i+1}^{n} |x_j^{k+1} - x_j^k|.
\end{aligned}
$$

Using induction hypothesis (33) and the previous inequality we have

$$|x_i^{k+1} - x_i^k| \leq C|x_i^k - x_i^*| + \sum_{j=i+1}^{n} c_j |x_j^k - x_j^*|,$$

where $C = M_1 L$ and for $i = n-1, ..., 2$

$$c_n = C(1 + M_1 M_5)^{n-i},$$

$$c_j = C(1 + M_1 M_5)^{j-i}, \;\; j = i+1, i+2, ..., n.$$

Hence, (32) holds for all $i = n, n-1, ..., 2$ and this inequality will be used later.

Now we are ready to prove that (31) holds for every $i = n, n-1, ..., 2, 1$. For $i = n$

$$
\begin{aligned}
x_n^{k+1} - x_n^* &= x_n^k - x_n^* - \left(\left.\frac{\partial \tilde{\phi_n}}{\partial x_n}\right|_{x_n^k}\right)^{-1} \tilde{\phi_n}(x_n^k) \\
&= -(u_{nn}^{k,\mu_k})^{-1} [\tilde{\phi_n}(x_n^k) - \tilde{\phi_n}(x_n^*) - \left(\left.\frac{\partial \tilde{\phi_n}}{\partial x_n}\right|_{x_n^k}\right)(x_n^k - x_n^*)] \\
&= -(u_{nn}^{k,\mu_k})^{-1} [(\tilde{\phi_n}(x_n^k) - \tilde{\phi_n}(x_n^*) - u_{nn}^{k,0}(x_n^k - x_n^*)) \\
&\quad + (u_{nn}^{k,0} - \left(\left.\frac{\partial \tilde{\phi_n}}{\partial x_n}\right|_{x_n^k}\right))(x_n^k - x_n^*)]
\end{aligned}
$$

follows from the Algorithm 1 and (20).

Then using (22), (23), (26) and (29)

$$
\begin{aligned}
|x_n^{k+1} - x_n^*| &= |(u_{nn}^{k,\mu_k})^{-1}| [|\tilde{\phi_n}(x_n^k) - \tilde{\phi_n}(x_n^*) - u_{nn}^{k,0}(x_n^k - x_n^*)| \\
&\quad + |u_{nn}^{k,0} - \left(\left.\frac{\partial \tilde{\phi_n}}{\partial x_n}\right|_{x_n^k}\right)| \cdot |x_n^k - x_n^*|] \\
&\leq M_1 \left[o(|x_n^k - x_n^*|) + |u_{nn}^{k,0} - u_{nn}^{k,\mu_k}||x_n^k - x_n^*| \right] \\
&\leq M_1 \left[o(|x_n^k - x_n^*|) + \delta_k |x_n^k - x_n^*| \right] \\
&= o(|x_n^k - x_n^*|)
\end{aligned}
$$

holds.

So, (31) holds for $i = n$.

We can prove that (31) holds for i, $i = n-1, n-2, ..., 1$. The Algorithm 1 implies

$$\begin{aligned}
x_i^{k+1} - x_i^* &= s_i(\bar{\mathbf{x}}^{k+1,i+1}) - x_i^* \\
&= x_i^k - x_i^* - \left(\frac{\partial \tilde{\phi}_i}{\partial x_i}\bigg|_{\bar{\mathbf{x}}^{k,i}} \right)^{-1} [\tilde{\phi}_i(\bar{\mathbf{x}}^{k,i}) \\
&\quad + \sum_{j=i+1}^{n} (\frac{\partial \tilde{\phi}_i}{\partial x_j}\bigg|_{\bar{\mathbf{x}}^{k,i}})(x_j^{k+1} - x_j^k)] \\
&= (-\frac{\partial \tilde{\phi}_i}{\partial x_i}\bigg|_{\bar{\mathbf{x}}^{k,i}})^{-1}[\tilde{\phi}_i(\bar{\mathbf{x}}^{k,i}) - (\frac{\partial \tilde{\phi}_i}{\partial x_i}\bigg|_{\bar{\mathbf{x}}^{k,i}})(x_i^k - x_i^*) \\
&\quad + \sum_{j=i+1}^{n} (\frac{\partial \tilde{\phi}_i}{\partial x_j}\bigg|_{\bar{\mathbf{x}}^{k,i}})(x_j^{k+1} - x_j^k)].
\end{aligned}$$

Therefore,

$$\begin{aligned}
|x_i^{k+1} - x_i^*| &\leq |(u_{ii}^{k,\mu_k})^{-1}||\tilde{\phi}_i(\bar{\mathbf{x}}^{k,i}) - \tilde{\phi}_i(\bar{\mathbf{x}}^{*,i}) - u_{ii}^{k,\mu_k}(x_i^k - x_i^*) \\
&\quad + \sum_{j=i+1}^{n} u_{ij}^{k,\mu_k}(x_j^{k+1} - x_j^k)| \\
&\leq |(u_{ii}^{k,\mu_k})^{-1}|[|\tilde{\phi}_i(\bar{\mathbf{x}}^{k,i}) - \tilde{\phi}_i(\bar{\mathbf{x}}^{*,i}) - \mathbf{u}_i^0(\bar{\mathbf{x}}^{k,i} - \bar{\mathbf{x}}^{*,i})| \\
&\quad + |\mathbf{u}_i^0(\bar{\mathbf{x}}^{k,i} - \bar{\mathbf{x}}^{*,i}) - u_{ii}^{k,\mu_k}(x_i^k - x_i^*)| \\
&\quad + \sum_{j=i+1}^{n} |u_{ij}^{k,\mu_k}||x_j^{k+1} - x_j^k|]. \qquad (34)
\end{aligned}$$

Using (23) and (27) we can state

$$\begin{aligned}
&|\mathbf{u}_i^0(\bar{\mathbf{x}}^{k,i} - \bar{\mathbf{x}}^{*,i}) - u_{ii}^{k,\mu_k}(x_i^k - x_i^*)| = \\
=\ & |\sum_{j=i}^{n} u_{ij}^{k,0}(x_j^k - x_j^*) - u_{ii}^{k,\mu_k}(x_i^k - x_i^*)| \\
\leq\ & |u_{ii}^{k,0} - u_{ii}^{k,\mu_k}||x_i^k - x_i^*| + \sum_{j=i+1}^{n} |u_{ij}^{k,0}||x_j^k - x_j^*| \\
\leq\ & \delta_k|(x_i^k - x_i^*)| + M_2 \sum_{j=i+1}^{n} |x_j^k - x_j^*|. \qquad (35)
\end{aligned}$$

From (22), (26), (29), (34) and (35) follows

$$
\begin{aligned}
|x_i^{k+1} - x_i^*| &\leq M_1[o(\|\bar{\mathbf{x}}^{k,i} - \bar{\mathbf{x}}^{*,i}\|) + \delta_k |x_i^k - x_i^*| \\
&\quad + M_2 \sum_{j=i+1}^{n} |x_j^k - x_j^*| + M_5 \sum_{j=i+1}^{n} |x_j^{k+1} - x_j^k|] \\
&\leq M_1[\sum_{j=i}^{n} o(|x_j^k - x_j^*|) + \delta_k |x_i^k - x_i^*| \\
&\quad + M_2 \sum_{j=i+1}^{n} |x_j^k - x_j^*| + M_5 \sum_{j=i+1}^{n} |x_j^{k+1} - x_j^k|] \\
&\leq M_1[o(|x_i^k - x_i^*|) + \sum_{j=i+1}^{n} o(|x_j^k - x_j^*|) + \delta_k |x_i^k - x_i^*| \\
&\quad + M_2 \sum_{j=i+1}^{n} |x_j^k - x_j^*| + M_5 \sum_{j=i+1}^{n} |x_j^{k+1} - x_j^k|] \\
&= o(|x_i^k - x_i^*|) + \sum_{j=i+1}^{n} o(|x_j^k - x_j^*|) \\
&\quad + M_1 M_2 \sum_{j=i+1}^{n} |x_j^k - x_j^*| + M_1 M_5 \sum_{j=i+1}^{n} |x_j^{k+1} - x_j^k|.
\end{aligned}
$$

Applying (32) on the previous inequality we have

$$|x_i^{k+1} - x_i^*| \leq o(|x_i^k - x_i^*|) + \sum_{j=i+1}^{n} c_j |x_j^k - x_j^*|,$$

where

$$c_{i+1} = M_1 M_2 + M_1 M_5 C + \theta_k, \quad \text{for} \ \ i = n-1, n-2, ..., 1,$$

$$c_j = M_1 M_2 + M_1 M_5 C \sum_{l=0}^{j-1-i} (1 + M_1 M_5)^l + \theta_k, \quad \text{for} \ \ j = i+2, ..., n-1,$$

$$c_n = M_1 M_2 + M_1 M_5 C \left(1 + \sum_{l=1}^{n-1-i} (1 + M_1 M_5)^l \right) + \theta_k,$$

while $\lim_{k \to \infty} \theta_k = 0$ and $C = M_1 L$.
So, it is proved that (31) holds for $i = n, n-1, ..., 2, 1$, i.e.

$$|x_i^{k+1} - x_i^*| \leq r_k |x_i^k - x_i^*| + \sum_{j=i+1}^{n} c_j |x_j^k - x_j^*|, \tag{36}$$

for $i = n, n-1, ..., 2, 1$, where $lim_{k\to\infty} r_k = 0$.
Let us denote

$$\tilde{e}_i^{k+1} = |x_i^{k+1} - x_i^*| \quad \text{for} \quad i = 1, 2, ..., n,$$

and let $\sigma_s^{(k)}$ be an elementary symmetric polynomial of degree s , $s = k - (n - i - 1), ..., k + 1$; $i = 1, ..., n$, of the $k + 1$ variables r_0, r_1,..., r_k, which are elements of the sequence $\{r_k\}$. Using (36) it can be proved for $i = 1, ..., n$ that

$$\tilde{e}_i^{k+1} \le \bar{\sigma}_{k+1,i}, \tag{37}$$

where

$$\bar{\sigma}_{k+1,i} = \sigma_{k+1}^{(k)} C_0 + \sigma_k^{(k)} C_1 + \cdots + \sigma_{k-(n-i-1)}^{(k)} C_{n-i},$$

with constants $C_j > 0, j = 0, 1, ..., n - i$. Since $lim_{k\to\infty} r_k = 0$ then $\lim_{k\to\infty} \bar{\sigma}_{k+1,i} = 0$. The fact that

$$\lim_{k\to\infty} \frac{\bar{\sigma}_{k+1,i}}{\bar{\sigma}_{k,i}} = 0$$

and relation (37) imply that the sequence $\{x^k\}$ is well defined and converges r-superlinearly to x^*. $\square$

4 Hybrid method

The superlinear convergence of JSB method is proved under the assumption of strictly complementary solution $\mathbf{x}^*$ of NCP. If $\mathbf{x}^*$ is a degenerate solution i.e. if $x_i^* = F_i(\mathbf{x}^*) = 0$ holds for some index $i \in \{1, 2, ..., n\}$, then function Φ is not differentiable at $\mathbf{x}^*$, so we define hybrid method in a similar way as in Chen [5]. This method is a combination of Brown's and Newton's method with smoothing, so we call it the Jacobian smoothing Brown-Newton method.

Let N_Φ be the set where Φ is not differentiable and W be a set such that $N_\Phi \subseteq W$. The set $W_\tau = \{\mathbf{x} \in R^n,\ dist(\mathbf{x}, W) \le \tau\}$ is defined for $\tau > 0$. The line segment between $\mathbf{x}$ and $\mathbf{y}$ is denoted by $\overline{\mathbf{xy}}$.

In addition, it is assumed that there is a positive number $L > 0$ such that for any $\mu > 0$ holds

$$\|\Phi_\mu'(\mathbf{x}) - \Phi_\mu'(\mathbf{y})\| \le L\|\mathbf{x} - \mathbf{y}\|, \quad \text{if} \quad \overline{\mathbf{xy}} \cap W_\tau = \emptyset. \tag{38}$$

The hybrid method is described as follows.

Algorithm 2: Jacobian smoothing Brown-Newton's method (HJSBN)

S0: Let $\mathbf{x}^0 \in R^n$, $\gamma > \tau > 0$, $W_\gamma = \{\mathbf{x} \in R^n,\ dist(\mathbf{x}, W) \leq \gamma\}$ be given. Let $\{\mu_k\}$ be a sequence of real positive numbers.

S1: Compute $\mathbf{x}^1$ from the Newton equation

$$\begin{aligned} \Phi'_{\mu_0}(\mathbf{x}^0)\mathbf{s}^0 &= -\Phi(\mathbf{x}^0), \\ \mathbf{x}^1 &= \mathbf{x}^0 + \mathbf{s}^0, \qquad k := 1. \end{aligned}$$

S2: If $\overline{\mathbf{x}^k\mathbf{x}^{k-1}} \cap W_\gamma \neq \emptyset$ then compute $\mathbf{x}^{k+1}$ from the Newton equation

$$\begin{aligned} \Phi'_{\mu_k}(\mathbf{x}^k)\mathbf{s}^k &= -\Phi(\mathbf{x}^k), \\ \mathbf{x}^{k+1} &= \mathbf{x}^k + \mathbf{s}^k, \end{aligned}$$

else compute $\mathbf{x}^{k+1}$ from Brown's method (take step S1 of Algorithm 1).

S3: Set $k := k + 1$ and return to the step S2. ♣

The next Theorem is about local convergence of HJSBN method.

Theorem 2 *Let $\mathbf{x}^*$ be the solution of the system $\mathbf{x} = G(\mathbf{x})$, all elements of $\partial_C\Phi(\mathbf{x}^*)$ be nonsingular and additional assumption (38) be satisfied. Then there exist positive constants ε, μ such that for $\|\mathbf{x}^0 - \mathbf{x}^*\| \leq \varepsilon$ and a sequence $\{\mu_k\} \leq \mu$ of positive numbers which satisfies $\lim_{k\to\infty} \mu_k = 0$, there follows that the sequence $\{\mathbf{x}^k\}$ generated by Algorithm 2 is well defined and converges r-superlinearly to $\mathbf{x}^*$.*

Proof. Since function Φ_μ has the Jacobian consistence property, i.e. $\Phi^0(\mathbf{x}^*) \in \partial_C\Phi(\mathbf{x}^*)$ and all elements of $\partial_C\Phi(\mathbf{x}^*)$ are nonsingular, then Lemma 1 implies that there exist $\mathcal{N}_0(\mathbf{x}^*, \varepsilon_0)$ and constants $M, M_1 > 0$ such that $\Phi^0(\mathbf{x})$ is nonsingular for any $\mathbf{x} \in \mathcal{N}_0(\mathbf{x}^*, \varepsilon_0)$ and $\|\Phi^0(\mathbf{x})^{-1}\| \leq M$ and there exists $\bar{\mu} > 0$ such that for $\mu \in (0, \bar{\mu})$ holds

$$\|\Phi'_\mu(\mathbf{x})^{-1}\| \leq M_1. \tag{39}$$

Let δ_k be a sequence of positive numbers such that

$$\lim_{k\to\infty} \delta_k = 0. \tag{40}$$

We can distinguish three cases:

1. $\mathbf{x}^* \in intW_\gamma$,
2. $\mathbf{x}^* \in R^n \backslash W_\gamma$,
3. $\mathbf{x}^* \in \widetilde{W}_\gamma = \{\mathbf{x} \in R^n,\ dist(\mathbf{x}, W) = \gamma\}$.

Case 1: If $\mathbf{x}^* \in intW_\gamma$ then there exists $\varepsilon > 0$ small enough such that $\mathcal{N}(\mathbf{x}^*, \varepsilon) \subseteq \mathcal{N}_0(\mathbf{x}^*, \varepsilon_0) \cap intW_\gamma$. For $\mathbf{x}^k \in \mathcal{N}(\mathbf{x}^*, \varepsilon)$ and given δ_k, from (4) and (39) there exists $\mu_k > 0$ such that

$$\|\Phi^0(\mathbf{x}^k) - \Phi'_{\mu_k}(\mathbf{x}^k)\| \leq \delta_k, \tag{41}$$

$$\|\Phi'_{\mu_k}(\mathbf{x}^k)^{-1}\| \leq M_1. \tag{42}$$

Since $\Phi^0(\mathbf{x}^k) \in \partial_C \Phi(\mathbf{x}^k)$, from (5) follows

$$\|\Phi(\mathbf{x}^k) - \Phi(\mathbf{x}^*) - \Phi^0(\mathbf{x}^k)(\mathbf{x}^k - \mathbf{x}^*)\| = o(\|\mathbf{x}^k - \mathbf{x}^*\|) \tag{43}$$

for $\mathbf{x}^k \in \mathcal{N}(\mathbf{x}^*, \varepsilon)$. By the Algorithm 2, the Newton method is applied in $\mathcal{N}(\mathbf{x}^*, \varepsilon)$, so using (40)-(43) we get

$$\begin{aligned}
\|\mathbf{x}^{k+1} - \mathbf{x}^*\| &= \|\mathbf{x}^k - \mathbf{x}^* - \Phi'_{\mu_k}(\mathbf{x}^k)^{-1}\Phi(\mathbf{x}^k)\| \\
&= \| - \Phi'_{\mu_k}(\mathbf{x}^k)^{-1}[\Phi(\mathbf{x}^k) - \Phi(\mathbf{x}^*) \pm \Phi^0(\mathbf{x}^k)(\mathbf{x}^k - \mathbf{x}^*) \\
&\quad -\Phi'_{\mu_k}(\mathbf{x}^k)(\mathbf{x}^k - \mathbf{x}^*)]\| \\
&\leq \| - \Phi'_{\mu_k}(\mathbf{x}^k)^{-1}\|[\|\Phi(\mathbf{x}^k) - \Phi(\mathbf{x}^*) - \Phi^0(\mathbf{x}^k)(\mathbf{x}^k - \mathbf{x}^*)\| \\
&\quad +\|\Phi^0(\mathbf{x}^k) - \Phi'_{\mu_k}(\mathbf{x}^k)\|\|\mathbf{x}^k - \mathbf{x}^*\|] \\
&\leq M_1 \left[o(\|\mathbf{x}^k - \mathbf{x}^*\|) + \delta_k\|\mathbf{x}^k - \mathbf{x}^*\|\right] \\
&= o(\|\mathbf{x}^k - \mathbf{x}^*\|).
\end{aligned}$$

Then

$$\|\mathbf{x}^{k+1} - \mathbf{x}^*\| \leq o(\|\mathbf{x}^k - \mathbf{x}^*\|) \leq \|\mathbf{x}^k - \mathbf{x}^*\| \leq \varepsilon$$

holds for $\mathbf{x}^k \in \mathcal{N}(\mathbf{x}^*, \varepsilon)$, i.e. $\mathbf{x}^{k+1} \in \mathcal{N}(\mathbf{x}^*, \varepsilon)$, which implies that $\{\mathbf{x}^k\}$ is well defined, and q-superlinear and also r-superlinear convergence is obtained.

Case 2: If $\mathbf{x}^* \in R^n \backslash W_\gamma$, then there exists $\varepsilon > 0$ small enough such that $\mathcal{N}(\mathbf{x}^*, \varepsilon) \subseteq \mathcal{N}_0(\mathbf{x}^*, \varepsilon_0) \cap (R^n \backslash W_\gamma)$ and additional assumption (38) holds in $\mathcal{N}(\mathbf{x}^*, \varepsilon)$. This assumption implies differentiability of Φ in $\mathbf{x}^*$. Since $\mathcal{N}(\mathbf{x}^*, \varepsilon) \cap W_\gamma = \emptyset$, by the Algorithm 2 Brown's method is applied in $\mathcal{N}(\mathbf{x}^*, \varepsilon)$, so from Theorem 1 follows r-superlinear convergence.

Case 3: If $\mathbf{x}^* \in \widetilde{W}_\gamma = \{\mathbf{x},\ dist(\mathbf{x}, W) = \gamma\}$, then there exists $\varepsilon > 0$ small enough such that $\mathcal{N}(\mathbf{x}^*, \varepsilon) \subseteq \mathcal{N}_0(\mathbf{x}^*, \varepsilon_0) \cap (R^n \backslash W_\tau)$ and additional assumption (38) holds in $\mathcal{N}(\mathbf{x}^*, \varepsilon)$, which implies the differentiability of Φ at $\mathbf{x}^*$. By the Algorithm 2, either Brown's or Newton's method is applied in $\mathcal{N}(\mathbf{x}^*, \varepsilon)$ in each iteration. Let $\mathbf{x}^k \in \mathcal{N}(\mathbf{x}^*, \varepsilon)$. If $\mathbf{x}^{k+1}$ is obtained by Brown's method, then from Theorem 1 there follows r-superlinear convergence. If $\mathbf{x}^{k+1}$ is obtained by the Newton method then conclusion follows from Case 1. $\square$

5 Numerical experiments

Some numerical results obtained by JSB method are presented in this section. Local superlinear convergence of JSB method is proved in the case of strictly complementary solution, while we define a hybrid method for a degenerate solution, for which superlinear convergence is also proved. It is important to notice that, in practice, JSB method is successful even in the case of degenerate solution, thus exceeding theoretical expectations.

Algorithms are implemented in Mathematica 5.0.

The main stopping criteria are

$$\|\mathbf{x}^k - \mathbf{x}^{k-1}\| \leq 10^{-6} \text{ and } \|\Phi(\mathbf{x}^k)\| \leq 10^{-6},$$

but if they are not satisfied, the algorithms are stopped after $k_{max} = 100$ iterations.

The sequence of smoothing parameters is defined in this way

$$\begin{aligned} \mu_0 &= \|\Phi(\mathbf{x}^0)\|, \\ \mu_{k+1} &= \frac{1}{4}\,\mu_k, \quad k = 0, 1, ... \end{aligned}$$

We compare Jacobian smoothing Brown's method (JSB) with Jacobian smoothing Newton's method (JSN) using different starting approximations $\mathbf{x}^0$.

First, we show some results obtained by testing NCP with function F defined by the following examples 1 and 2.

Example 1. Function $F : R^4 \to R^4$ is given by

$$F_1(\mathbf{x}) = 3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4 - 6$$

$$F_2(\mathbf{x}) = 2x_1^2 + x_1 + x_2^2 + 3x_3 + 2x_4 - 2$$

$$F_3(\mathbf{x}) = 3x_1^2 + x_1x_2 + 2x_2^2 + 2x_3 + 3x_4 - 1$$

$$F_4(\mathbf{x}) = x_1^2 + 3x_2^2 + 2x_3 + 3x_4 - 3$$

$(\mathbf{x}^0)^\top$	JSN	JSB
$(1, 0, 1, 0)$	6	6
$(1, 0, 0, 1)$	5	5
$(1, 0.2, 0.5, 1)$	5	5
$(1, 0.5, 0.5, 1)$	5	5
$(1.5, -0.5, 4.5, -1)$	6	7
$(1.1, -0.1, 3.1, -0.1)$	6	6
$(0.85, 0.2, 0.5, 1)$	5	5
$(1.1, 0.2, 0.2, 0.4)$	5	5
$(1.5, -0.5, 0.5, 1)$	6	5

Table 1. Example 1

Example 2. Function $F : R^4 \to R^4$ is given by

$$F_1(\mathbf{x}) = 3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4 - 6$$

$$F_2(\mathbf{x}) = 2x_1^2 + x_1 + x_2^2 + 10x_3 + 2x_4 - 2$$

$$F_3(\mathbf{x}) = 3x_1^2 + x_1x_2 + 2x_2^2 + 2x_3 + 9x_4 - 9$$

$$F_4(\mathbf{x}) = x_1^2 + 3x_2^2 + 2x_3 + 3x_4 - 3$$

$(\mathbf{x}^0)^\top$	JSN	JSB
$(1.1, 0.2, 0.2, 0.4)$	$(8, x^*_{SC})$	$(8, x^*_{SC})$
$(1.1, -0.1, 3.1, -0.1)$	$(3, x^*_{SC})$	$(4, x^*_{SC})$
$(0.5, 0, 3.5, 0)$	$(5, x^*_{SC})$	$(6, x^*_{SC})$
$(1, 0.2, 0.5, 1)$	$(18, x^*_D)$	$(8, x^*_{SC})$
$(1.2, 0.01, 0.01, 0.4)$	$(18, x^*_D)$	$(20, x^*_D)$

Table 2. Example 2

NCP with function F given in Example 1 has strictly complementary solution $\mathbf{x}^* = (\frac{1}{2}\sqrt{6}, 0, 0, 0.5)^\top$, while NCP with function F from Example 2 has two solutions, the degenerate solution $\mathbf{x}^*_D = (\frac{1}{2}\sqrt{6}, 0, 0, 0.5)^\top$ and strictly complementary solution $\mathbf{x}^*_{SC} = (1, 0, 3, 0)^\top$. For both methods Table 1 and Table 2 present number of iterations needed for convergence, while the solution to which the method converges is also marked in Table 2.

Beside these two examples of dimension $n = 4$, we tested another five examples from Lukšan [12] and Spedicato and Huang [13]. Test problems are generated in the usual way proposed by Gomes-Ruggiero et al [14].

Let $f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), ..., f_n(\mathbf{x}))^\top$ be a differentiable nonlinear mapping from R^n to R^n and let $\mathbf{x}^* = (1, 0, 1, 0, ...)^\top \in R^n$. For $i = 1, 2, ..., n$ set

$$F_i(\mathbf{x}) = \begin{cases} f_i(\mathbf{x}) - f_i(\mathbf{x}^*), & \text{if} \quad i \quad \text{odd or} \quad i > r \\ f_i(\mathbf{x}) - f_i(\mathbf{x}^*) + 1, & \text{otherwise} \end{cases}$$

where $r \geq 0$ is an integer. For function F defined in this way, vector $\mathbf{x}^*$ is a solution of NCP, but not necessarily its unique solution. If $r < n$, $\mathbf{x}^*$ is a degenerate solution of NCP, while for $r = n$ it is a strictly coplementary solution. Function f is defined as follows:

Example 3. Lukšan [12], problem 4.7

Example 4. Lukšan [12], problem 4.8

Example 5. Lukšan [12], problem 4.14

Example 6. Lukšan [12], problem 4.17

Example 7. Spedicato and Huang [13], problem 2.

All examples are tested with three dimensions $n = 4$, $n = 10$, $n = 100$ and starting iterations suggested in Lukšan [12] and Spedicato and Huang [13]. For each dimension we consider a degenerate solution ($r = n/2$) and a strictly complementary one ($r = n$).

The obtained results are compared using three indices: the index of robustness, the efficiency index and the combined robustness and efficiency index, which are given in Bogle and Perkins [15].

The robustness index is defined by

$$R_j = \frac{t_j}{n_j},$$

the efficiency index is

$$E_j = \sum_{i=1, r_{ij} \neq 0}^{m} (\frac{r_{ib}}{r_{ij}})/t_j,$$

and the combined index is

$$E_j \times R_j = \sum_{i=1, r_{ij} \neq 0}^{m} (\frac{r_{ib}}{r_{ij}})/n_j,$$

where r_{ij} is the number of iterations required to solve the problem i by the method j, $r_{ib} = \min_j r_{ij}$, t_j is the number of successes by method j and n_j is the number of problems attempted by method j.

Following tables report the results of the two methods.

	JSN	JSB
R	0.978723	0.978723
E	0.988544	0.990554
E × R	0.967511	0.969478

Table 3. Strictly complementary solution ($r = n$)

	JSN	JSB
R	0.9375	0.9375
E	0.979731	0.979038
E × R	0.918498	0.917848

Table 4. Degenerate solution ($r = n/2$)

By the results presented in Tables 1,2,3 and 4 we can notice very similar behaviour of both methods. Numerical results confirmed theoretical expectations in the sense of superlinear convergence of both methods, while they exceeded theoretical results for JSB method, because this method can be applied successively in practice even in the case of degenerate solution.

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