Mathematical Model for Efficient Water Flow Management

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Abstract

A mathematical model for optimization of pump scheduling in water distribution system is proposed. The water system we consider works without large water storage facility and hence the mass balance principle is the base for the model. A result concerning pressure drop after T-junctions is the main simplification we propose. An optimization problem is formulated and solved by Branch and Bound method for a real water distribution system using data from a small city in Serbia.
Key words: water distribution system, modelling water flow, linear programming, branch and bound method.

1 Introduction

Water distributional system is the most important element of urban planning and requires significant research and investment. Water distributional systems are becoming larger and very complex, especially in big cities.

Usually, the main source consists of a set of pumps of different capacity. These pumps work in combination and are supposed to give sufficient amount of water. Thus, in any time period some pumps are in use while others are idle. In other words optimization of operative control mean choosing the right combination of pumps that will be working at one time period.

There have been several attempts in recent years to develop optimal control algorithms to assist in the operation of a complex water distributional system.

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Mays [8] lists and classifies several algorithms that have been developed to solve the control problem in a water distributional system. Various algorithms have been proposed for determining least cost pump scheduling polices and they are based on linear programming, nonlinear programming, dynamic programming, genetic algorithms. However, the success of these algorithms has been very limited and very few have been actually applied to a real water distribution system.

In order to optimize the total cost of a water distributional system one needs to analyze the components. The major part of the total budget depends on pumping of treated water, [8].

Lansey et al. [7] introduced the number of pump switches as a way to evaluate pumps maintenance cost. Baran et al. [1] proposed reservoir level variation and maximum power peak as new constrains in pump scheduling optimization.

We make our model for the water factory in Zrenjanin. Zrenjanin is a city in Serbia with approximately 80000 inhabitants. Its water distribution system has been gradually built since 1945 and it has nearly $270km$ of pipeline with diameter ranging from $50mm$ to $600mm$. Dominant pipeline material is asbestos-cement. The majority of the pipes are older than 20 years and more than half of the pipes have diameter smaller then $100mm$. There is one main source with 32 wells, see Figure 1. From the main source the city is supplied with two transit pipelines with diameter of $500mm$ and $600mm$, and in these pipelines we are monitoring pressure and flow rate. In these pipelines the pressure is usually around $3bar$ and during summer around $2bar$. The capacity of the main source is about $520l/s$. Zrenjanin doesn't have a large water storage. So, the water system functions on the principle:

$$wells \rightarrow pumps \rightarrow pipes.$$

Given all these characteristics of the water system we have chosen the mass balance mathematical model. This model is based on the equilibrium between the amount of water that comes from pumps and the amount water that is used in the city. The water demand curve is obtained from a statistical study of historical data.

The present work is organized as follows. Section 2 contains a description of $Q - H$ characteristics and calculation of water flow at each pump. In Section 3 loss coefficients are introduced. These coefficients are calculated using a model for $T-$junction and splitting of pipes. We also state and prove a theorem about pressure after $T-$junction which gives significant simplification in further modelling of a water distributional system. In Section 4 we formulate the optimization problem for operative control of pumps and discuss application of Branch and Bound algorithm for its solution. Finally, Section 5 contains the implementation details of the algorithm in four different test and experimental results.

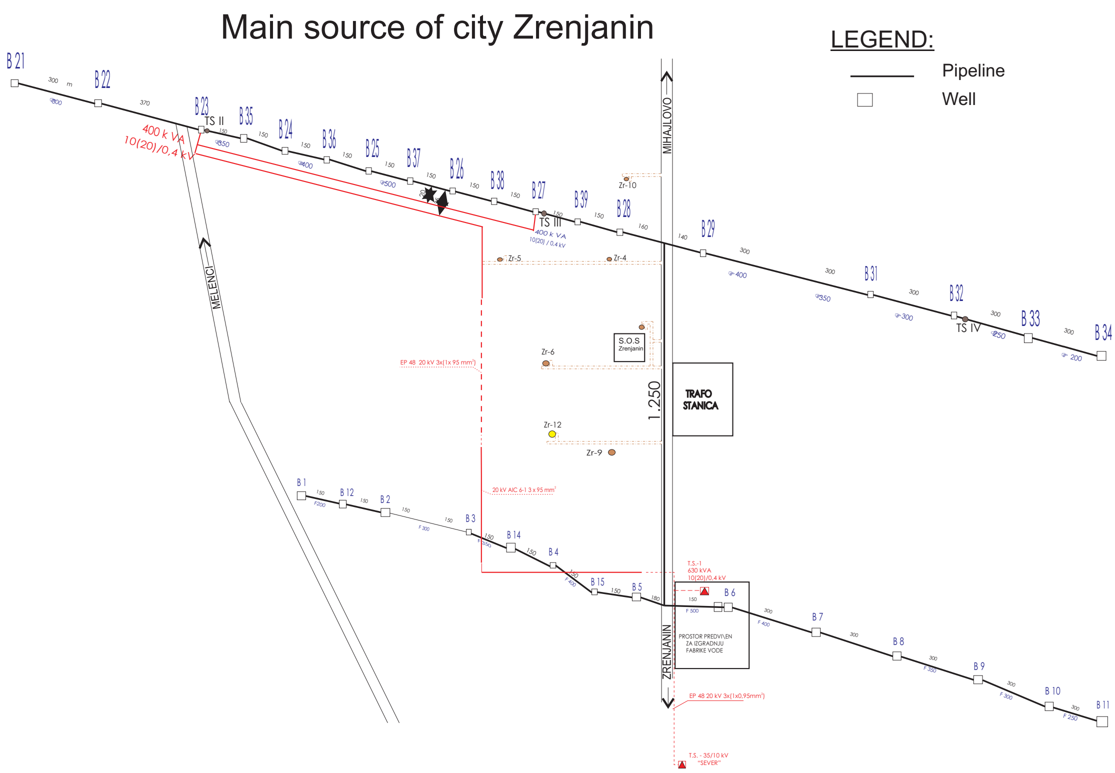

Figure 1: The main source in Zrenjanin.

2 Q-H pump characteristic

Water head is a term commonly used with pumps. Head represents the height of a vertical column of water. Pressure p and water head H are mutually connected concepts and in our model we use the equation

$$H = \frac{p}{\rho g},$$

where ρ is fluid density and g is the gravity constant. The relationship between water flow Q and water head H (pressure p) is given by the so-called $Q-H$ characteristic for each pump. Each $Q-H$ curve can be approximated with a quadratic function

$$H = aQ^2 + bQ + c,\ a < 0.$$

For each pump we have the data for the water level in wells and the water flow. Fitting that data one can get an approximate $Q-H$ characteristic. Two illustrative measurements for a pump in a well $B36$ are given in Table 1, see also Figure 1.

$H[m]$	0	10	20	30	40	50	60	70	80	90
$Q[l/s]$	25.50	24.80	23.55	22.30	20.75	19.10	17.35	15.35	11.88	8.80
$Q[l/s]$	25.61	24.57	23.33	22.08	20.54	18.95	17.20	15.05	11.63	8.40

Table 1. Two measurements for the pump in a well $B36$.

The main question is how to calculate the water flow at each pump. Water flow depends on the pressure at each pump. For each pump we distinguish dynamic and static levels of the water. Dynamic pressure (dynamic level of the water), H_{dyn} is the difference between the level of water in the well and ground level, while static pressure H_{sta}, represents pressure in the main pipeline of water system. Clearly

$$Q = Q(H),$$

where $H = H_{dyn} + H_{sta}$.
The level of water head in Table 1 represents the dynamic level of water H_{dyn} plus the static level of water H_{sta}. In our model we assume that the pressure in the main pipeline is approximately $3bar$.

Dynamic level of water depends on the well. But, for the majority of wells in our model the dynamic level is from $20m$ to $25m$. For such $H = H_{dyn} + H_{sta}$, and with a given $Q-H$ characteristic for each pump we can calculate the water flow on each pump.

As we suppose that the maximal dynamic level of water is $25m$ and the minimal level is $20m$, and that $H_{sta} = 30m$, we get $H_{max} = 25 + 30 = 55m$, $H_{min} = 20 + 30 = 50m$.

For such values H_{max} i H_{min} we get Q_{min} and Q_{max}, respectively. In our model we suppose that the pumps work with a fixed charge and water flow at each pump is calculated as

$$Q = \frac{Q_{min} + Q_{max}}{2}.$$

The power data for each pump is given in Table 2.

well	$Q_{min}\,[l/s]$	$Q_{max}\,[l/s]$	$Q\,[l/s]$	$P\,[kW]$
B1	6.28949	6.69163	6.4906	7.5
B2	11.1979	12.0593	11.6286	9.2
B3	8.06727	8.386	8.2266	7.5
B4	7.95415	8.3207	8.1374	7.5
B5	5.51555	6.05393	5.7847	5.5
B6	9.27804	10.7004	9.9892	9.2
B7	4.93792	5.0428	4.9904	5.5
B8	9.75562	10.6943	10.2250	9.2
B9	12.8577	13.5878	13.2228	13
B10	4.67011	4.80571	4.7379	5.5
B11	7.26195	7.74699	7.5045	9.2
B12	14.0959	15.5554	14.8257	15
B14	12.564	13.1828	12.8734	15
B15	11.8411	12.7045	12.2728	13
B21	18.3652	18.9134	18.6393	18.5
B22	17.5501	18.4094	17.9798	18.5
B23	14.3785	15.3727	14.8756	13
B24	18.2656	18.8031	18.5344	18.5
B25	20.1319	20.9004	20.5162	22
B26	10.6657	11.63	11.1479	9.2
B27	7.39161	7.74515	7.5684	7.5
B28	4.57655	4.71105	4.6438	5.5
B29	8.05973	8.44248	8.2511	7.5
B31	7.71703	8.04934	7.8832	7.5
B32	12.9039	13.5293	13.2166	15
B33	4.94405	5.04779	4.9959	5.5
B34	14.8598	15.6736	15.2667	15
B35	18.3441	19.1965	18.7703	22
B36	18.0719	18.9435	18.5077	18.5
B37	13.738	15.187	14.4625	15
B38	4.96378	5.09192	5.0278	5.5
B39	9.09228	10.8459	9.9691	9.2

Table 2. Flow and power of each pump.

3 Loss coefficient and T-junction

The loss coefficient is defined as the adimensional difference in the total pressure between the ends of a straight pipe or respectively other pipe geometries.

We will consider the loss coefficient for a viscous pipe, sudden contraction and $T-$junction.

In a straight viscous pipe the loss of water head is given by the following equation, [2]

$$h_{12} = f\frac{L}{D}\frac{v^2}{2g} = \frac{\triangle p}{\rho g}$$

where $\triangle p$ is pressure drop, f friction coefficient, D pipe diameter, L pipe length, v mean velocity and g gravity acceleration. The previous equation can be written in the form

$$h_{12} = K_{12}\frac{v^2}{2g}, \tag{1}$$

or

$$\triangle p = p_1 - p_2 = \frac{1}{2}\rho K_{12} v^2, \tag{2}$$

where K_{12} is the loss coefficient for a straight viscous pipe. From previous equations we can conclude that loss coefficient has the form

$$K_{12} = f\frac{L}{D}.$$

Now, consider the flow in a pipe with sudden contraction, where fluid flows from the wider section to the narrow smaller section. The loss coefficient for sudden contraction is, [2]

$$K_{12} = 1 - \frac{A_2}{A_1} \tag{3}$$

where A_1 is the cross section area of the wider part and A_2 is the cross section area of the narrow part.

Let us now consider a combining $T-$junction, that is the junction where we have two inflows and one outflow. We are interested in calculating the outflow pressure given the flow rate and two inflow pressures. We will use the loss coefficient for $T-$junction to express the loss coefficient for an elbow pipe.

For such pipe geometry the empirically determined loss coefficient between *entry1* and *exit3*, Figure 2, has the following form, [2]

$$K_{13} = 0.61\left(\frac{v_1}{v_3}\right)^2 + 1 - 2\left(\left(\frac{v_1}{v_3}\right)\frac{Q_1}{Q_3}\cos\alpha' + \left(\frac{v_2}{v_3}\right)\frac{Q_2}{Q_3}\cos\beta'\right), \tag{4}$$

where Q_1, Q_2, Q_3 represents flows and v_1, v_2, v_3 velocities in the pipes denoted in Figure 2 respectively with *entry1*, *entry2*, *exit3*. Until now we know just the mass balance equation

$$Q_1 + Q_2 = Q_3.$$

For α' and β' the following equations hold

$$\alpha' = 1.41\alpha - 0.00594\alpha^2,$$

$$\beta' = 1.41\beta - 0.00594\beta^2.$$

Since α and β are equal 90^o, we have $\alpha' = 78.786^o$, $\beta' = 78.786^o$.

Similarly, between the points *entry2* and *exit3*, see Figure 2, we can write the loss coefficient in the form

$$K_{23} = 0.61\left(\frac{v_2}{v_3}\right)^2 + 1 - 2\left(\left(\frac{v_1}{v_3}\right)\frac{Q_1}{Q_3}\cos\alpha' + \left(\frac{v_2}{v_3}\right)\frac{Q_2}{Q_3}\cos\beta'\right).$$

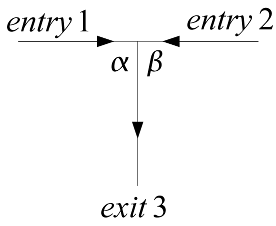

Figure 2: T-junction

In our model we use the modified loss coefficient

$$K_{13} = 0.61\left(\frac{v_1}{v_3}\right)^2 + 1 - 2\left(\left(\frac{v_1}{v_3}\right)\frac{Q_1}{Q_3}\cos\alpha'\right).$$

This modified loss coefficient can be obtained from equation (4) putting $Q_2 = 0$ when we are dealing with an elbow pipe. Clearly $Q_1 = Q_3$ in this case.

The main pipe that brings water from the other wells is denoted by *entry1*, its water flow is Q_1, and the cross section area A, while the water flow from the pipe from observed entrance well is denote by Q_2 and its cross section area by A_2.

In this model we split the pipe into one sudden contraction and one elbow pipe as shown at Figure 3.

The loss coefficient for elbow is

$$K_{23} = 0.61\Big(\frac{v_2}{v_3}\Big)^2 + 1 - 2\Big(\Big(\frac{v_2}{v_3}\Big)\frac{Q_2}{Q_3}\cos\alpha'\Big),$$

where $Q_2 = Q_3$.

We will use the equivalent form

$$K_{23} = 0.61\Big(\frac{A-x}{A_2}\Big)^2 + 1 - 2\Big(\Big(\frac{A-x}{A_2}\Big)\cos\alpha'\Big),$$

since $v_2 = \dfrac{Q_2}{A_2}$ and $v_3 = \dfrac{Q_3}{A-x}$.

Let us now consider the model for the loss coefficient of sudden contraction. The loss coefficient for this geometry is

$$K_{13} = 1 - \frac{x}{A},$$

where A is the cross section area of the main pipe, and x denote the cross section area of the split pipe.

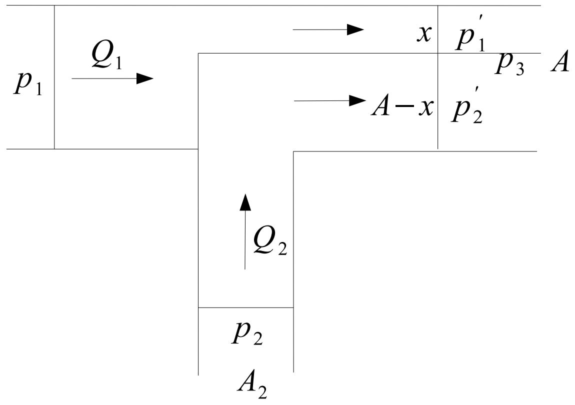

Figure 3: Splitting T-junction into one elbow pipe and one sudden contraction

The pressure loss can be calculated using the following two equations

$$p_1 - p_1' = \frac{1}{2}\rho K_{13}\Big(\frac{Q_1}{x}\Big)^2,$$

$$p_2 - p_2' = \frac{1}{2}\rho K_{23}\Big(\frac{Q_2}{A-x}\Big)^2.$$

The first equation represents the pressure loss at a sudden contraction and the second equation pressure drop in an elbow pipe.

Since $p_1' = p_2'$, (that is the pressure at the same place in the main pipe), we have the equation

$$p_1 - \frac{1}{2}\rho K_{13}\Big(\frac{Q_1}{x}\Big)^2 = p_2 - \frac{1}{2}\rho K_{23}\Big(\frac{Q_2}{A-x}\Big)^2,$$

or

$$p_1 - \frac{1}{2}\rho\Big(1-\frac{x}{A}\Big)\Big(\frac{Q_1}{x}\Big)^2 =$$

$$p_2 - \frac{1}{2}\rho\,\Big(0.61\Big(\frac{A-x}{A_2}\Big)^2 + 1 - 2\Big(\Big(\frac{A-x}{A_2}\Big)\frac{Q_1}{Q_3}\cos\alpha'\Big)\Big)\Big(\frac{Q_2}{A-x}\Big)^2,$$

which we solve for x. For such x, $p_1' = p_2' = p_3$, where p_3 is the pressure after a T-junction.

For our T-junction we have $A = 0.15^2 \cdot \pi$ $[m^2]$, $A_2 = 0.075^2 \cdot \pi$ $[m^2]$. We also apply the condition $p_2 > p_1$, on the pressure since the pressure at a pump is greater than the pressure in the main pipe. Then

$$p_3 = p_1 - \frac{1}{2}\rho\Big(1 - \frac{x}{A}\Big)\Big(\frac{Q_1}{x}\Big)^2$$

and

$$p_3 = p_2 - \frac{1}{2}\rho\Big(0.61\Big(\frac{A-x}{A_2}\Big)^2 + 1 - 2\Big(\frac{A-x}{A_2}\Big)\cos\alpha'\Big)\Big(\frac{Q_2}{A-x}\Big)^2.$$

Now we are ready to state the theorem.

Theorem 3.1 *Let $p_2 > p_1$ and let the water flow Q_2 at each pump be such that $Q_2 > \sqrt{\frac{2(p_2-p_1)}{\rho\cdot 1843}}$. Then there exists a unique $x \in (0, A)$ such that the pressure p_3 is smaller than the pressures p_1 and p_2.*

Proof.

We need to show that there exists a unique $x \in (0, A)$, such that $p_3 < \min\{p_1, p_2\}$ and the following equation

$$p_1 - \frac{1}{2}\rho\Big(1 - \frac{x}{A}\Big)\Big(\frac{Q_1}{x}\Big)^2 = p_2 - \frac{1}{2}\rho\Big(0.61\Big(\frac{A-x}{A_2}\Big)^2 + 1 - 2\Big(\frac{A-x}{A_2}\Big)\cos\alpha'\Big)\Big(\frac{Q_2}{A-x}\Big)^2 \tag{5}$$

is satisfied. First we will show that $p_3 < p_1$ and $p_3 < p_2$. The inequality $p_3 < p_1$ follows from

$$p_3 = p_1 - \frac{1}{2}\rho\Big(1 - \frac{x}{A}\Big)\Big(\frac{Q_1}{x}\Big)^2 < p_1,$$

which is obviously true for all $x \in (0, A)$. The second condition which must be satisfied is

$$p_1 - \frac{1}{2}\rho\Big(1 - \frac{x}{A}\Big)\Big(\frac{Q_1}{x}\Big)^2 < p_2.$$

That is also true because $x \in (0, A)$ and $p_2 > p_1$.

Let us show that for $x \in (0, A)$ the following two inequalities are satisfied

$$p_3 = p_2 - \frac{1}{2}\rho\Big(0.61\Big(\frac{A-x}{A_2}\Big)^2 + 1 - 2\Big(\frac{A-x}{A_2}\Big)\cos\alpha'\Big)\Big(\frac{Q_2}{A-x}\Big)^2 < p_1, \tag{6}$$

and

$$p_3 = p_2 - \frac{1}{2}\rho\Big(0.61\Big(\frac{A-x}{A_2}\Big)^2 + 1 - 2\Big(\frac{A-x}{A_2}\Big)\cos\alpha'\Big)\Big(\frac{Q_2}{A-x}\Big)^2 < p_2. \tag{7}$$

The inequality (6) is equivalent to

$$p_2 - \frac{1}{2}\rho\Big(0.61\frac{Q_2^2}{A_2^2} + \frac{Q_2^2}{(A-x)^2} - 2\frac{Q_2^2\cos\alpha'}{(A-x)A_2}\Big) < p_1 \tag{8}$$

or

$$\frac{1}{2}\rho\Big(0.61\frac{Q_2^2}{A_2^2} + \frac{Q_2^2}{(A-x)^2} - 2\frac{Q_2^2\cos\alpha'}{(A-x)A_2}\Big) - (p_2 - p_1) > 0.$$

Let

$$y = \frac{1}{A-x}.$$

As $x \in (0, A)$ then $y \in \Big(\frac{1}{A}, \infty\Big)$. Inequality (8) can be written in the form

$$\frac{1}{2}\rho \left(0.61\frac{Q_2^2}{A_2^2} + Q_2^2 y^2 - 2\frac{Q_2^2 \cos\alpha'}{A_2} y\right) - (p_2 - p_1) > 0.$$

The left side of previous inequality is denoted by $f(y)$,

$$f(y) = \frac{1}{2}\rho \left(0.61\frac{Q_2^2}{A_2^2} + Q_2^2 y^2 - 2\frac{Q_2^2 \cos\alpha'}{A_2} y\right) - (p_2 - p_1).$$

The first derivative of $f(y)$ is

$$f'(y) = \rho\, Q_2^2 y - \rho\frac{Q_2^2 \cos\alpha'}{A_2}.$$

Obviously, one zero of this equation is $y = \cos\alpha'/A_2$. Therefore $f'(y) > 0$ for $y > \cos\alpha'/A_2$ i.e. function f is increasing. As we know that $\cos\alpha'/A_2 < A^{-1}$, then function f is increasing for $y \in (A^{-1}, \infty)$. Now we will find the necessary condition for function f to be positive on interval (A^{-1}, ∞). Clearly

$$f\Big(\frac{1}{A}\Big) = \frac{1}{2}Q_2^2 \left(0.61\frac{1}{A_2^2} + \frac{1}{A^2} - 2\frac{\cos\alpha'}{A \cdot A_2}\right) - (p_2 - p_1).$$

For the considered T-junction we know that $A = 0.15^2 \cdot \pi[m^2]$ and $A_2 = 0.075^2 \cdot \pi[m^2]$ so

$$\left(0.61\frac{1}{A_2^2} + \frac{1}{A^2} - 2\frac{\cos\alpha'}{A \cdot A_2}\right) > 1843,$$

or

$$\frac{1}{2}\rho Q_2^2 \cdot 1843 > p_2 - p_1.$$

The assumption

$$Q_2 > \sqrt{\frac{2(p_2 - p_1)}{\rho \cdot 1843}},$$

implies $f(A^{-1}) > 0$ and function f is increasing for $y \in (A^{-1}, \infty)$, so $f(y) > 0$ for $y \in (A^{-1}, \infty)$, and we conclude that equation (6) is satisfied.

Obviously we can deduce

$$-\frac{1}{2}\rho\Big(0.61\Big(\frac{A-x}{A_2}\Big)^2 + 1 - 2\Big(\frac{A-x}{A_2}\Big)\cos\alpha'\Big)\Big(\frac{Q_2}{A-x}\Big)^2 < 0$$

and inequality (7) is satisfied for $x \in (0, A)$. Let us now prove the uniqueness of $x \in (0, A)$ such that

$$p_1 - \frac{1}{2}\rho\Big(1 - \frac{x}{A}\Big)\Big(\frac{Q_1}{x}\Big)^2 = p_2 - \frac{1}{2}\rho\Big(0.61\Big(\frac{A-x}{A_2}\Big)^2 + 1 - 2\Big(\frac{A-x}{A_2}\Big)\cos\alpha'\Big)\Big(\frac{Q_2}{A-x}\Big)^2.$$

First we define the function g by

$$g(y) = p_1 - \frac{1}{2}\rho\Big(\frac{1}{Ay}\Big)\Big(\frac{Q_1}{A-\frac{1}{y}}\Big)^2 - p_2 + \frac{1}{2}\rho\Big(0.61\frac{Q_2^2}{A_2^2} + Q_2^2y^2 - 2\frac{Q_2^2\cos\alpha'}{A_2}y\Big).$$

Clearly $\lim_{y\to\infty} g(y) = \infty$ and $\lim_{y\to\frac{1}{A}} g(y) = -\infty$. Function g is continuous on (A^{-1},∞) so it has at least one zero in that interval.

Now we will show that function g has only one zero in (A^{-1},∞). The equivalent form of g is

$$g(y) = p_1 - \frac{1}{2}\rho\Big(\frac{1}{Ay}\Big)\Big(\frac{Q_1}{A-\frac{1}{y}}\Big)^2 - p_2 + (f(y) + p_2 - p_1),$$

or

$$g(y) = -\frac{1}{2}\rho\Big(\frac{1}{Ay}\Big)\Big(\frac{Q_1}{A-\frac{1}{y}}\Big)^2 + f(y).$$

Function g is increasing on (A^{-1},∞) because

$$g'(y) = f'(y) - \frac{1}{2}\rho\frac{1}{A}Q_1^2\cdot\frac{d}{dy}\left(\frac{1}{y}\left(\frac{1}{\frac{Ay-1}{y}}\right)^2\right),$$

$$g'(y) = f'(y) - \frac{1}{2}\rho\frac{1}{A}Q_1^2\Big(\frac{1-A^2y^2}{(Ay-1)^4}\Big).$$

As we know that $f'(y) > 0$ for (A^{-1},∞), inequality $g'(y) > 0$ is satisfied if and only if

$$1 - A^2y^2 < 0,$$

which is true because $y > A^{-1}$. Therefore there exists one and only one zero of g and the equation (5) has one and only one solution in the interval $(0, A)$. $\square$

This theorem allows a significant simplification of the water distributional system model. Next two tables show that the pressure after $T-$junction p_3 is indeed smaller than p_1 and p_2 and that pressure drop is less than $0.001bar$.

Calculation for that T-junction is represented in next two tables. In Table 3 we put a fixed flow rate from the observed well and from the pipe while we change the pressure. Table 4 shows the pressure p_3 if we change the flow rate while pressures p_1 and p_2 are fixed.

$p_1[bar]$	$p_2[bar]$	p_3[bar]
3.00	3.10	2.9998547494
3.00	3.30	2.9999191259
3.00	3.50	2.9999379925
3.00	4.00	2.9999565840
3.00	5.00	2.9999695054

Table 3. Pressure p_3 if $Q_1 = 50l/s$ and $Q_2 = 12l/s$.

$Q_1[l/s]$	$Q_2[l/s]$	$p_3[bar]$
50	5	2.9999746553
50	15	2.9999218052
100	5	2.9998986291
100	15	2.9996872974
10	20	2.9999957641

Table 4. Pressure p_3 if $p_1 = 3.00\,bar$ and $p_2 = 3.50\,bar$.

From Table 3 and Table 4 we can see that the pressure loss at the T-junction is less than $0.001\,bar$, i.e. the pressure loss is negligible. As in this model we have 32 wells, the pressure loss at all T-junctions is around $28 \cdot 0.001\,bar$, because the maximal number of T-junctions is 28 if all pumps work at the same time.

4 The Optimization Model

Pump scheduling is a process of choosing which of the available pumps are to be used and for which periods of a day the pumps are to be in use.

The model consists of 32 pumps which are presented in Figure 1. The input data for this model is the water demand curve. Pumping capacities are supposed to be constant during any time interval, without any additional costs. Also, for the time period of $1hour$, each pump combination gives a fixed discharge, and uses a fixed amount of electric energy and fixed power.

For this model, the following assumptions are introduced:

A1 Water source supplies enough water at any time and without additional costs;

A2 Pressure in the main pipeline is always between minimal and maximal. This assumption is justified by the results of Section 3;

A3 The water demand curve and the characteristics of pumps (discharge and power) are considered.

The objective function in our optimization model includes electrical energy cost and maintenance cost. Electrical energy cost E is the price of consumed energy by all pumps during the optimization period. In our model the electrical energy cost is replaced by equivalent measure - the power of pumps. The main maintenance cost is modeled by a switch on/off of a pump and our objective is to keep the number of switches as small as possible. Pumps maintenance cost, denoted by M, can be equally important as the electrical energy cost.

So, our problem is to minimize cost function C

$$\min\ C = \min(E + M).$$

We assume that all pumps work with a fixed flow Q_i, $i = 1, ..., 32$. The power vector of all pumps is $\mathbf{P} = (P_1, P_2, ..., P_{32})$. The mass balance implies the constraint

$$Q_{min} \leq \sum_{i=1}^{32} Q_i \leq Q_{max}.$$

Our problem is to find the combination of pumps which will be working in a specific time interval such that the total water flow is in (Q_{min}, Q_{max}) with minimal cost.

We make an optimization model for one day. In our model we suppose that the shortest period for each combination of pumps is one hour, i.e. a pump can be switched off/on after being active/inactive for at least one hour.

We divide one day into k time periods. For each period we require

$$Q_{min}^j \leq \sum_{i=1}^{32} Q_i \leq Q_{max}^j, \quad j = 1, ..., k,$$

where Q_{min}^j and Q_{max}^j is respectively, minimal and maximal water flow in each period $j = 1, ..., k$. These values are obtained from the water demand curve based on historical data.

Pumps can be turned on or off only at the beginning of each time interval. Different time interval could be considered if needed, we assumed that 1 hour is the minimal time for one pump combination. In this way number of possible pump combinations is 2^{32}. But due to the problem constraints, a large number of possible combinations are not feasible.

For the first optimization period we know

$$Q_{min}^1 \leq \sum_{i=1}^{32} Q_i \leq Q_{max}^1$$

where Q_{min}^1 and Q_{max}^1 is minimal and maximal water flow for the first time period. Therefore we are looking for the solution of

$$\min \ \mathbf{P}^\mathsf{T} \mathbf{c}^1,$$

where $\mathbf{c}^1$ represent the corresponding pump combination.

The pump combination $\mathbf{c}^1 = (c_1^1, c_2^1, ..., c_{32}^1)$, is

$$c_i^1 = \begin{cases} 0 & , & \text{switched off} \\ 1 & , & \text{switched on} \end{cases}, i = 1, ..., 32.$$

So, for the first time period we have the constrained linear programming problem

$$\begin{aligned} \min \ \mathbf{P}^\mathsf{T} \mathbf{c}^1 \\ \text{s. t. } \mathbf{A}\mathbf{c}^1 \leq \mathbf{b}_1, \\ c_i^1 \in \{0, 1\}, i = 1, ..., 32. \end{aligned} \tag{9}$$

where $\mathbf{A}$ is the matrix which has $Q_1, ..., Q_{32}$ in the first row and $-Q_1, ..., -Q_{32}$ in the second row while $\mathbf{b}_1 = \left[Q^1_{max}, -Q^1_{min}\right]^T$. Solution (9) is then taken as the initial approximation for the second period.

In the second period we have an additional condition for pump combination $\mathbf{c}^2$. Since the desired water amount could be achieved in many different ways, maintenance cost calls for the smallest possible number of switch on/of i.e. we require that $\|\mathbf{c}^2 - \mathbf{c}^1\|_1$ be as small as possible. the components of $\mathbf{c}^2$ are again

$$c_i^2 = \begin{cases} 0 & , \text{ pump doesn't work} \\ 1 & , \text{ pump works} \end{cases}, i = 1, ..., 32.$$

Let $\mathbf{y}^2 = \mathbf{c}^2 - \mathbf{c}^1$. Clearly there are 4 possible cases for y_i^2 as shown in Table 5.

c_i^1	c_i^2	y_i^2
0	1	1
1	0	-1
0	0	0
1	1	0

Table 5. All possible combinations of $\mathbf{c}^1$ and $\mathbf{c}^2$.

In order to state the minimal maintenance costs we define $\mathbf{S} = (s_1, s_2, ..., s_{32})$ whose components are

$$s_i = \begin{cases} \sigma & , \quad c_i^1 = 0 \\ -\sigma & , \quad c_i^1 = 1 \end{cases}$$

where $\sigma > 0$ represent the maintenance cost. We assume that the maintenance factor is equal for all pumps. So we will minimize the function $\mathbf{S}^\mathsf{T}\mathbf{y}^2$. Putting both aims together, we have to minimize the function

$$\mathbf{P}^\mathsf{T}\mathbf{c}^2 + \mathbf{S}^\mathsf{T}(\mathbf{c}^2 - \mathbf{c}^1),$$

Since $\mathbf{S}^\mathsf{T}\mathbf{c}^1$ is a constant we have the following optimization problem

$$\begin{aligned} \min\ (\mathbf{P}^\mathsf{T}\mathbf{c}^2 + \mathbf{S}^\mathsf{T}\mathbf{c}^2) \\ \mathbf{A}\mathbf{c}^2 \le \mathbf{b}_2 \\ c_i^2 \in \{0,1\}, i = 1, ..., 32. \end{aligned} \tag{10}$$

where $\mathbf{A}, \mathbf{b}$ are the same as in (9).

If $Q^2_{min} > Q^1_{min}$ then some additional pumps are switched but maintenance costs imply that all pumps which were working in the first time period will continue working in the second time period. Similarly, if $Q^2_{min} < Q^1_{min}$ then some pumps will be switched off and no new pumps will be switched on.

For all other periods of optimization we have the minimization problem

$$\begin{aligned} \min\ (\mathbf{P}^\mathsf{T}\mathbf{c}^{j+1} + \mathbf{S}^\mathsf{T}\mathbf{c}^{j+1}) \\ \mathbf{A}\mathbf{c}^{j+1} \le \mathbf{b}_{j+1} \end{aligned} \tag{11}$$

$$c_i^{j+1} \in \{0,1\}, i = 1, ..., 32.$$

with $\mathbf{b}_{j+1} = \left[Q_{max}^{j+1} - Q_{min}^{j+1}\right]^T$.

The sequence of problems (9)-(11) is solved in MATLAB 7.0 using the built-in function bintprog. This function is performing Branch and Bound method. Let us remark that a solution for a particular day depend on the solution in the first period, i.e. depend on the pump combination $\mathbf{c}^1$. The value of the maintenance parameter in all tests was $\sigma = 100$. The same result was obtained with $\sigma = 10$ and $\sigma = 10^6$ and therefore the particular value of this parameter does not influence the performance of the algorithm.

5 Experimental Results

In this section we test the model from Section 4. Water demand curve is obtained from historical data, see Figure 4.

We tested the model using four different slicing of a day.

- Test 1. We divide a day into 24 periods. The minimal period for one combination of pumps is one hour. The minimal necessary water flow for one hour is the minimal water flow for that hour. We start optimization with minimal registered flow, from 3 a.m to 4 a.m.
- Test 2. A day is divided into 24 periods. Optimization starts from midnight. Like in the previous test, the minimal necessary water flow for one hour is the minimal water flow for that hour, Table 6.
- Test 3. We divide a day into 24 periods. In this test the minimal necessary water flow for one hour is the average registered water flow for that hour.
- Test 4. We split a day into two periods, 8-23 and 23-8. The water demand for 8-23 is much larger then for 23-08. The minimal water flow for these periods are the average flows.

The results are shown at Figures 5-8. Each figure a) shows the minimal and maximal demand curves together with the water flow obtained while b) figures show active pumps (denoted by blue boxes) during 24 hours. The numbering of pumps is the same as at Figure 1. The optimization procedure is obviously yielding good results. For the first three tests the generated amount of water is clearly satisfying the constraints while in the fourth test there is a slight shortage at transition times due to the large difference in the required amount of water. The total power used in each test is given in Table 7 together with the number of working pumps for each test. The amounts of used power in Test 1 and Test 2 are quite similar while Test 3 and Test 4 require larger amounts of power. Given the fact that there is a slight shortage of water in Test 4, a split into two periods only is clearly not the best option. The smallest amount of power is needed in the policy implied by Test 2 and additional 39.2kW, 348.5kW and 293kW are needed for Test 1,3 and 4 respectively. If counting on yearly level these amounts yield differences in hundreds of MW and hence the policy imposed by Test 2 is the best one in terms of energy consumption. The energy cost is roughly proportional to used power although some other factors also pay a role in determining the total amount in money for industrial consumers.

The cost of operation is not just the electricity but also the maintenance that is in our model proportional to the total number of switching each pump on or off. It is quite obvious from Figures 5-8 that the frequency of switching on/off is reasonably similar in all tests. Such behavior is due to the presence of $S^T c^{j+1}$ in the objective function of (11). One can also observe that the policy imposed by Test 2, which is optimal in terms of energy consumption, is requiring the largest amount of active pumps - in one day 25 pumps are supposed to work, while the number of active pumps in other tests is slightly smaller. Therefore the true optimal policy is dependent on the actual relationship between the energy and maintenance costs.

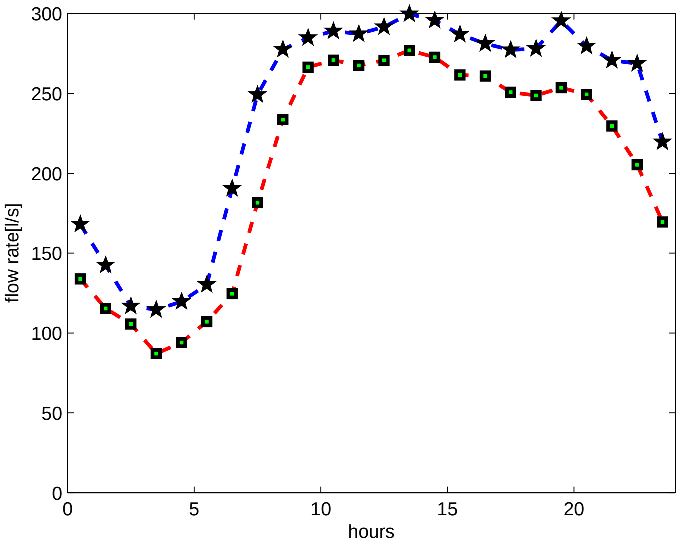

Figure 4: Minimal and maximal water demand

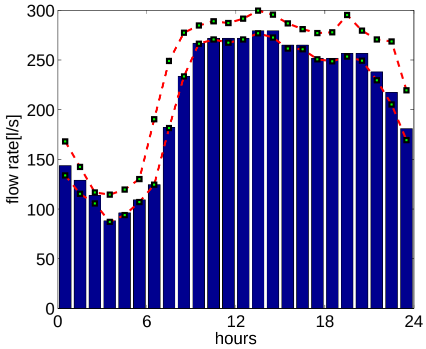

Figure 5a: Test 1 - Demand curve and water flow

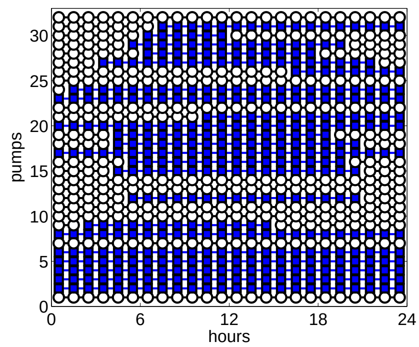

Figure 5b: Active pumps for Test 1

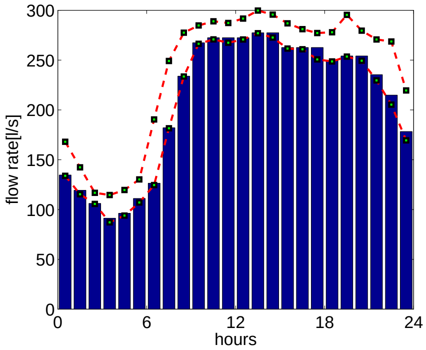

Figure 6a: Test 2 - Demand curve and water flow

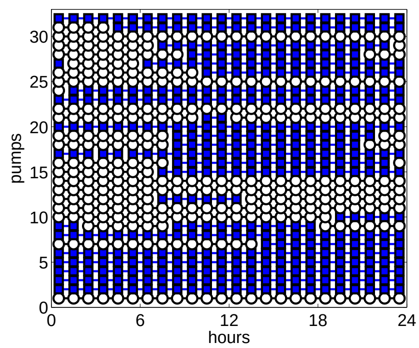

Figure 6b: Active pumps for Test 2

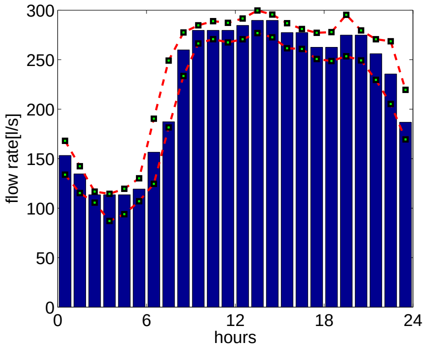

Figure 7a: Test 3 - Demand curve and water flow

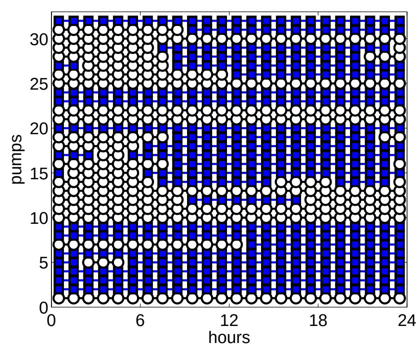

Figure 7b: Active pumps for Test 3

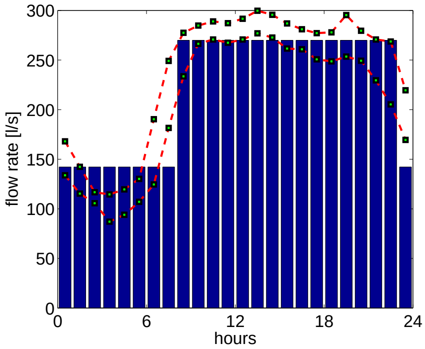

Figure 8a: Test 4 - Demand curve and water flow

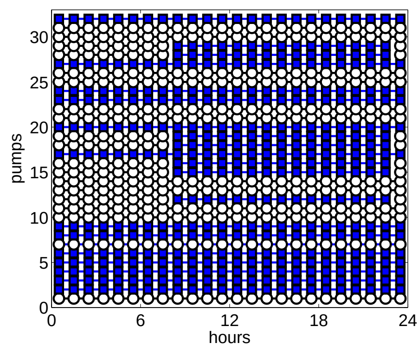

Figure 8b: Active pumps for Test 4

test	used power $[kW]$	number of pumps
test 1	4861.2	23
test 2	4822	25
test 3	5170.5	24
test 4	5115	21

Table 7. Used power and number of pumps

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