Higher Commutators
- Some Results and Open Problems -

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Centralizing Property and Commutators

Definition. Let $\mathbf{A}$ be an algebra, $\alpha_1, \alpha_2, \eta \in \operatorname{Con} \mathbf{A}$. Then we say that α_1 centralize α_2 modulo η if for all polynomials $f(\mathbf{x}_1, \mathbf{x}_2)$ and $\mathbf{a}_1, \mathbf{b}_1, \mathbf{u}, \mathbf{v}$ vectors from $\mathbf{A}$ such that: $\mathbf{a}_1 \equiv \mathbf{b}_1 \pmod{\alpha_1}$, $\mathbf{u} \equiv \mathbf{v} \pmod{\alpha_2}$ and

$$f(\mathbf{a}_1, \mathbf{u}) \equiv f(\mathbf{a}_1, \mathbf{v}) \pmod{\eta},$$

we have

$$f(\mathbf{b}_1, \mathbf{u}) \equiv f(\mathbf{b}_1, \mathbf{v}) \pmod{\eta}.$$

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Definition. (Freese, Gumm, Hagemann, Herrmann, Hobby, Kiss, McKenzie,...) $[\alpha_1, \alpha_2] := \bigwedge\{\eta \in \operatorname{Con} \mathbf{A} \mid C(\alpha_1, \alpha_2; \eta)\}$

Another Definition: Matrix Form

Definition. (R.Freese, R.N.McKenzie) Let $\mathbf{A}$ be an algebra, $\mathbf{a}_1, \mathbf{b}_1 \in \mathbf{A}^n$, $\mathbf{a}_2, \mathbf{b}_2 \in \mathbf{A}^m$ and $\alpha_1, \alpha_2 \in \operatorname{Con} \mathbf{A}$. Then $M_{\mathbf{A}}(\alpha_1, \alpha_2)$ is the subalgebra of $\mathbf{A}^{2\times 2}$ generated by:

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$$\text{if } (x_{11}, x_{12}) \in \eta \text{ then } (x_{21}, x_{22}) \in \eta$$

for all $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in M_{\mathbf{A}}(\alpha_1, \alpha_2)$.

Ternary Case $C(\alpha, \beta, \gamma; \eta)$

Definition. Let $\mathbf{A}$ be an algebra and $\alpha, \beta, \gamma, \eta$ be congruences of $\mathbf{A}$. Then we say that α, β centralize γ modulo η if for every polynomial $f(\mathbf{x}, \mathbf{y}, \mathbf{z})$ and $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{u}, \mathbf{v}$ vectors from $\mathbf{A}$ such that: $\mathbf{a} \equiv \mathbf{b} \pmod{\alpha}$, $\mathbf{c} \equiv \mathbf{d} \pmod{\beta}$, $\mathbf{u} \equiv \mathbf{v} \pmod{\gamma}$ and

$$f(\mathbf{a}, \mathbf{c}, \mathbf{u}) \equiv f(\mathbf{a}, \mathbf{c}, \mathbf{v}) \pmod{\eta}$$

$$f(\mathbf{a}, \mathbf{d}, \mathbf{u}) \equiv f(\mathbf{a}, \mathbf{d}, \mathbf{v}) \pmod{\eta}$$

$$f(\mathbf{b}, \mathbf{c}, \mathbf{u}) \equiv f(\mathbf{b}, \mathbf{c}, \mathbf{v}) \pmod{\eta},$$

we have $f(\mathbf{b}, \mathbf{d}, \mathbf{u}) \equiv f(\mathbf{b}, \mathbf{d}, \mathbf{v}) \pmod{\eta}$.

Higher Centralizing Property and Higher Commutators

Definition. (Bulatov $C(\alpha_1, \ldots, \alpha_n; \eta)$) Let $\mathbf{A}$ be an algebra, $\alpha_1, \ldots, \alpha_n, \eta \in \operatorname{Con} \mathbf{A}$. Then we say that $\alpha_1, \ldots, \alpha_{n-1}$ centralize α_n modulo η if for all polynomials $f(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ and $\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}, \mathbf{b}_1 \ldots, \mathbf{b}_{n-1}, \mathbf{u}, \mathbf{v}$ vectors from $\mathbf{A}$ such that: $\mathbf{a}_i \equiv \mathbf{b}_i \pmod{\alpha_i}$, $1 \leq i \leq n$, $\mathbf{u} \equiv \mathbf{v} \pmod{\alpha_n}$ and

$$f(\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}, \mathbf{u}) \equiv f(\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}, \mathbf{v}) \pmod{\eta},$$

for all $(\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}) \in \{\mathbf{a}_1, \mathbf{b}_1\} \times \cdots \times \{\mathbf{a}_{n-1}, \mathbf{b}_{n-1}\}$ and $(\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}) \neq (\mathbf{b}_1, \ldots, \mathbf{b}_{n-1})$, we have

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Definition. (Bulatov $C(\alpha_1, \ldots, \alpha_n; \eta)$) Let **A** be an algebra, $\alpha_1, \ldots, \alpha_n, \eta \in \mathrm{Con}\,\mathbf{A}$. Then we say that $\alpha_1, \ldots, \alpha_{n-1}$ centralize α_n modulo η if for all polynomials $f(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ and $\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}, \mathbf{b}_1 \ldots, \mathbf{b}_{n-1}, \mathbf{u}, \mathbf{v}$ vectors from **A** such that: $\mathbf{a}_i \equiv \mathbf{b}_i \pmod{\alpha_i}$, $1 \le i \le n$, $\mathbf{u} \equiv \mathbf{v} \pmod{\alpha_n}$ and

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Higher Commutator Relations

Definition. (J. Shaw, 2008) Let $\mathbf{A}$ be an algebra and $\alpha_1, \ldots, \alpha_n \in \operatorname{Con} \mathbf{A}$. Then $M_{\mathbf{A}}(\alpha_1, \ldots, \alpha_n)$ is the subalgebra of $\mathbf{A}^{2^{n-1} \times 2}$ generated by:

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$$\begin{pmatrix} a_1 & a_1 \\ \vdots & \vdots \\ a_1 & a_1 \\ b_1 & b_1 \\ \vdots & \vdots \\ b_1 & b_1 \end{pmatrix}, \ldots, \begin{pmatrix} a_{n-1} & a_{n-1} \\ b_{n-1} & b_{n-1} \\ \vdots & \vdots \\ \vdots & \vdots \\ a_{n-1} & a_{n-1} \\ b_{n-1} & b_{n-1} \end{pmatrix}, \begin{pmatrix} a_n & b_n \\ \vdots & \vdots \\ \vdots & \vdots \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix}$$

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such that $(a_i, b_i) \in \alpha_i$ for all $i \in \{1, \ldots, n\}$.

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$$(x_{i1}, x_{i2}) \in \eta \text{ for all } i \in \{1, \ldots, 2^{n-1} - 1\}, \text{ then } (x_{2^{n-1}1}, x_{2^{n-1}2}) \in \eta$$

$$\text{for all } \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ \vdots & \vdots \\ \vdots & \vdots \\ x_{2^{n-1}1} & x_{2^{n-1}2} \end{pmatrix} \in M_{\mathbf{A}}(\alpha_1, \ldots, \alpha_n).$$

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Two definitions are equivalent!

Absorbing Polynomials

Definition. Let **A** be an algebra, let $k \in \mathbb{N}$, let $p : A^k \longrightarrow A$, let $(a_0, \ldots, a_{k-1}) \in A^k$, and let $o \in A$. Then p is absorbing at $(a_0, \ldots, a_{k-1})$ with value o if for all $(x_0, \ldots, x_{k-1}) \in A^k$ we have: if there is an $i \in \{0, 1, \ldots, k-1\}$ such that $x_i = a_i$, then $p(x_0, \ldots, x_{k-1}) = p(a_0, \ldots, a_{k-1})$, and $p(a_0, \ldots, a_{k-1}) = o$.

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Definition. Let **V** be an expanded group and $n \in \mathbb{N}$. A polynomial $p \in \mathrm{Pol_n}\mathbf{V}$ is absorbing if

$$p(0, x_2, \dots, x_n) = p(x_1, 0, \dots, x_n) = \cdots = p(x_1, x_2, \dots, 0) = 0,$$

for all $x_1, \dots, x_n \in V$.

Definition With Absorbing Polynomials

Proposition. [2] Let **A** be a Mal'cev algebra with a Mal'cev term m, $\alpha_0, \ldots, \alpha_n$ congruences of **A** and $n \geq 0$. Then $[\alpha_0, \ldots, \alpha_n]$ is generated as a congruence by the set

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$$T = \{\big(c(b_0, \ldots, b_n), c(a_0, \ldots, a_n)\big) \mid b_0, \ldots, b_n, a_0 \ldots, a_n \in A,$$

$$\forall i : a_i \equiv_{\alpha_i} b_i, c \in \mathrm{Pol}_{n+1}\mathbf{A} \text{ and}$$

$$c|_{\{a_0,b_0\}\times\cdots\times\{a_n,b_n\}} \text{ is absorbing at } (a_0, \ldots, a_n)\}.$$

Ideals and Congruences

Definition. An ideal of expanded group $(V, +, -, 0, F)$ is a normal subgroup I of the group $(V, +)$ such that $f(\mathbf{a} + \mathbf{i}) - f(\mathbf{a}) \in I$, for all $k \in \mathbb{N}$, all k-ary fundamental operations $f \in F$ and all $\mathbf{a} \in V^k, \mathbf{i} \in I^k$.

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Proposition. Let $\mathbf{V}$ be an expanded group and let $I \in \mathrm{Id}\mathbf{V}$. Then

$$\gamma_V(I) := \{(v_1, v_2) \in V^2 \mid v_1 - v_2 \in I\}$$

is an isomorphism from $(\mathrm{Id}\mathbf{V}, \cap, +)$ to $(\mathrm{Con}\,\mathbf{V}, \wedge, \vee)$

Ternary Case in Expanded Groups

Definition. (S. Scott) If $A, B \in \operatorname{Id} \mathbf{V}$, $\mathbf{V} = \langle V, +, F \rangle$ then the ideal $[A, B]$ is generated by the set

$$\{p(a, b) \mid a \in A, b \in B, p \in \operatorname{Pol}_2 \mathbf{V}$$

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If $A, B, C \in \operatorname{Id} \mathbf{V}$, $\mathbf{V} = \langle V, +, F \rangle$ then the ideal $[A, B, C]$ is generated by the set

$$\{p(a, b, c) \mid a \in A, b \in B, c \in C, p \in \operatorname{Pol}_3 \mathbf{V}$$

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Higher Commutator Ideals

Definition. Let $n \in \mathbb{N}$. In an expanded group $\mathbf{V}$ for $A_1, \ldots, A_n \in \mathrm{Id}\mathbf{V}$ we define the n-ary commutator ideal of $A_1, \ldots, A_n$, in abbreviation $[A_1, \ldots, A_n]_{\mathbf{V}}$, as an ideal of $\mathbf{V}$ generated by

$$\{p(a_1, \ldots, a_n) \,|\, (a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n, p \text{ is absorbing}\}.$$

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Theorem. [2] Let $\mathbf{V}$ be an expanded group and $A_1, \ldots, A_n \in \mathrm{Id}\mathbf{V}$ and $\gamma_V(A_1), \ldots, \gamma_V(A_n) \in \mathrm{Con}\mathbf{V}$ the corresponding congruences of $\mathbf{V}$. Then

$$\gamma_V([A_1, \ldots, A_n]) = [\gamma_V(A_1), \ldots, \gamma_V(A_n)].$$

Generalized Properties of Binary Commutator

Proposition. [2] If **A** is in a congruence permutable variety then

(HC1) $[\alpha_1, \ldots, \alpha_n] \leq \bigwedge_{i=1}^{n} \alpha_i$

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(HC6) If $\eta \leq \alpha_1, \ldots, \alpha_n$, then $[\alpha_1/\eta, \ldots, \alpha_n/\eta] = ([\alpha_1, \ldots, \alpha_n] \vee \eta)/\eta$

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(HC7) $\bigvee_{i\in I}[\alpha_1, \dots, \alpha_{j-1}, \rho_i, \alpha_{j+1}, \dots, \alpha_n] = [\alpha_1, \dots, \alpha_{j-1}, \bigvee_{i\in I} \rho_i, \alpha_{j+1}, \dots, \alpha_n]$.

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In general the equality in (HC8) is not true!

Higher Commutators in Groups

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Proposition. (P. Mayr, 2009) Let $\mathbf{G} = (G, \cdot, ^{-1}, 1)$ and $n \geq 2$. If $N_1, \ldots, N_n$ are normal subgroups of $\mathbf{G}$, then

$$[N_1, \ldots, N_n] = \prod_{\pi \in S_n} [\ldots[[N_{\pi(1)}, N_{\pi(2)}], N_{\pi(3)}], \ldots, N_{\pi(n)}].$$

Higher Commutators in Rings

Proposition. Let $\mathbf{R} = (R, +, \cdot, -, 0)$ be a ring, let $n \geq 2$ and let $J_1, \ldots, J_n$ be ideals of $\mathbf{R}$. Then:

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Proposition. (P. Mayr, 2009) Let $\mathbf{R} = (R, +, \cdot, -, 0)$ be a ring, let $n \geq 1$ and let $J_1, \ldots, J_n$ be ideals of $\mathbf{R}$. Then:

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Important Example

Higher commutators can not be ontained by composing binary commutators in general!

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Example:

$$[V,[V,V]] \neq [V,V,V] \text{ for } \mathbf{V} = \langle \mathbb{Z}_4, +_4, 2xyz \rangle$$

Multilinear Expanded Groups

Definition. Let $(V, +, -, 0, F)$ be an expanded group and $k \in \mathbb{N}$. An operation $f : V^k \to V$ is called multilinear if

$$f(x_1, \ldots, x_{i-1}, y + z, x_{i+1}, \ldots, x_k) =$$

$$= f(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_k) + f(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_k)$$

for every $i \in \{1, \ldots, k\}$, and all $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k, y, z \in V$.

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for every $i \in \{1, \ldots, k\}$, and all $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k, y, z \in V$.

Definition. For $k \geq 2$, a multilinear expanded group of degree k is an expanded group $(V, +, -, 0, F)$, where all $f \in F$ are multilinear operations and all operations have at most k arguments.

Commutator Algebra

Definition. For $n \geq 2$ we define $\mathcal{L}$ to be the language with operation symbols $\mathsf{f}_2, \ldots, \mathsf{f}_n$, where each f_i has arity i. We abbreviate $\mathsf{f}_k(\mathsf{x}_1, \ldots, \mathsf{x}_k)$ by $[\mathsf{x}_1, \ldots, \mathsf{x}_k]$ for all $k \in \{2, \ldots, n\}$.

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We define an algebra $\mathbf{I}(\mathbf{V})$ on the language $\mathcal{L}$ whose universe is the set $\mathrm{Id}\mathbf{V}$ such that:

$$\mathsf{f}_k^{\mathbf{I}(\mathbf{V})}(A_1, \ldots, A_k) := [A_1, \ldots, A_k]$$

for each $k \in \{2, \ldots, n\}$ and for all $A_1, \ldots, A_k \in \mathrm{Id}\mathbf{V}$.

Higher Commutators in Multilinear Expanded Groups

Example: $t = \Big[x_3, x_1, \Big[\Big[x_4, [x_7, x_2], [x_6, x_9, x_8], x_{10}\Big], x_5\Big]\Big]$

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Theorem. [3] Let $\mathbf{V}$ be a multilinear expanded group of degree k, let $n \geq 2$, and let $A_1, \ldots, A_n$ be ideals of $\mathbf{V}$. Let T be the set of those terms $\mathrm{t} \in \mathcal{L}$ with the following properties:

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- In t, each of the variables $\mathrm{x}_1, \ldots, \mathrm{x}_n$ occurs exactly once.
- t contains only operation symbols in $\{\mathrm{f}_i \mid i \leq k\}$.

Then $[A_1, \ldots, A_n]$ is the join of all ideals

$$\{\mathrm{t}^{\mathbf{I}(\mathbf{V})}(A_1, \ldots, A_n) \mid \mathrm{t} \in T\}.$$

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An algebra is k-supernilpotent if k is the smallest natural number with the property: $[\underbrace{1,\dots,1}_{k+1}]=0$.

Variety of Supernilpotent Algebras

Theorem. (E. Aichinger, N. Mudrinski - unpublished) The class of all k-supernilpotent algebras is a variety for all $k \in \mathbb{N}$.

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Theorem. [2] Let $\mathbf{A}$ be a finite nilpotent algebra of finite type that generates a congruence modular variety. Then, $\mathbf{A}$ factors as a direct product of algebras of prime power cardinality if and only if $\mathbf{A}$ is a supernilpotent Mal'cev algebra.

Supernilpotent in Multilinear Expanded Groups

Theorem. [3] Let $n, k \in \mathbb{N}$ and let $\mathbf{V}$ be a multilinear expanded group of degree n that is nilpotent of class k. Then, $\mathbf{V}$ is n^k-supernilpotent.

Polynomial Completeness in Groups

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Theorem. (P.Mayr, 2009) Let **G** be a finite group all whose Sylow subgroups are abelian. Then $f : G^k \to G$, $k \in \mathbb{N}$ is polynomial iff f preserves all subgroups of $\mathbf{G}^{max\{4,|G|\}}$ that contain $\{(g,\ldots,g) \in G^{max\{4,|G|\}} \mid g \in G\}$.

Polynomial Interpolation in Rings

Approach: We check weather f can be interpolated by a polynomial function in arbitrarily many places.

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Theorem. (P. Mayr, 2009) Let $\mathbf{R}$ be a finite local ring with 1, and let $n \in \mathbb{N}_0$ be such that Jacobson radical J satisfies $J^{n+1} = 0$. Then a function $f : R^k \to R$ is a polynomial on $\mathbf{R}$ iff for all $S \subseteq R^k$ with $|S| \leq |R|^n$ there exists a polynomial function p on $\mathbf{R}$ such that $f|_S = p|_S$.

Decidability of Affine Completeness

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An algebra $\mathbf{A}$ is affine complete if it is k-affine complete for every $k \geq 1$.

Theorem. [2] There is an algorithm that decides whether a finite nilpotent algebra of finite type that is a product of algebras of prime power order and generates a congruence modular variety is affine complete.

Identity Checking Problem

Let **A** be an algebra.

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Theorem. [2] The polynomial equivalence problem for a finite nilpotent algebra **A** of finite type that is a product of algebras of prime power order and generates a congruence modular variety has polynomial time complexity in the length of the input terms.

Constantive Mal'cev Clones

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If R is a set of relations on A, we denote the set of all the operations on A that preserve all relations from the set R by $\mathrm{Comp}(A, R)$.

Finitely Related

Theorem. [1] Let **A** be a finite Mal'cev algebra. If there exists an $n \geq 2$ such that $[\underbrace{1, \ldots, 1}_{n}] = 0$, then

$$\operatorname{Pol} \mathbf{A} = \operatorname{Comp}(A, \operatorname{Inv}^{|A|^n}(A, \operatorname{Pol} \mathbf{A})).$$

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Theorem. [1] Let $\mathbf{A}$ be a finite Mal'cev algebra whose congruence lattice is of height at most two. We define $n \geq 2$ to be the smallest natural number such that $[\underbrace{1, \ldots, 1}_{n}] = 0$ if such n exists, otherwise $n := 1$. Then,

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Gumm's Theorem

Theorem. (H.P.Gumm) Let **A** be a Mal'cev algebra. Then **A** is Abelian iff there exist a ring R and **A** is polynomially equivalent to a left R-module.

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Theorem. Let **A** be an n-supernilpotent Mal'cev algebra. Then the polynomial clone of **A** is generated by all polynomials of arity at most $n-1$ and the Mal'cev term.

Special 4-ary Relations

Definition. Let **A** be a Mal'cev algebra, m a Mal'cev polynomial on **A** and $\alpha, \beta, \eta \in \operatorname{Con} \mathbf{A}$

$$\rho(\alpha, \beta, \eta, m) := \{(a, b, c, d) \in A^4 \mid a \equiv b \pmod{\alpha},$$
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$$\mathrm{Cen}(\mathbf{A}, m) := \{\rho(\alpha, \beta, \eta, m) \mid \alpha \text{ centralizes } \beta \text{ modulo } \eta\}$$

Commutator Preserving Functions

Lemma. Let **A** be a Mal'cev algebra, m a Mal'cev polynomial on **A** and $f : A^k \to A$. Then the following are equivalent:

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Corollary. Let **A** be a Mal'cev algebra and m a Mal'cev polynomial on **A**. Then all commutator preserving functions of A form a clone.

Some Open Problems

Is it the same true for higher commutators:

Do the functions that preserve the higher commutators of a Mal'cev algebra form a clone?

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Do the functions that preserve the higher commutators of a Mal'cev algebra form a clone?

Is there a generalization of the set of relations $\mathrm{Cen}(\mathbf{A}, m)$ for higher commutators?

Partial solution in Expanded Groups

Theorem. (E.Aichinger, N.Mudrinski - unpublished) Let $\mathbf{V} = (V, +, -, 0, F)$ be the expanded group such that $(V, +)$ is an Abelian group and Con$\mathbf{V}$ is the three element chain $\{0, \alpha, 1\}$. Then the following are equivalent:

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(2) for all $f \in \mathrm{Pol}(\mathbf{V})$, f preserves ρ where

$$\rho = \{(v_1, \ldots, v_8) \mid -v_1 + v_4 - v_5 + v_8 \equiv 0 \pmod{\alpha}$$

$$-v_1 + v_2 - v_7 + v_8 \equiv 0 \pmod{\alpha}$$

$$-v_1 + v_2 - v_3 + v_4 \equiv 0 \pmod{\alpha}$$

$$v_1 - v_2 + v_3 - v_4 + v_5 - v_6 + v_7 - v_8 = 0\}.$$

Congruence Modular Varieties

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Open: (HC4)-(HC8)?

Maximality

Theorem. (R.Freese, R.N.McKenzie) Let **A** be an algebra in congruence modular variety. Then binary commutator operation $[,] : \mathrm{Con}\,\mathbf{A} \times \mathrm{Con}\,\mathbf{A} \to \mathrm{Con}\,\mathbf{A}$ is the largest operation that satisfies:

- $[\alpha, \beta] \leq \alpha \wedge \beta$
- $[(\alpha \vee \theta)/\theta, (\beta \vee \theta)/\theta] = ([\alpha, \beta] \vee \theta)/\theta.$

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Is it true in congruence modular varieties?

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